q-Virasoro/W algebra at root of unity limit and parafermion

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2. Introduction

2d-4d connection: relation between 2d CFT and 4d gauge theory

2d coset CFT

conformal block

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_u}{\widehat{\mathfrak{sl}}(n)_{r+u}}$$

$\Longleftrightarrow$

SU(n) on $\mathbf{R}^4/\mathbf{Z}_r$

instanton pt. fn.

super Virasoro/W_n (r = 2)
parafermion (general r)
CFT

Ex. n = 2 , r = 2 — Belavin-Feigin, Nishioka-Tachikawa, Bonelli-Maruyoshi-Tanzini, Belavin-Belavin-Bershtein

n = 2, r = 4 — Wyllard, Alfimov-Tarnopolsky parameters q and t

2d CFT q-Virasoro/W block	Awata-Yamada ⇔	**5d SU(n)** instanton pt. fn.
$q \to 1,\ t \to 1$		
2d CFT Virasoro/W block	Alday-Gaiotto-Tachikawa,Wyllard ⇔	**4d SU(n)** instanton pt. fn.
$\boxed{q \to \omega,\ t \to \omega,\ \omega^r=1}$ today's talk		
2d coset CFT coset block	⇔	**SU(n) on $\mathbf{R}^4/\mathbf{Z}_r$** instanton pt. fn.

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_u}{\widehat{\mathfrak{sl}}(n)_{r+u}}$$

The parameter u relates to omega-background.

standpoint: We regard the 2d/5d connection as a parent one.
The 2d/4d connections are obtained from 2d/5d at the root of unity limit of q and t. Itoyama-Oota-R.Y.

Contents

2. q-Virasoro algebra

$q,\ t=q^{\beta},\ p=q/t$

Definition

$\mathcal{T}(z)$: q-Virasoro operator

$$f(w/z)\mathcal{T}(z)\mathcal{T}(w)-f(z/w)\mathcal{T}(w)\mathcal{T}(z)=\frac{(1-q)(1-t^{-1})}{(1-p)}\Big[\delta(pz/w)-\delta(p^{-1}z/w)\Big],$$

$$f(z)=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}z^n\right),\qquad \delta(z)=\sum_{n\in\mathbb{Z}}z^n$$

Shiraishi-Kubo-Awata-Odake, Frenkel-Reshetikhin

- q-deformed Heisenberg algebra

$$[\alpha_n,\alpha_m]=-\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}\delta_{n+m,0},\qquad (n\neq 0),$$

$$[\alpha_n,Q]=\delta_{n,0},$$

- realization

$$\mathcal{T}(z)=:\exp\left(\sum_{n\neq 0}\alpha_n z^{-n}\right):p^{1/2}q^{\sqrt{\beta}\alpha_0}+:\exp\left(-\sum_{n\neq 0}\alpha_n (pz)^{-n}\right):p^{-1/2}q^{-\sqrt{\beta}\alpha_0},$$

$$\underset{q=e^{-h/\sqrt{\beta}}\to 1,}{\Longrightarrow}\qquad \mathcal{T}(z)=2+h^2\left(z^2L(z)+\frac{Q_E^2}{4}\right)+O(h^4)\qquad Q_E=\sqrt{\beta}-\frac{1}{\sqrt{\beta}}$$

$L(z)$: Virasoro operator

q-deformed boson

Using the q-deformed Heisenberg operators, we define the q-deformed boson,

- q-deformed boson

$$\widetilde{\varphi}(z) = \beta^{1/2} Q + 2\beta^{1/2}\alpha_0 \log z + \sum_{n\neq 0} \frac{(1+p^{-n})}{(1-q^n)} \alpha_n z^{-n}$$

- Introduce the deformed screening current and screening charge, defined by

$$S(z) =: e^{\widetilde{\varphi}(z)} : \qquad Q_{[a,b]} = \int_a^b \mathrm{d}_q z\, S(z)$$

Jackson integral: $\displaystyle \int_0^a \mathrm{d}_q z\, f(z) = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k$

3. q → ω and t → ω limit, $\omega^r=1$

- This limit is realized by

$$q=\omega \mathrm{e}^{-(1/\sqrt{\beta})h},\quad t=\omega \mathrm{e}^{-\sqrt{\beta}h},\quad p=q/t=\mathrm{e}^{Q_E h}.\qquad h\to 0$$

$$t=q^{\beta}\ \Rightarrow\ \beta=\frac{rm_-+1}{rm_++1},\qquad m_\pm:\text{non-negative integer}$$

- decompose the q-boson into two parts,

$$\widetilde{\varphi}(z)=\widetilde{\varphi}_0(z)+\widetilde{\varphi}_R(z)$$

$$\widetilde{\varphi}_0(z)=\beta^{1/2}Q+\frac{2}{r}\beta^{1/2}\alpha_0\log z^r+\sum_{n\neq 0}\frac{(1+p^{-nr})}{(1-q^{nr})}\alpha_{nr}\,z^{-nr},$$

$$\widetilde{\varphi}_R(z)=\sum_{\ell=1}^{r-1}\sum_{n\in\mathbb{Z}}\frac{(1+p^{-nk-\ell})}{1-q^{nr+\ell}}\alpha_{nr+\ell}\,z^{-nr-\ell}.$$

- $h\to 0$ limit $\quad \widetilde{\varphi}_0(z)=\beta^{1/2}\phi(w)+O(h),\quad \widetilde{\varphi}_R(z)=\varphi(w)+O(h),\qquad w=z^r$

$$\phi(w)=Q_0+a_0\log w-\sum_{n\neq 0}\frac{a_n}{n}w^{-n},\quad \varphi(w)=\sum_{\ell=1}^{r-1}\varphi^{(\ell)}(w)=\sum_{\ell=1}^{r-1}\sum_{n\in\mathbb{Z}}\frac{\tilde{a}_{n+\ell/r}}{n+\ell/k}w^{-n-\ell/r}.$$

$$[a_m,a_n]=m\delta_{m+n,0},\quad [a_n,Q_0]=\delta_{n,0},$$

$$[\tilde{a}_{n+\ell/r},\tilde{a}_{-m-\ell'/r}]=(n+\ell/k)\delta_{m,n}\delta_{\ell,\ell'}.$$

- the limit of the deformed screening charge

$$\lim_{h\to 0}\frac{(1-q^r)}{(1-q)}Q_{[a,b]}=\int_{a^r}^{b^r}dw\psi_1(w):e^{\sqrt{\beta}\phi(w)}$$ up to normalization

- we have defined

$$\psi_1(w)=\frac{A_r}{w^{(r-1)/r}}\sum_{\ell=0}^{r-1}\omega^\ell:\exp\left\{\sqrt{\frac{2}{r}}\phi^{(\ell)}(w)\right\}:\qquad \phi^{(\ell)}(w)\equiv\varphi(e^{2\pi i\ell}w)$$

A_r: normalization constant

Successively, we can construct $\psi_2(w),\cdots,\psi_{r-1}(w)$

$$\psi_{\ell+1}(w)\equiv\lim_{w'\to w}\frac{(w-w')^{2\ell/r}}{c_{1,\ell}}\psi_1(w')\psi_\ell(w)\qquad(\ell=1,2,\cdots,r-2)$$

$c_{1,\ell}$: constant

In particular, $\psi_1^\dagger(w)\equiv\psi_{r-1}(w)=\dfrac{B_r}{w^{(r-1)/r}}\displaystyle\sum_{\ell=0}^{r-1}\omega^\ell\exp\left\{-\sqrt{\frac{2}{r}}\phi^{(\ell)}(w)\right\}$

B_r: constant

Z_r-parafermion

- ψ_ℓ satisfies the defining relation of $\mathbf{Z}_r$-parafermion

$$\psi_\ell(w)\psi_{\ell'}(w') = \frac{c_{\ell,\ell'}}{(w-w')^{2\ell\ell'/r}} \{\psi_{\ell+\ell'}(w') + \mathcal{O}(w-w')\}, \quad \ell+\ell' < r,$$

$$\psi_\ell(w)\psi^\dagger_{\ell'}(w') = c_{\ell,r-\ell'}(w-w')^{-2\ell(r-\ell')/r} \{\psi_{\ell-\ell'}(w') + \mathcal{O}(w-w')\}, \quad \ell' < \ell$$

$$\psi_\ell(w)\psi^\dagger_\ell(w') = (w-w')^{-2\Delta^{(r)}_\ell} \left\{1 + \frac{2\Delta^{(r)}_\ell}{c^{(r)}}(w-w')^2 T_{\mathrm{PF}}(w') + \mathcal{O}((w-w')^3)\right\}$$

Fateev-Zamolodchikov

T_{PF} is the stress tensor for parafermions.

$$\psi^\dagger_\ell(w) = \psi_{r-\ell}(w)$$

conformal dimension central charge

$$\Delta^{(r)}_\ell = \frac{\ell(r-\ell)}{r}, \qquad c^{(r)} = \frac{2(r-1)}{r+2}, \qquad c^2_{\ell\ell'} = \frac{(\ell+\ell')!(r-\ell)!(r-\ell')!}{\ell!\ell'!(r-\ell-\ell')!r!}$$

- The constants A_r and B_r can be determined by

$$\langle\psi_1(w)\psi^\dagger_1(w')\rangle = \frac{1}{(w-w')^{2(r-1)/r}}.$$

For r=2 case
NS fermion

Stress tensor

- We have

boson (coupled to Q_E) $\quad \phi(w) \quad \Rightarrow \quad T_B(w)$

parafermions $\quad \psi_\ell(w) \quad \Rightarrow \quad T_{\mathrm{PF}}(w)$

The stress tensor of whole system is

$$T(w) = T_B(w) + T_{\mathrm{PF}}(w)$$

For r=2, we have confirmed that the superconformal stress tensor can be obtained from q-Virasoro operator at the limit.

- The central charge is

$$c^{(r)} = 1 - \frac{6Q_E^2}{r} + \frac{2(r-1)}{r+2} = \frac{3r}{r+2} - \frac{6Q_E^2}{r}$$

β is restricted to the rational number $\beta = \dfrac{rm_- + 1}{rm_+ + 1}$

When $m_- = m_+ + 1$, $\quad c^{(r,m)} = \dfrac{3r}{r+2} - \dfrac{6r}{m(m+r)} \quad m = rm_+ + 1$

we can reproduce the unitary series.

Zamolodchikov

4. generalization to q-W_n algebra

q-bosons for q-W_n algebra

Awata-Yamada

We consider the simple Lie algebra $\mathfrak{sl}(n)$

$\mathfrak{h}$: Cartan subalgebra $\qquad C_{ab}$: Cartan matrix

- For the q-W_n algebra, introduce a $\mathfrak{h}$-valued q-deformed boson $\widetilde{\varphi}(z)$ which is defined by

$$\langle e_a, \widetilde{\varphi}(z)\rangle \equiv \widetilde{\varphi}_a(z) = \widetilde{\varphi}_{0,a}(z) + \widetilde{\varphi}_{R,a}(z),$$

e_a: simple root

$a = 1, \cdots, n-1$

$\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbf{C}$

$$\widetilde{\varphi}_{0,a}(z) = \beta^{\pm\frac{1}{2}} Q_a + \frac{1}{r}\beta^{\frac{1}{2}}\alpha_{0,a}\log z^r + \sum_{n\neq 0}\frac{1}{q^{nr/2}-q^{-nr/2}}\alpha_{nr,a}z^{-nr}$$

$$\widetilde{\varphi}_{R,a}(z) = \sum_{\ell=1}^{r-1}\widetilde{\varphi}_{\ell,a}(z) = \sum_{\ell=1}^{r-1}\sum_{n\in\mathbf{Z}}\frac{1}{q^{(nr+\ell)/2}-q^{-(nr+\ell)/2}}\alpha_{nr+\ell,a}z^{-(nr+\ell)}$$

$$[Q_a, \alpha_{0,b}] = C_{ab}$$

$$[\alpha_{n,a}, \alpha_{m,b}] = \frac{1}{n}(q^{n/2}-q^{-n/2})(t^{n/2}-t^{-n/2})C_{ab}(p)\delta_{n+m,0}$$

$$[Q_a, Q_b] = 0, \qquad [\alpha_{0,a}, \alpha_{0,b}] = 0$$

We take the $q \to \omega^k$ and $t \to \omega^k$ limit, then

$$\omega = e^{\frac{2\pi i}{r}}, \quad k, r : \text{coprime}$$

$$\phi_a(w) = Q_{0,a} + a_{0,a} \log w - \sum_{n \neq 0} \frac{1}{n} a_{n,a} w^{-n}$$

$$\beta = \frac{rm_- + k}{rm_+ + k}$$

$$\varphi_a(w) = \sum_{\ell=1}^{r-1} \varphi_{\ell,a}(w), \qquad \varphi_{\ell,a}(w) = \sum_{\ell=1}^{r-1} \sum_{n \in \mathbf{Z}} \frac{1}{n + \ell/r} \widetilde{a}_{n+\ell/r,a} w^{-(n+\ell/r)}$$

- parafermion: for each simple root,

$$\psi_{e_a}(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{\ell=0}^{r-1} \omega^\ell : \exp\left[\sqrt{\frac{1}{r}} \phi_a^{(\ell)}(w)\right] :$$

$$\Rightarrow \quad \psi_\alpha \sim \prod_{a=1}^{n-1} \psi_{e_a}^{n_a}, \quad \text{for} \quad \alpha = \sum_{a=1}^{n-1} n_a e_a \in Q \quad (\text{root lattice})$$

$$\langle \psi_\alpha(w) \psi_{-\alpha}(w') \rangle = (w - w')^{-2 + \frac{\alpha^2}{r}}$$

$$\psi_\alpha^r \sim 1, \qquad \psi_{-\alpha} \sim \psi_\alpha^{r-1}$$

In the case of $\mathfrak{sl}(2)$, $\psi_1(w) = \psi_{e_1}(w)$ is the first $\mathbf{Z}_r$-parafermion.

Stress tensor (general r and k)

- We have

boson (coupled to Q_E)	$\phi_a(w)$	$\Rightarrow$	$T_B(w)$
parafermions	$\psi_\alpha(w)$	$\Rightarrow$	$T_{\rm PF}(w)$

The stress tensor of whole system is

$$T(w) = T_B(w) + T_{\rm PF}(w)$$

- The central charge is

$$\begin{aligned} c_n^{(r)} &= (n-1)\left(1 - n(n+1)\frac{Q_E^2}{r}\right) + \frac{n(n-1)(r-1)}{r+n} \\ &= \frac{r(n^2-1)}{r+n} - n(n^2-1)\frac{Q_E^2}{r}. \end{aligned}$$

$$Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}, \quad \beta = \frac{rm_- + k}{rm_+ + k}$$

Set m=r m_+ +k, m_- = m_+ + s

We have

$$c_n^{(r,m,s)} = \frac{r(m/s-n)(n^2-1)(m/s+n+r)}{m/s(r+n)(m/s+r)}.$$

This is the central charge of the coset model,

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_u}{\widehat{\mathfrak{sl}}(n)_{u+r}}, \qquad u = \frac{m}{s} - n \qquad \text{for} \quad s \neq 0$$

For $s = 0 (\beta = 1)$,

$$c = \frac{r(n^2-1)}{r+n}$$: central charge of $\widehat{\mathfrak{sl}}(n)_r$

4d side

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_u}{\widehat{\mathfrak{sl}}(n)_{u+r}} \quad \Leftrightarrow \quad SU(n) \text{ on } \mathbf{R}^4/\mathbf{Z}_r$$

Because of $\beta = -\frac{\epsilon_1}{\epsilon_2} \quad \Rightarrow \quad \begin{array}{l} \epsilon_1 = \epsilon(u+n+r) \\ \epsilon_2 = -\epsilon(u+n) \end{array}$

- Seiberg-Witten limit $(\epsilon_1, \epsilon_2 \to 0)$

  corresponds to $\epsilon \to 0$

- Nekrasov-Shatashvili limit $(\epsilon_1 \to 0 \ \text{ or } \ \epsilon_2 \to 0)$

  corresponds to $u + r \to -n \ \text{ or } \ u \to -n$

  critical level limit

5. summary

We considered the root of unity limit of the q-Virasoro/W_n algebra.

- the sl(n) type parafermions are obtained.
- the central charge agrees with that of the coset model

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_u}{\widehat{\mathfrak{sl}}(n)_{r+u}}$$

- the parameter p is related to the omega beckground

$$\epsilon_1 = \epsilon(u+n+r), \qquad \epsilon_2 = \epsilon(u+n)$$