# Minimal surfaces in $q$-deformed $AdS_5 \times S^5$

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# Ⅰ. Introduction

## AdS/CFT correspondence

type IIB superstring on $AdS_5 \times S^5$ $\Longleftrightarrow$ $\mathcal{N}=4$ SU($N$) SYM ( large $N$ limit )

A remarkable feature : an integrable structure behind AdS/CFT

- The integrability plays an important role in testing the conjectured relations in the AdS/CFT

e.g. anomalous dimensions, Wilson loops ….

In this talk, we will focus on the classical integrability on the string-theory side

- Type IIB superstring on $AdS_5 \times S^5$ is realized as a coset sigma model

[Metsaev-Tseytlin, '98]

- The existence of Lax pair $\Longrightarrow$ the classical integrablity

[Bena-Polchinski-Roiban, '03]

## Next step

Integrable deformations of the AdS/CFT

- Preserving the integrability while deforming the background (symmetry) in a non-trivial way

- It would be significant to reveal a deeper integrable structure behind gauge/gravity dualities beyond the conformal invariance

- Here we focus on a *q*-deformation of $AdS_5 \times S^5$ superstring

[Delduc-Magro-Vicedo, '13]

# II. $q$-deformation of $AdS_5 \times S^5$ superstring

# Integrable deformations : Yang-Baxter sigma models

## Deformed principle chiral models

$$S = -\frac{1}{2}\int d\tau d\sigma \gamma^{\alpha\beta}\,\mathrm{Tr}\left[g^{-1}\partial_\alpha g\,\underline{\frac{1+\eta^2}{1-\eta R}}\,(g^{-1}\partial_\beta g)\right]$$

[Klimcik, '02,'08]

Integrable deformation

$\eta$ : deformation parameter

$R$ : a solution of modified classical Yang-Baxter equation (mCYBE)

mCYBE (non-split) : $[RX,RY] - R([RX,Y] + [X,RY]) = [X,Y], \qquad \forall X,Y \in \mathfrak{g}$

- The existence of a Lax pair $\Longrightarrow$ classical integrability
- Generalized to symmetric coset models [Delduc-Magro-Vicedo, '13]
- type IIB superstring on $AdS_5 \times S^5$ [Delduc-Magro-Vicedo, '13]

NOTE : Another kind of integrable deformations based on (non-modified) CYBE

[Kawaguchi-Matsumoto-Yoshida, '14] [Matsumoto-Yoshida,'15]

Many *r*-matrices have been identified with solutions of type IIB SUGRA

# $q$-deformed superstring action

[Delduc-Magro-Vicedo, '13,'14]

$$S = -\frac{(1+\eta^2)^2}{2(1-\eta^2)} \int d\tau \int_0^{2\pi} d\sigma\, P_-^{\alpha\beta}\, \mathrm{Str}\left[(g^{-1}\partial_\alpha g)\, d \circ \underline{\frac{1+\eta^2}{1-\eta R_g \circ d}}\, (g^{-1}\partial_\beta g)\right]$$

Integrable deformation

Group element: $g \in SU(2,2|4)$

Deformation parameter : $\eta \in [0,1)$

$$d \equiv P_1 + \frac{2}{1-\eta^2} P_2 - P_3 \qquad R_g = \mathrm{Ad}g^{-1} \circ R \circ \mathrm{Ad}g$$

*R*-operator:
( Drinfeld-Jimbo type )

$$R(X) = \begin{cases} -iX & \text{(if } X \text{ is a positive root)} \\ 0 & \text{(if } X \text{ is a Cartan)} \\ +iX & \text{(if } X \text{ is a negative root)} \end{cases}$$

- The existence of Lax pairs $\Rightarrow$ classical integrablity
- SU(2,2|4) symmetry $\Rightarrow$ $q$-deformed SU(2,2|4)
- kappa-invariance

# A $q$-deformed $AdS_5 \times S^5$ background

- The $q$-deformed metric (in the string frame) and the B-field were derived

[Arutyunov-Borsato-Frolov, '13]

$$ds^2_{\mathrm{AdS}_5} = R^2(1+C^2)^{\frac{1}{2}}\left[\underline{\frac{1}{1-C^2\sinh^2\rho}}(-\cosh^2\rho\, dt^2 + d\rho^2)\right.$$
$$\left. + \sinh^2\rho\left(\frac{1}{1+C^2\sinh^4\rho\sin^2\zeta}(d\zeta^2+\cos^2\zeta\, d\psi_1^2)+\sin^2\zeta\, d\psi_2^2\right)\right]$$

Deformation parameter : $C \equiv \dfrac{2\eta}{1-\eta^2} \in [0,\infty)$

- Some arguments towards the complete SUGRA solution

[Lunin-Roiban-Tseytlin,'14] [Arutyunov-Borsato-Frolov,'15] [Hoare-Tseytlin,'15]

RR couplings fail to satisfy eom of IIB SUGRA, despite the presence of $\kappa$-symmetry

- A possible gauge-theory dual has not been uncovered yet

- A singularity surface (curvature singularity) exists at $\rho_s = \mathrm{arcsinh}\,\dfrac{1}{C}$

## An interesting issue

Revealing the nature of the singularity surface

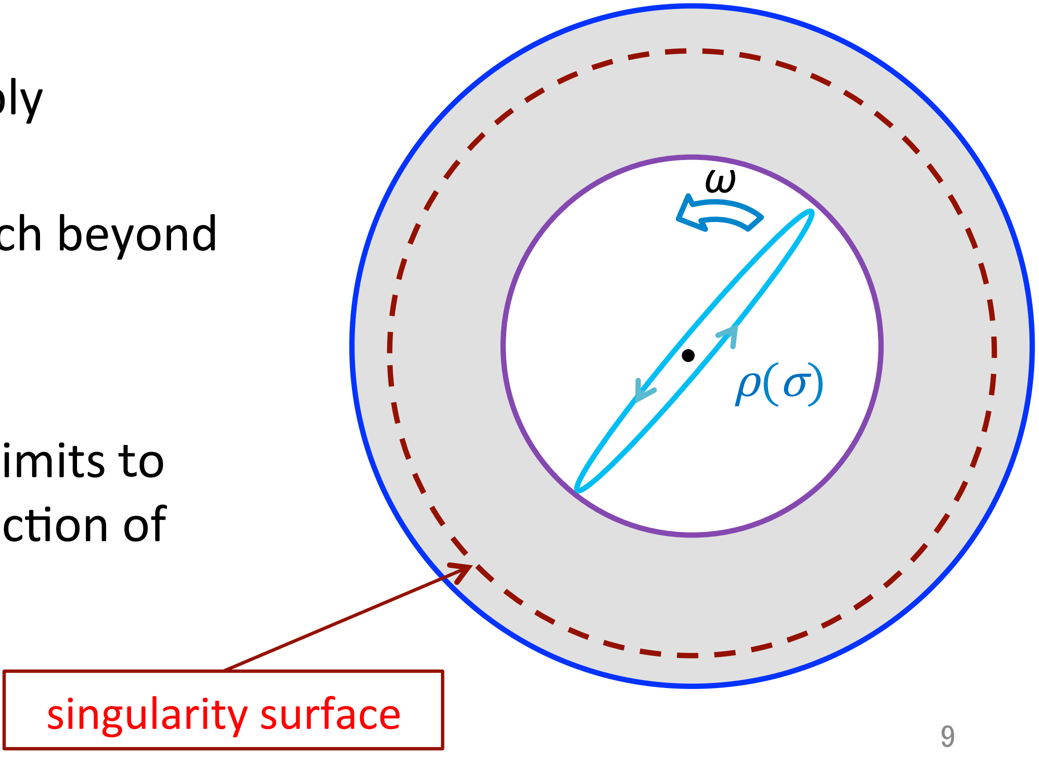
- GKP-like rotating string solutions have been considered as probes.

[Frolov, IGST14] [T.K., Yoshida, '14]

- The Virasoro constraints imply

" GKP-like strings never stretch beyond the singularity surface "

- We considered two kinds of limits to express the energy $E$ as a function of the spin $S$ explicitly

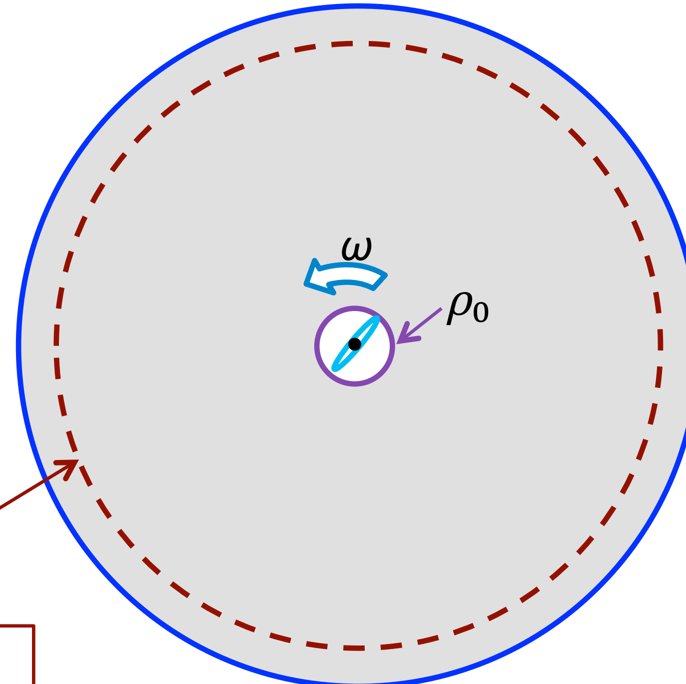


# A short string limit

In the large $\omega$ case : $\omega \gg \kappa$

- The string is confined to a narrow region near the origin of deformed AdS

- Spin behaves as $S/\sqrt{\lambda} \ll 1$




singularity surface

$$E^2 = 2\sqrt{\lambda}(1+C^2)^{\frac{1}{2}}\, S\left[1 + (1+C^2)^{\frac{1}{2}}\frac{2S}{\sqrt{\lambda}} + \ldots\right] \quad \text{with} \quad S/\sqrt{\lambda} \ll 1$$

The undeformed limit $C \to 0$

$$E^2 = 2\sqrt{\lambda}\, S\left[1 + \frac{2S}{\sqrt{\lambda}} + \ldots\right]$$

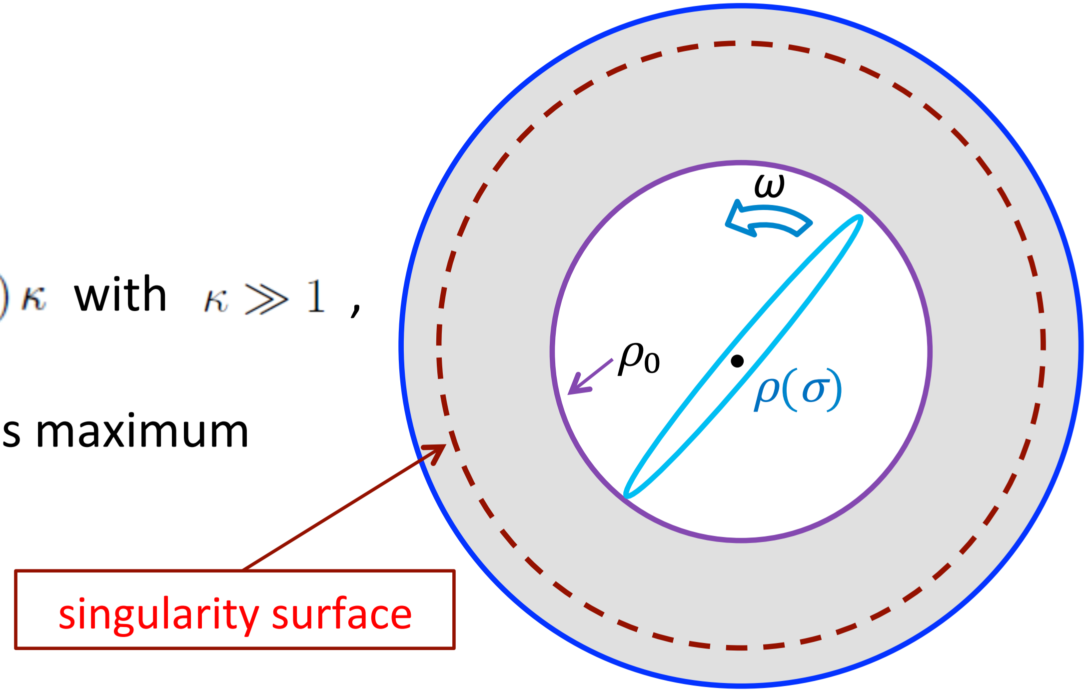
The undeformed result is reproduced precisely

[Gubser-Klebanov-Polyakov, '02]

# A long string limit

In the limit : $\omega \to (\sqrt{1+C^2}+C)\kappa$ with $\kappa \gg 1$ ,

the length of the string becomes maximum




$$E-(\sqrt{1+C^2}+C)S=\frac{2\sqrt{\lambda}}{\pi}\frac{\sqrt{1+C^2}}{C}\Bigg[(\sqrt{1+C^2}+C)\mathrm{arcsinh}\Big(\sqrt{1+\frac{1}{C^2}}-1\Big)^{\frac{1}{2}} -\mathrm{arctanh}\Big(1+\frac{C}{\sqrt{1+C^2}}\Big)^{-\frac{1}{2}}+\mathcal{O}(\epsilon)\Bigg].$$

The result is quite different from the GKP relation

$$E-S=\frac{\sqrt{\lambda}}{\pi}\log\Big[\frac{2\pi}{\sqrt{\lambda}}S\Big]+\ldots.$$ [Gubser-Klebanov-Polyakov, '02]

# III. A holographic setup for $q$-deformed geometry

## Observation

- Classical string solutions such as GKP-like strings cannot stretch beyond the singularity surface

- The causal structure around the singularity surface is very similar to the boundary of the global AdS space

  e.g. For massless particles, it takes infinite affine time to reach the singularity

The d.o.f. are confined into the region enclosed by the singularity surface?

**Our conjecture**

The singularity surface might be treated as the holographic screen

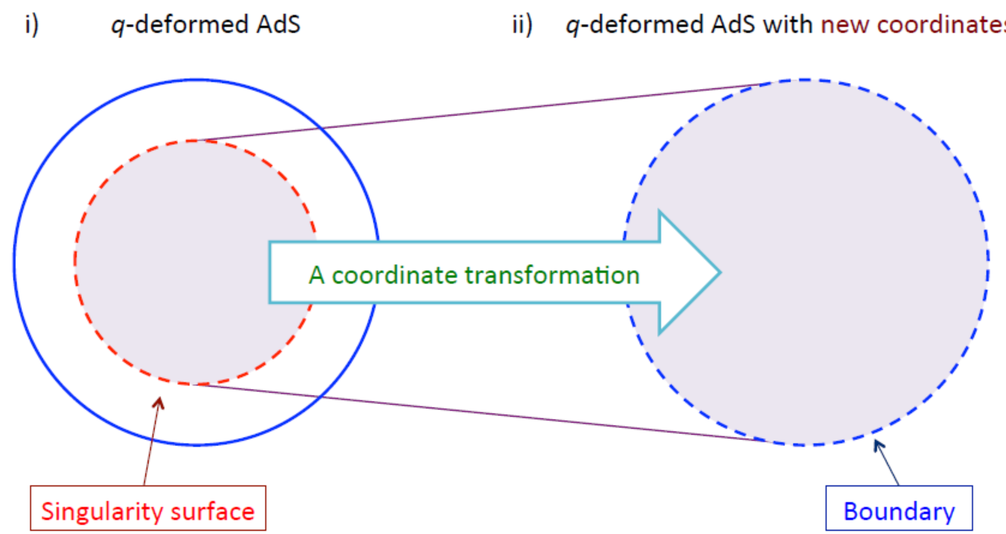
- It would be worth trying to look for a coordinate system which describes spacetime only inside the singularity surface

analogue : the tortoise coordinates for black holes

# Another coordinate system for $q$-deformed $\mathrm{AdS}_5$

- Performing the coordinate transformation : $\dfrac{\cosh\rho}{\sqrt{1-C^2\sinh^2\rho}} \equiv \cosh\chi$

$\rho \in [0, \mathrm{arcsinh}(1/C))$ is mapped to $\chi \in [0,\infty)$




$$ds^2_{\mathrm{AdS}_5} = R^2(1+C^2)^{\frac{1}{2}}\Big[-\cosh^2\chi\, dt^2 + \frac{d\chi^2}{1+C^2\cosh^2\chi} + \frac{(1+C^2\cosh^2\chi)\sinh^2\chi}{(1+C^2\cosh^2\chi)^2+C^2\sinh^4\chi\sin^2\zeta}(d\zeta^2+\cos^2\zeta\, d\psi_1^2) + \frac{\sinh^2\chi\sin^2\zeta\, d\psi_2^2}{1+C^2\cosh^2\chi}\Big]$$

- The singularity surface is now located at infinity of the radial direction

# Ⅳ.　Minimal surfaces

# Minimal surfaces for the *q*-deformed background

- Within the usual AdS/CFT case, Wilson loops are calculated by an area of an open string extending to the boundary of AdS ( minimal surface )
- For the deformed case, we consider minimal surfaces which end on the ‘‘boundary’’ ( singularity surface )
- These solutions reduce to usual solutions in the undeformed limit

To seek for the mysterious gauge-theory dual, minimal surfaces might be a good clue

- For this purpose, it is helpful to use Poincaré coordinates for *q*-deformed $\mathrm{AdS}_5$

$$ds^2_{\mathrm{AdS}_5} = R^2\sqrt{1+C^2}\Bigg[\frac{dz^2+dr^2}{z^2+C^2(z^2+r^2)} + \frac{C^2(z\,dz+r\,dr)^2}{z^2\big(z^2+C^2(z^2+r^2)\big)}$$
$$+\frac{\big(z^2+C^2(z^2+r^2)\big)r^2}{\big(z^2+C^2(z^2+r^2)\big)^2+C^2r^4\sin^2\zeta}\left(d\zeta^2+\cos^2\zeta\,d\psi_1^2\right) + \frac{r^2\sin^2\zeta\,d\psi_2^2}{z^2+C^2(z^2+r^2)}\Bigg]$$

The singularity surface is now located at $z = 0$ ( boundary )

# 1) $q$-deformed $AdS_2$ : a minimal surface with a circular boundary

- We constructed a minimal surface which ends at the boundary of the $q$-deformed AdS with the Poincaré coordinates

whose boundary ( $z$ = 0 ) shape is a circle ( radius = $a$ )

Ansatz : $$z = \sqrt{a^2 - r^2}\,, \qquad r = r(\sigma)\,, \qquad \psi_1 = \psi_1(\tau)\,, \qquad \psi_2 = \zeta = 0\,,$$

with the conformal gauge

Induced metric : $$ds^2_{\mathrm{AdS}_2} = \frac{R^2\sqrt{1+C^2}\,r^2}{(1+C^2)a^2 - r^2}\left[\frac{a^2\,dr^2}{r^2(a^2-r^2)} + d\psi_1^2\right]$$

Solution: $$z = a\tanh\sigma\,, \qquad r = \frac{a}{\cosh\sigma}\,, \qquad \psi_1 = \tau\,.$$

- Evaluating the classical Euclidean action ( area of the minimal surface )

$$
\begin{aligned}
S &= \frac{\sqrt{\lambda}}{4\pi}\sqrt{1+C^2}\int_0^{2\pi} d\tau \int_0^{\infty} d\sigma \left[\frac{2}{\sinh^2\sigma + C^2\cosh^2\sigma}\right] \\
&= \sqrt{\lambda}\frac{\sqrt{1+C^2}}{C}\,\mathrm{arccot}[C]
\end{aligned}
$$

- The minimal surface area can be computed without any regularization in contrast with the undeformed case

  The result would come from the finiteness of the space-like proper distance to the singularity surface

$\Rightarrow$ *q*-deformation may be regarded as a UV regularization

NOTE: An additional contribution (total derivative) coming from the boundary vanishes when $C \neq 0$

## 2) $q$-deformed $AdS_3 \times S^1$ : a cusped minimal surface

- The bc is two lines separated by $\pi - \phi$ on the boundary of the $q$-deformed AdS and $\theta$ on the sphere part
- The string solution fits inside $q$-deformed $AdS_3 \times S^1$ :

$$ds^2_{\mathrm{AdS}_3\times\mathrm{S}^1} = R^2\sqrt{1+C^2}\left[\frac{dz^2+dr^2+r^2d\varphi^2}{z^2+C^2(z^2+r^2)} + \frac{C^2(z\,dz+r\,dr)^2}{z^2\big(z^2+C^2(z^2+r^2)\big)} + d\vartheta^2\right]$$

- As world-sheet coordinates we can take $r$ and $\varphi$ and the ansatz for the other coordinates is

$$z = \frac{r}{f(\varphi)}\,, \qquad \vartheta = \vartheta(\varphi)$$

- The two conserved quantities are

$$p \equiv \frac{1}{E}\,, \qquad q \equiv \frac{J}{E} = \frac{1+C^2(1+f^2)}{f^2}\,\vartheta'$$

- The resulting equations are elliptic and the classical solution is expressed as elliptic integrals of first and third kind

- In the limit : ϕ → $\pi$, the two curves approach antiparallel lines

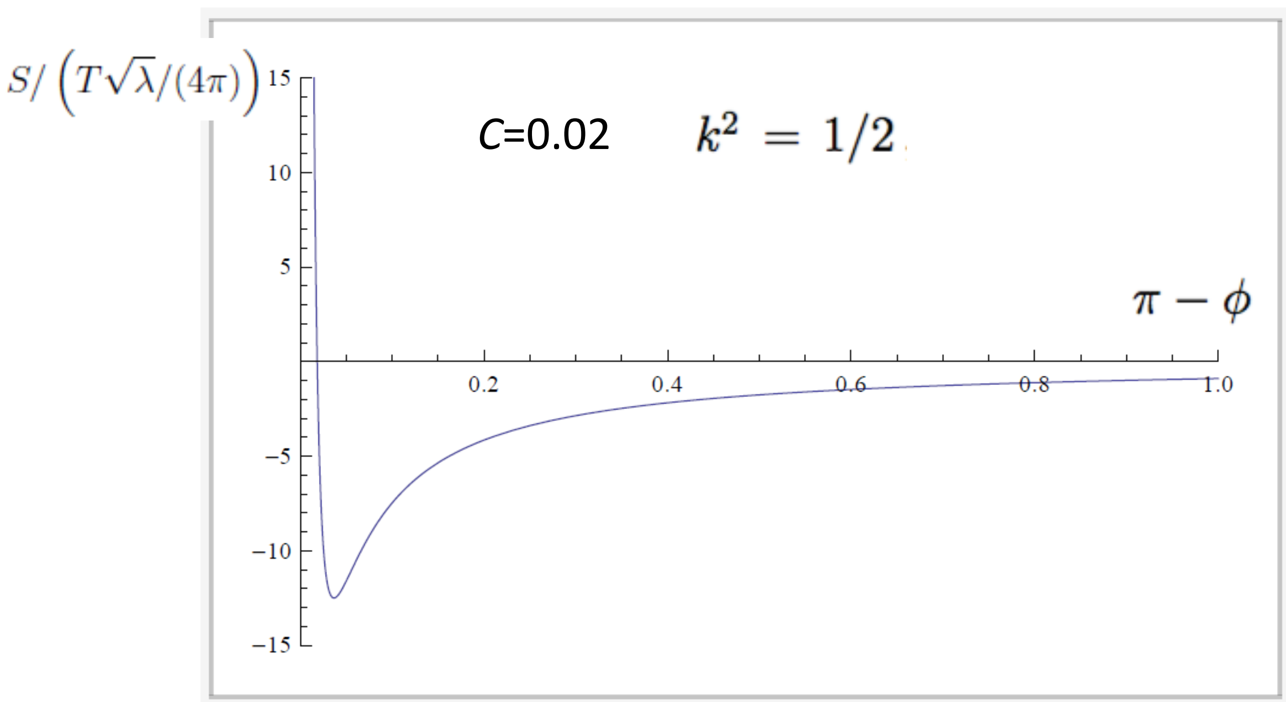
Undeformed case : [Drukker-Forini,'11]

- In the case $C$ << 1, the classical action leads to a repulsive potential

$$S = \frac{T\sqrt{\lambda}}{4\pi}\frac{\left[E\left(k^2\right)-(1-k^2)K\left(k^2\right)\right]^2}{k\sqrt{1-k^2}}\left[-\frac{8}{\pi-\phi}+\frac{16C}{(\pi-\phi)^2}\frac{E\left(k^2\right)-(1-k^2)K\left(k^2\right)}{k\sqrt{1-k^2}\,K\left(k^2\right)}\right]$$

- A strong repulsive force between quark and antiquark if they are close enough

analogy to gravity duals for non-commutative gauge theories



# V. Summary &Discussion

# Summary

We have discussed the nature of the singularity surface of the *q*-deformed $AdS_5\times S^5$ superstring and classical string solutions

- GKP-like strings cannot stretch beyond the singularity surface
- The singularity surface may be regarded as the holographic screen
- We have introduced a coordinate system which describes the spacetime only inside the singularity surface
- Area of minimal surfaces does not have a linear divergence, in contrast with the undeformed case
- A quark-antiquark potential from the *q*-deformed $AdS_5\times S^5$ has an analogy to gravity duals for non-commutative gauge theories

# Outlook

- A possible gauge-theory dual ?

- To find more support for the conjecture of the singularity surface acting as a holographic screen

- One-loop beta function ?

# THANK YOU