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Size scaling of self-gravitating polymers and strings

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arXiv:1506.01160 [hep-th] (to appear in PTEP)
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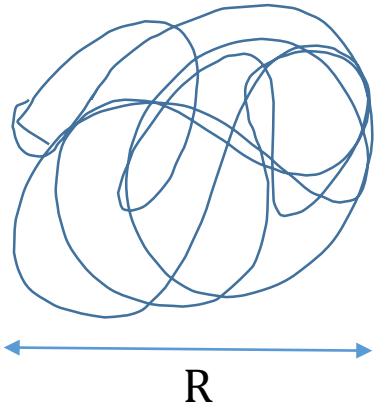
Long Strings and Black Holes

a **free** string of level \tilde{N}

[Mitchell-Turok, Mañes]

$$\langle R^2 \rangle_{\tilde{N}} \sim \int d\sigma \langle :X^2(0, \sigma): \rangle_{\tilde{N}} \sim \sqrt{\tilde{N}} \alpha' \sim \text{Length}$$

Free random walk of $N = \sqrt{\tilde{N}}$ steps



Self-interaction

String/black hole correspondence



Small black hole

$$S_{\text{string}} \cong S_{\text{BH}}$$

[Susskind, Horowitz-Polchinski]

$$R \cong \frac{\ell_s}{g_s^2 \sqrt{\tilde{N}}} \quad (d = 3)$$

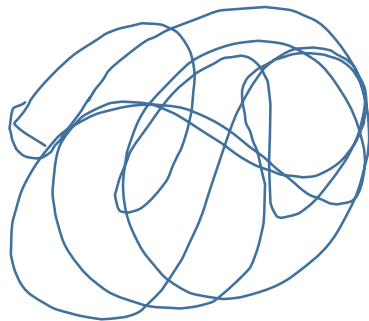
Random walks and Polymers

Typical configuration of long strings \longrightarrow (Interacting) random walks

(Self-)interacting 1D objects

- Cosmic strings
- Vortex lines
- **Polymer chains**
- ...

Coil-globule transition in polymer melt



Coil $\sim N^{\frac{3}{5}}$



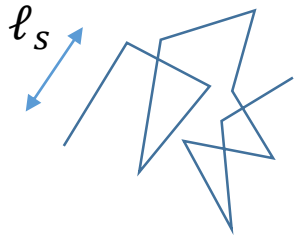
Interactions
(solvent, van der Waals, ...)



Θ -point

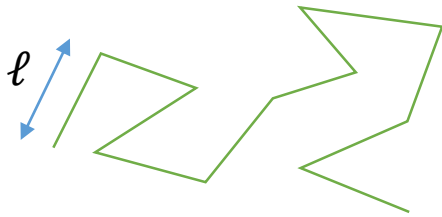
Globule $\sim N^{\frac{1}{3}}$

Strings as self-avoiding walks

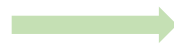


Fundamental strings are modelled by free (Gaussian) random walks with the bond length ℓ_s .

$$R_0 \cong \ell \sqrt{N}$$

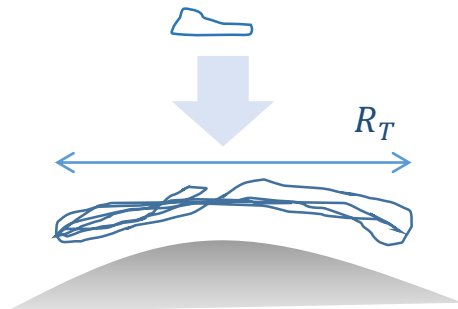


(Real) polymers are modelled by **self-avoiding** random walks with the (Kuhn) bond length ℓ .



$$R \cong \ell N^{\frac{3}{d+2}} \quad (\text{Flory's exponent for real polymers})$$

Note: Repulsive property may emerge in **high-density** regime nonperturbatively.



[Susskind-Lindesay, Ropotenko]

$$L \propto e^t$$

$$R_T \propto t$$



$$R_T \propto e^t$$

Effective Hamiltonian for Polymers

Edwards Hamiltonian

[Edwards-Muthukumar, Doi-Edwards]

$$\beta H = \frac{d}{2\ell^2} \int_0^N \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2 d\sigma + \int_0^N d\sigma_1 \int_0^N d\sigma_2 V(\mathbf{R}(\sigma_1), \mathbf{R}(\sigma_2))$$

Interaction term

$$V = \frac{-g^2 \ell^{d-2}}{|\mathbf{R}(\sigma_1) - \mathbf{R}(\sigma_2)|^{d-2}} + u \ell^d \delta^{(d)}(\mathbf{R}(\sigma_1) - \mathbf{R}(\sigma_2))$$

Newton Interaction

Repulsive force
(Self-Avoiding effect)

Evaluate the size (free case)

$$\langle \mathbf{R}^2 \rangle_{V=0} = \ell^2 N \equiv R_0^2$$

Free random walk: $R_0 = \ell\sqrt{N}$

Phenomenological Free Energy:

$$\beta F \sim \underbrace{-(d-1) \ln R}_{\text{diffusion}} + \underbrace{\frac{R^2}{N\ell^2}}_{\text{elasticity}} - \underbrace{\frac{g^2 \ell^{d-2} N^2}{R^{d-2}}}_{\text{gravity}} + \underbrace{\frac{u \ell^d N^2}{R^d}}_{\text{repulsive (excluded-volume effect)}}$$

Entropic force

Balance determines the size: Free case $R_0 = \ell\sqrt{N}$

Interaction becomes effective in free configurations:

$$\frac{g^2 \ell^{d-2} N^2}{R_0^{d-2}} \sim O(1)$$

$$g_o \sim N^{\frac{d-6}{4}}$$

[Horowitz-Polchinski, Khuri]

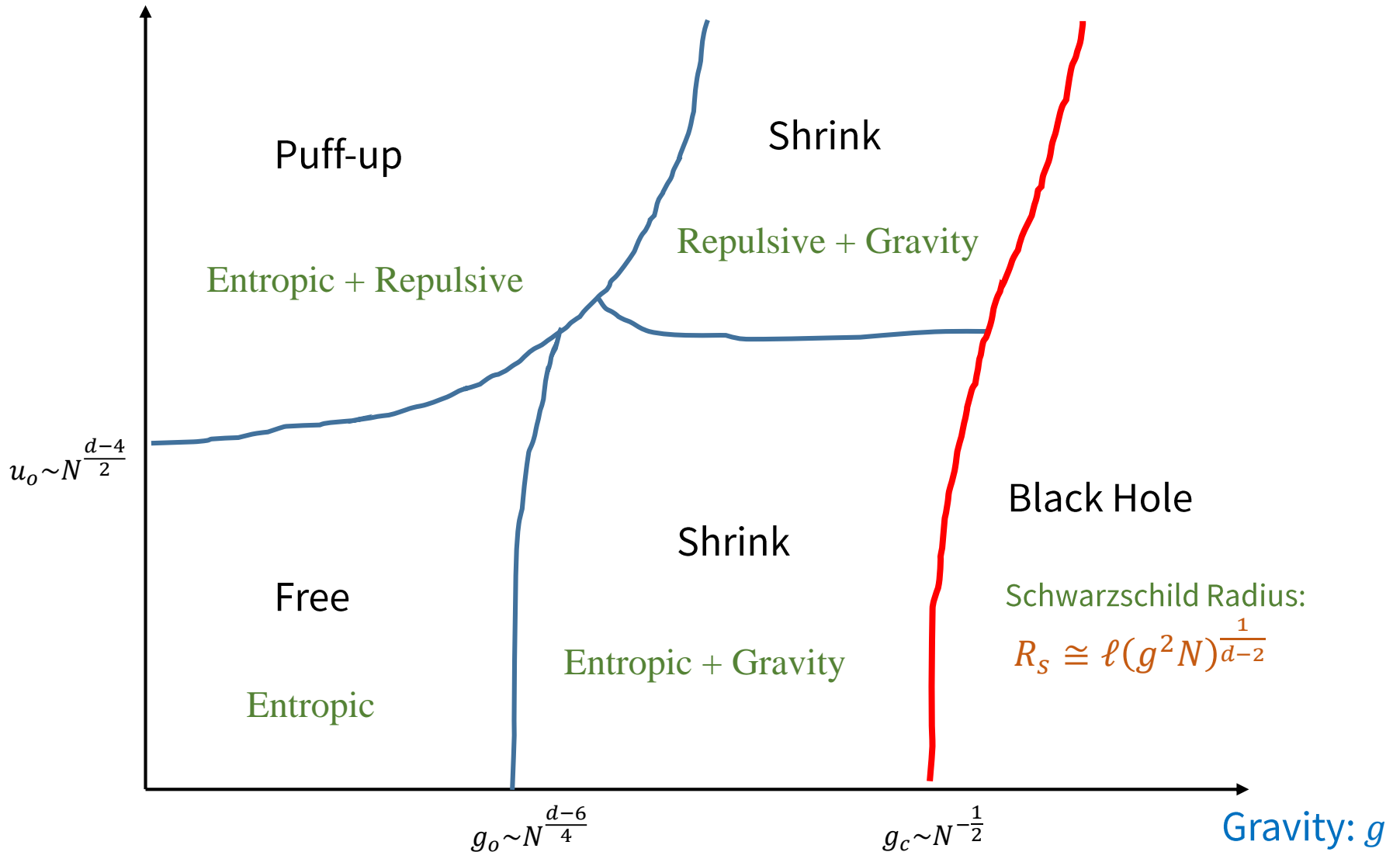
$$\frac{u \ell^d N^2}{R_0^d} \sim O(1)$$

$$u_o \sim N^{\frac{d-4}{2}}$$

($d = 4$ is critical dim.)

Today's Goal

Repulsive force: u



PLAN

- Introduction
- **Two** Approximation Methods and Size Evaluation
- Summary of Size Scaling
- Conclusion

[Doi-Edwards]

Variational Method

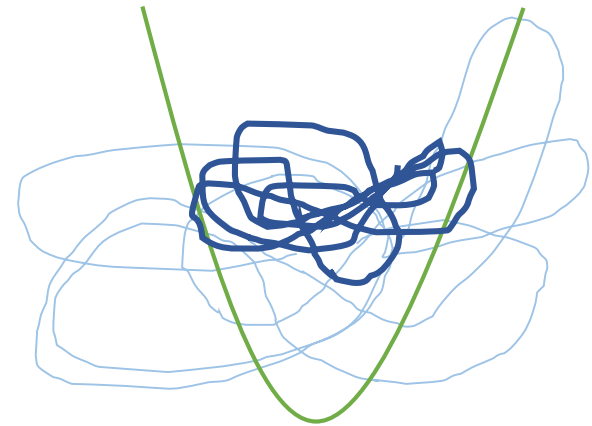
Harmonic Hamiltonian:
$$\beta H_0 = \frac{d}{2\ell^2} \int_0^N d\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2 + \frac{q^2 d}{2\ell^2} \int_0^N d\sigma \mathbf{R}(\sigma)^2$$

Convexity (Jensen's Inequality):
$$\beta F \leq \beta F_0(q) + \langle \beta(H - H_0) \rangle_0$$

$$\beta F \leq qN - N^2 g^2 q \frac{d}{2}^{-1} + N^2 u q \frac{d}{2}$$
 Tune q to minimize RHS

$$\langle R^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N$$

{	$\ell^2 N$	$(q_0 N \ll 1)$
	(free size)	
{	$\frac{\ell^2}{q_0}$	$(q_0 N \geq O(1))$
	(contraction)	



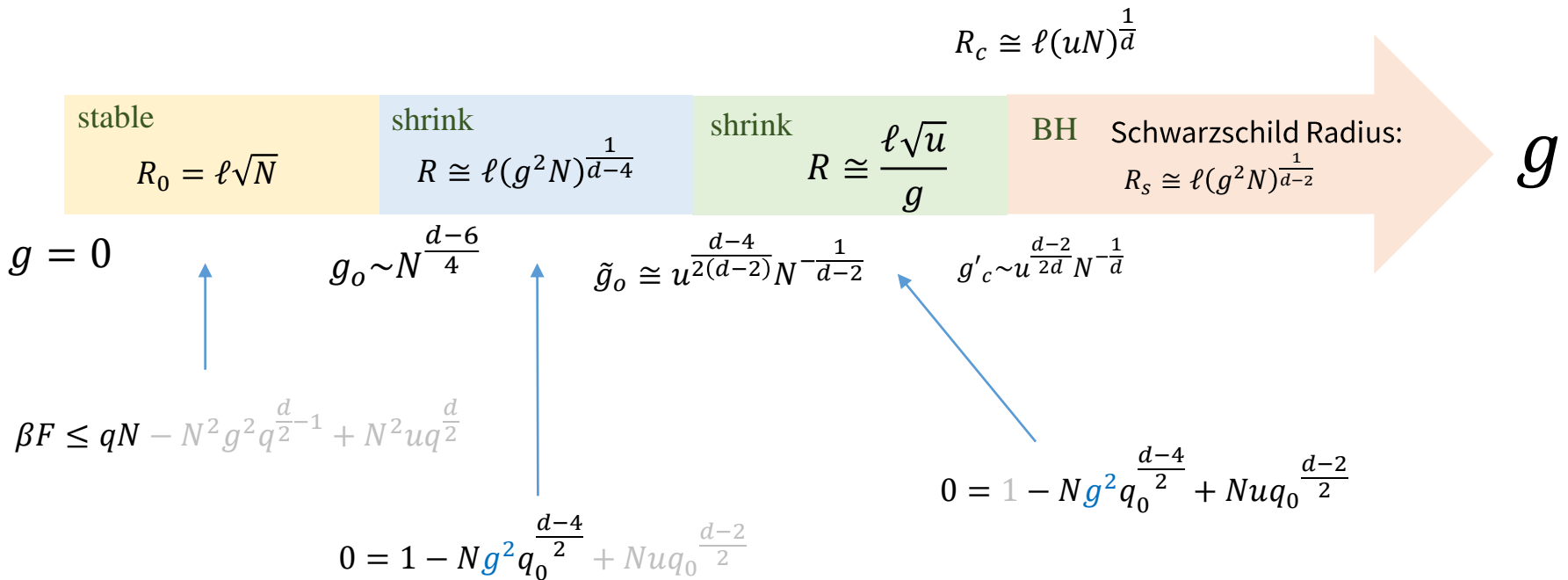
$$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$$

$$(2 < d < 4)$$

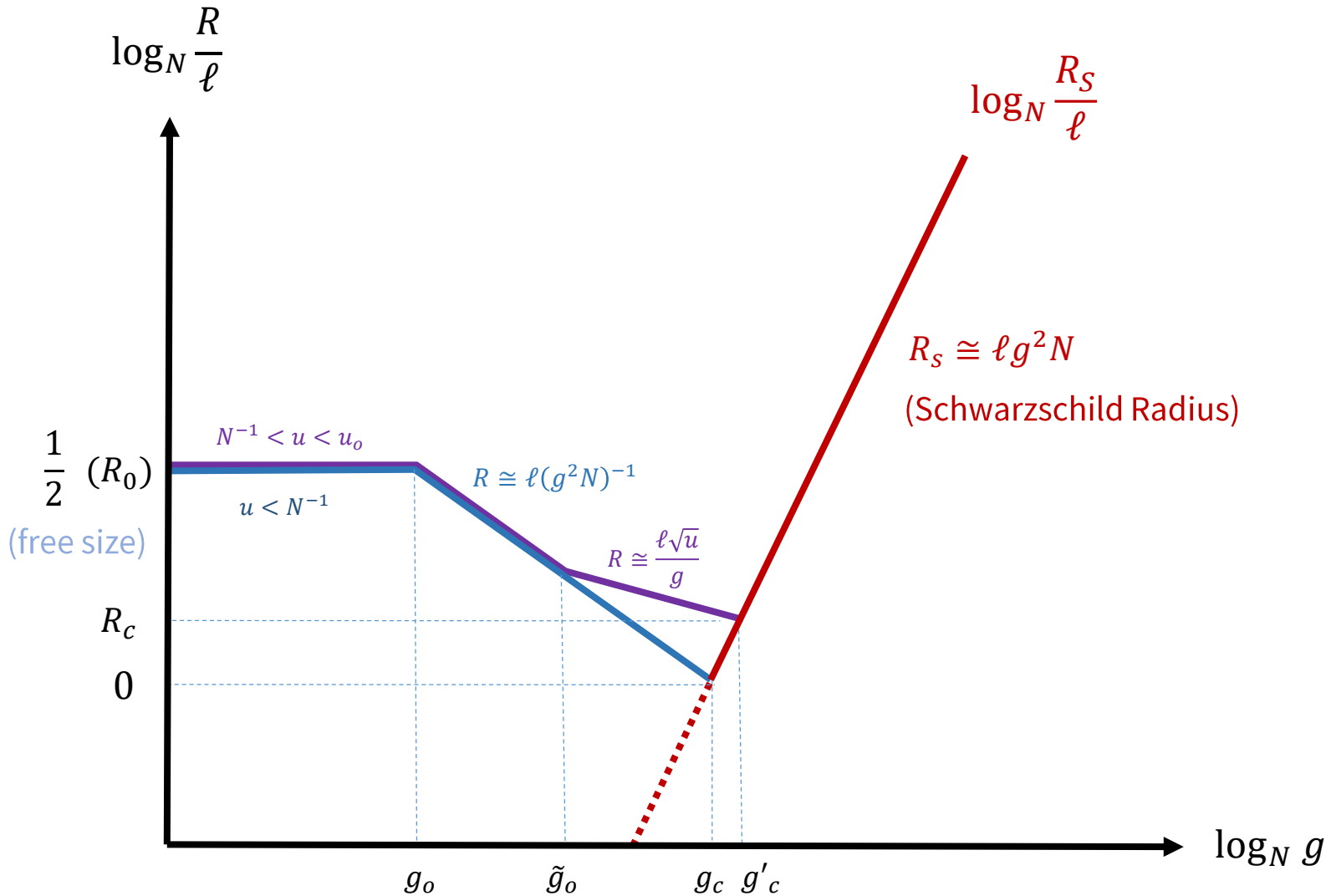
$$\langle R^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N & (q_0 N \ll 1) \\ \frac{\ell^2}{q_0} & (q_0 N \geq O(1)) \end{cases}$$

1. No gravity: $g = 0, u \geq 0$: $q_0 = 0 \longrightarrow R_0 = \ell\sqrt{N}$ (free size)

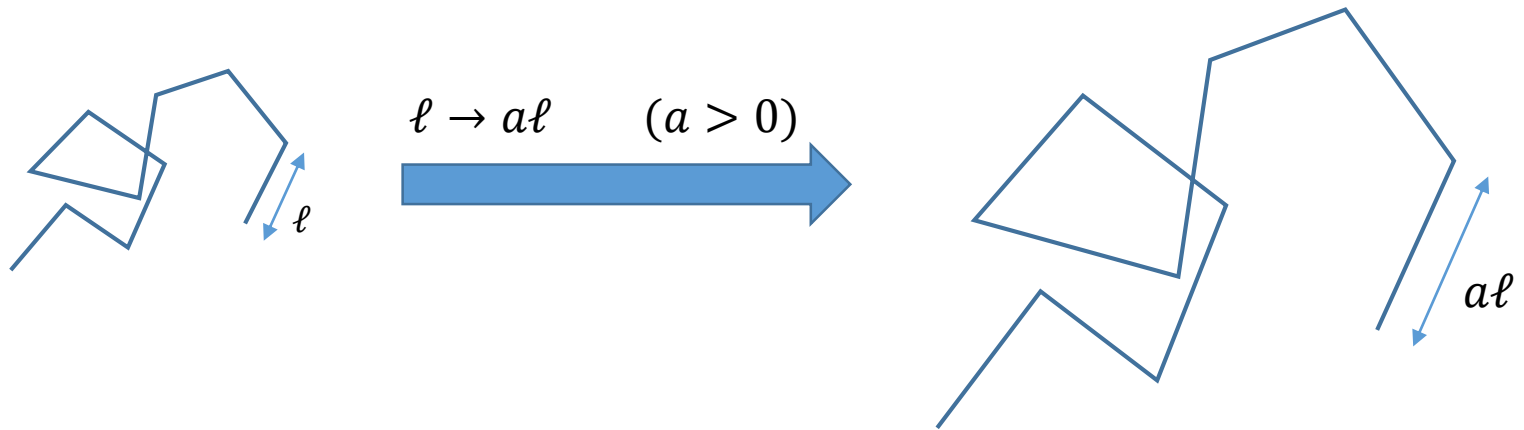
2. Generic Case $g > 0, u > 0$:



Size change ($d = 3$)



Uniform Expansion Model (UEM)



- Bond length is rescaled
- The configuration **remains free** walk one.

$$R = aR_0 = a\ell\sqrt{N}$$

Free (Gaussian) Hamiltonian with the bond length $a\ell$

$$\beta H' = \frac{d}{2a^2\ell^2} \int_0^N d\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2$$

Evaluate the mean-size-squared:

$$\langle \mathbf{R}^2 \rangle = \frac{\int (\mathbf{R}(N) - \mathbf{R}(0))^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\langle e^{-\beta(H-H')} (\mathbf{R}(N) - \mathbf{R}(0))^2 \rangle'}{\langle e^{-\beta(H-H')} \rangle'}$$

$$\langle A \rangle' \equiv \frac{1}{Z'} \int A e^{-\beta H'}$$

$$\langle \mathbf{R}^2 \rangle = \frac{\int (\mathbf{R}(N) - \mathbf{R}(0))^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\left\langle e^{-\beta(H-H')} (\mathbf{R}(N) - \mathbf{R}(0))^2 \right\rangle'}{\left\langle e^{-\beta(H-H')} \right\rangle'}$$

$$\cong \left\langle (\mathbf{R}(N) - \mathbf{R}(0))^2 \right\rangle' (1 + \langle \beta(H - H') \rangle') - \left\langle \beta(H - H') (\mathbf{R}(N) - \mathbf{R}(0))^2 \right\rangle' + O([\beta(H - H')]^2)$$

$$\cong \underline{Na^2 \ell^2} + \underline{\left[a^d (1 - a^2) + C_1 u N^{\frac{4-d}{2}} - C_2 g^2 N^{\frac{6-d}{2}} a^2 \right] N \ell^2 a^{2-d}}$$

Required Size

(free walk config.)

$$= 0$$

C_1, C_2 : Positive N independent constants

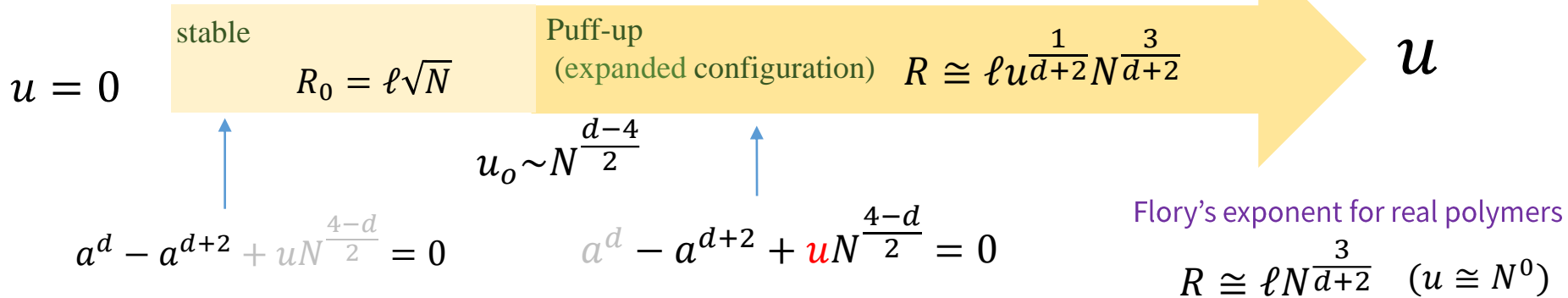
Consistency condition: $a^d - a^{d+2} + u N^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$

Size: $R = \ell a \sqrt{N}$

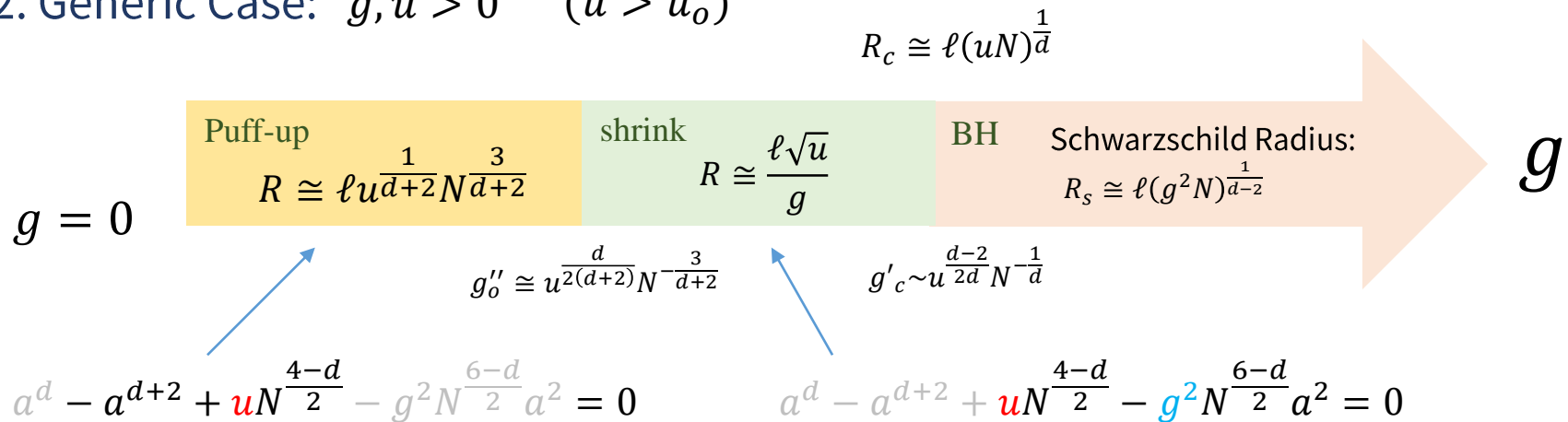
$$a^d - a^{d+2} + uN^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$$

$$R = \ell a\sqrt{N}$$

1. Pure repulsive: $g^2 = 0, u > 0$

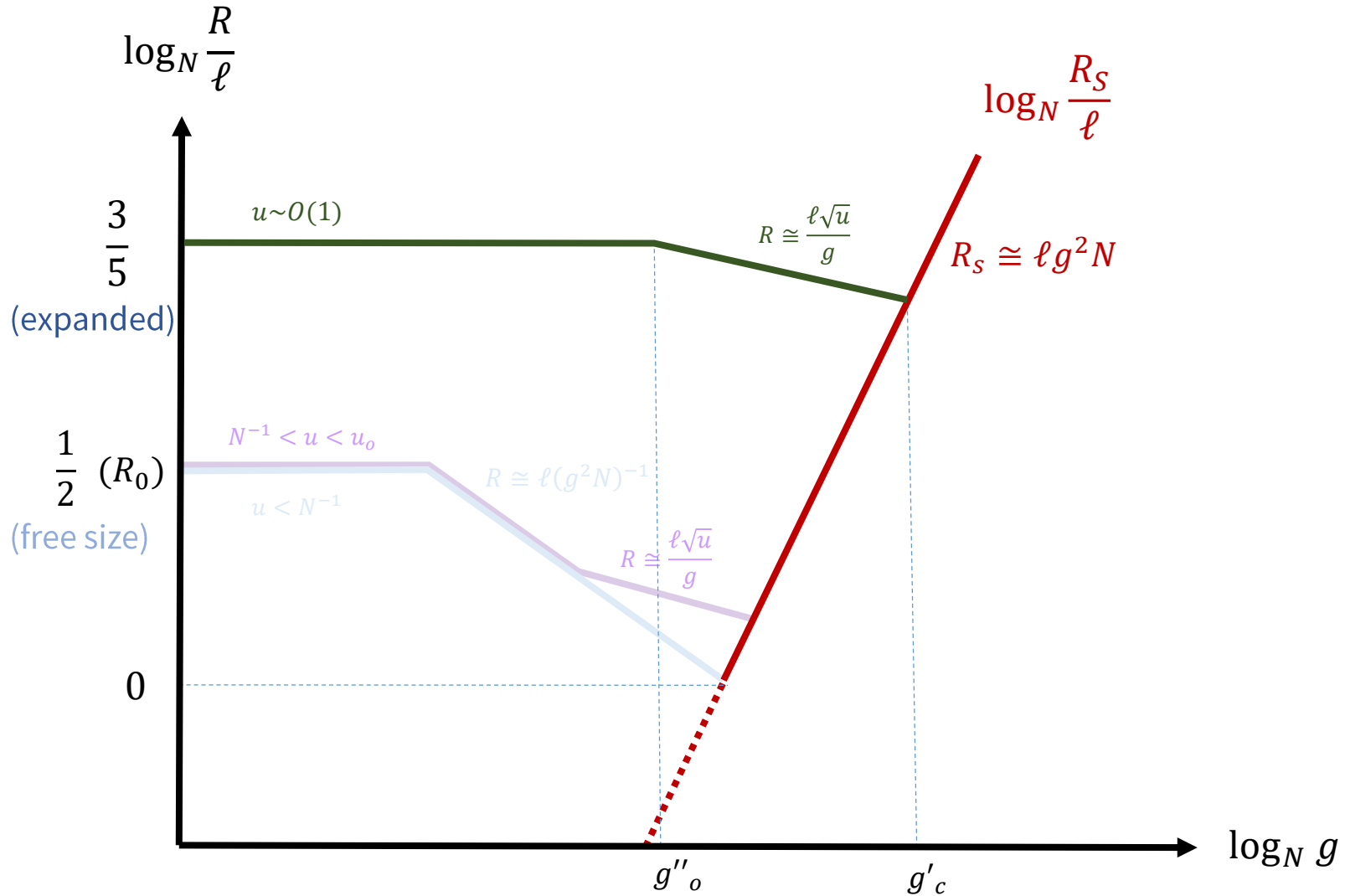


2. Generic Case: $g, u > 0$ ($u > u_o$)



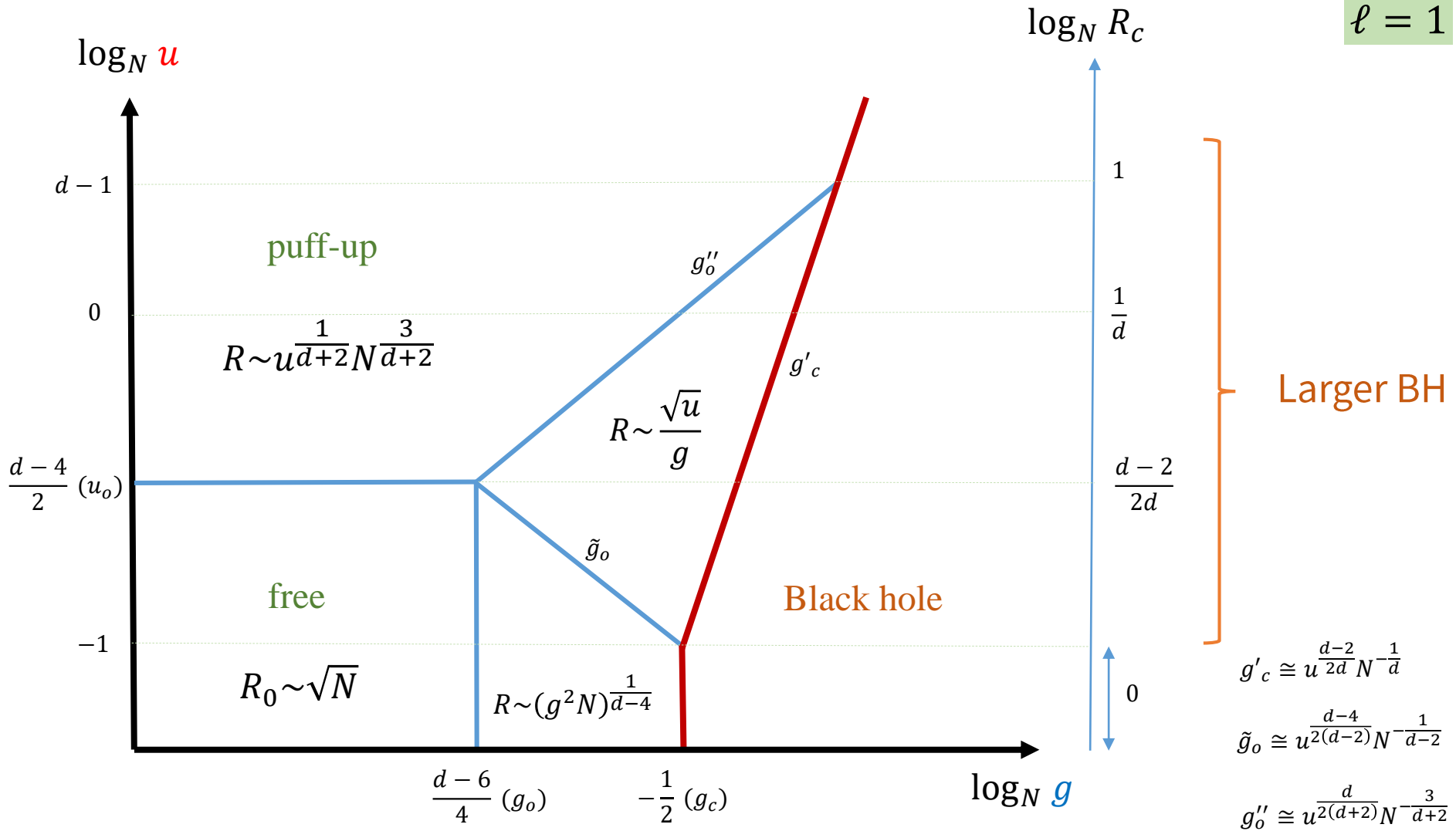
(Note: if $u < u_o$, no solution for small g)

Size change ($d = 3$)



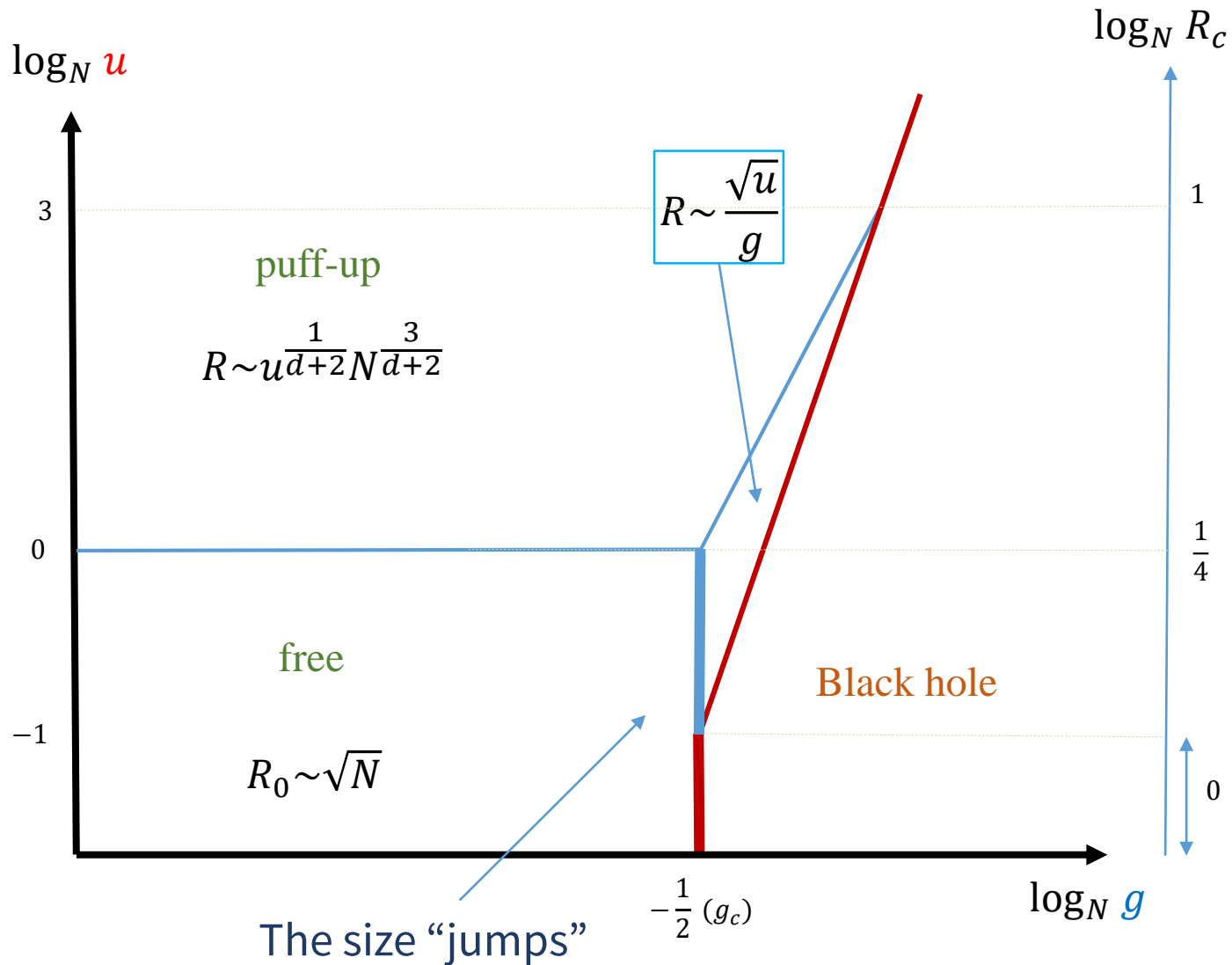
The Size Scaling ($2 < d < 4$)

$\ell = 1$



The Size Scaling ($d = 4$)

$\ell = 1$



(Note: $d > 4$ exhibits a pathological behavior)

Summary

- Self-gravitating polymers (self-avoiding walks)
→ Collapse to a **larger size** black hole
- Interesting size scaling behaviors are observed.

Next:

- Density distribution, elasticity (pressure), detailed gravitational collapse
(Need GR ? -- Tolman-Oppenheimer-Volkoff eq.)
- Possible source of self-avoiding property
Fermionic walk? Higher form field exchange?
- Beyond a mean field calculation. RG analysis
- Entropy? Corresponding point?

Variational Method (details)

Harmonic Hamiltonian:
$$\beta H_0 = \frac{d}{2\ell^2} \int_0^N d\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2 + \frac{q^2 d}{2\ell^2} \int_0^N d\sigma \mathbf{R}(\sigma)^2$$

Convexity:
$$e^{-\beta F} = \int d\mathbf{R} e^{-\beta H} = \int d\mathbf{R} e^{-\beta H_0} e^{-\beta(H-H_0)} \geq e^{\langle -\beta(H-H_0) \rangle_0} e^{-\beta F_0}$$

$$\beta F \leq \beta F_0(q) + \langle \beta(H - H_0) \rangle_0 \quad \text{Tune } q \text{ to minimize RHS}$$

Propagator:

$$G_0(\sigma, \sigma') = \left(\frac{qd}{2\pi\ell^2 \sinh q|\sigma - \sigma'|} \right)^{\frac{d}{2}} \exp \left(- \frac{qd [\mathbf{R}(\sigma)^2 + \mathbf{R}(\sigma')^2] \cosh q|\sigma - \sigma'| - 2\mathbf{R}(\sigma) \cdot \mathbf{R}(\sigma')}{2\ell^2 \sinh q|\sigma - \sigma'|} \right)$$

Evaluate $\beta F_0 = -\log Z_0$ and $\langle \beta(H - H_0) \rangle_0 = \left\langle \int_0^N d\sigma \int_0^N d\sigma' V - \frac{q^2 d}{2\ell^2} \int_0^N d\sigma \mathbf{R}(\sigma)^2 \right\rangle$

$$\beta F \leq \frac{d}{2} \ln(\cosh qN) - \frac{qdN}{4} \tanh qN$$

$$-2 \int_0^N d\sigma' \int_0^{\sigma'} d\sigma \left[\frac{g^2}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{qd}{2F_1(\sigma, \sigma'; q)} \right)^{\frac{d-2}{2}} - u \left(\frac{qd}{2F_2(\sigma, \sigma'; q)} \right)^{\frac{d}{2}} \right]$$

$$F_1(\sigma, \sigma'; q) = \frac{\sinh q\sigma \cosh q(N - \sigma) + \sinh q\sigma' \cosh q(N - \sigma') - 2 \sinh q\sigma \cosh q(N - \sigma')}{\cosh qN}$$

$$F_2(\sigma, \sigma'; q) = \frac{\sinh q\sigma \sinh q(\sigma' - \sigma)}{\sinh q\sigma'} + \frac{\cosh q(N - \sigma') \sinh q\sigma'}{\cosh qN} \left(1 - \frac{\sinh q\sigma}{\sinh q\sigma'} \right)^2$$

Simplification (an approximation):

$$\frac{e^{-qN}}{e^{-q\sigma}} \frac{e^{-q(N-\sigma')}}{e^{-q\sigma'}} \frac{e^{-q(N-\sigma)}}{e^{-q(\sigma'-\sigma)}} \ll 1$$

Bound:

$$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$$

Stationary Cond.:

$$0 = 1 - N g^2 q_0^{\frac{d-4}{2}} + N u q_0^{\frac{d-2}{2}}$$

Omit (positive N-independent) numerical factors

$$\langle \mathbf{R}^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N & (q_0 N \ll 1) & \text{(Free walk size)} \\ \frac{\ell^2}{q_0} & (q_0 N \geq O(1)) & \text{(shrink)} \end{cases}$$

Uniform Expansion Model (details)

Free (Gaussian) Hamiltonian with the bond length $a\ell$

$$\beta H' = \frac{d}{2a^2\ell^2} \int_0^N d\sigma \left(\frac{\partial \mathbf{R}}{\partial \sigma} \right)^2$$

Propagator:
$$G'(\sigma, \sigma') = \left(\frac{d}{2\pi a^2 \ell^2 |\sigma - \sigma'|} \right)^{\frac{d}{2}} \exp \left(-\frac{d}{2a^2 \ell^2 |\sigma - \sigma'|} (\mathbf{R}(\sigma) - \mathbf{R}(\sigma'))^2 \right)$$

Calculate the mean-size-squared:

$$\langle \mathbf{R}^2 \rangle = \frac{\int (\mathbf{R}(N) - \mathbf{R}(0))^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\langle e^{-\beta(H-H')} (\mathbf{R}(N) - \mathbf{R}(0))^2 \rangle'}{\langle e^{-\beta(H-H')} \rangle'}$$

$\langle A \rangle' \equiv \frac{1}{Z'} \int A e^{-\beta H'}$

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Required Size

$$= 0$$

C_1, C_2 : Positive N independent constants