Yukawa Institute Workshop Strings and Fields
“Developments in String Theory and Quantum Field Theory”

# Conic D-branes

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(arXiv:1505.04506)

# Cones are critical

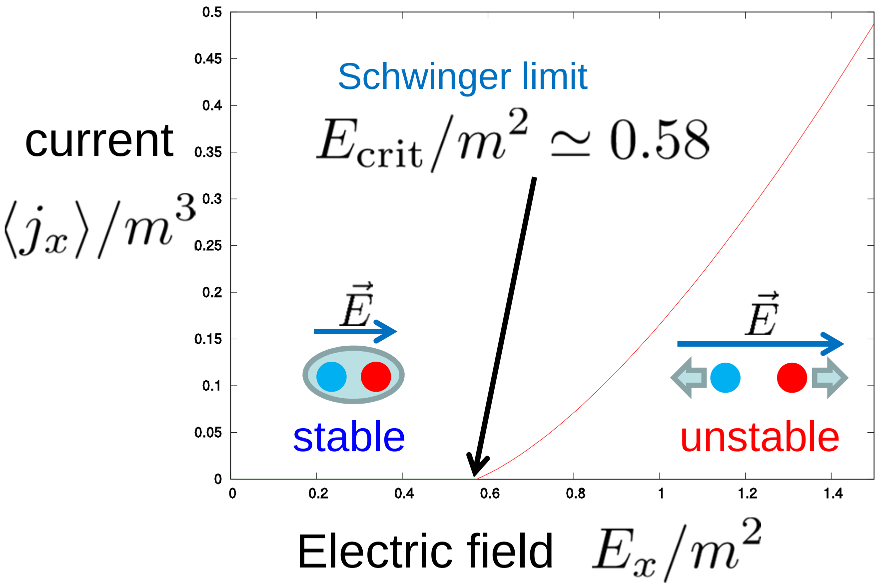
- Conic shapes widely appear in nature

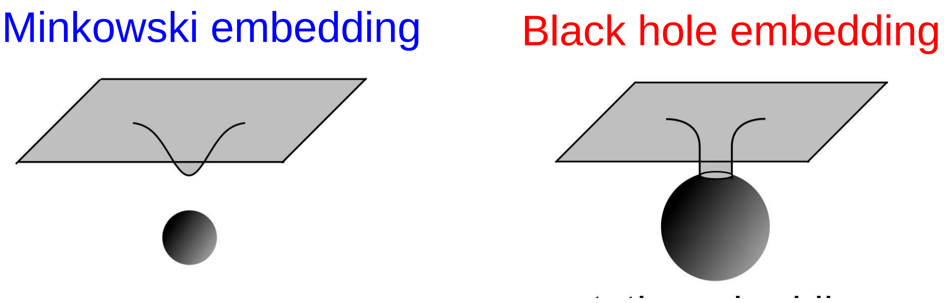
- Cones are critical geometries where the topology changes
  - Merger/fragmentation of liquid
  - Merger/fragmentation of black hole horizon
  - Phase boundary

# D3/D7 system

Karch, Katz (2002), Grana, Polchinski (2002), Bertolini *et al*. (2002)

- Holographic dual to $\mathcal{N} = 2$ SQCD
  - In large $N_c$ limit, a probe D7-brane is embedded in $\mathrm{AdS}_5 \times S^5$ geometry
  - Fluctuations of the D7-brane = "meson" excitations
- Phase transition by applying electric fields
  - Dielectric breakdown due to Schwinger effect

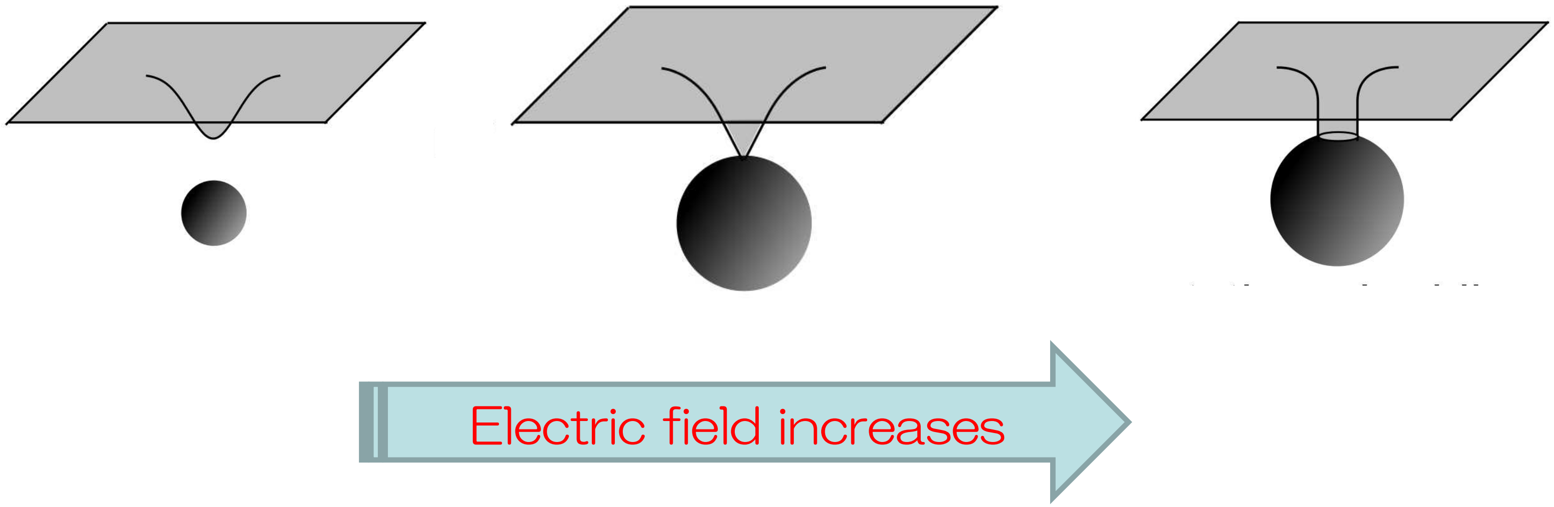




Karch, O'Bannon (2007)
Erdmenger, Meyer, Shock (2007)
Albash, Filev, Johnson, Kundu (2007)

# Critical embedding in the D3/D7 system

- A phase boundary between the Minkowski embeddings and the BH embeddings
  - Two series of the solutions merge
  - The shape of the D7-brane is conical



Minkowski embedding

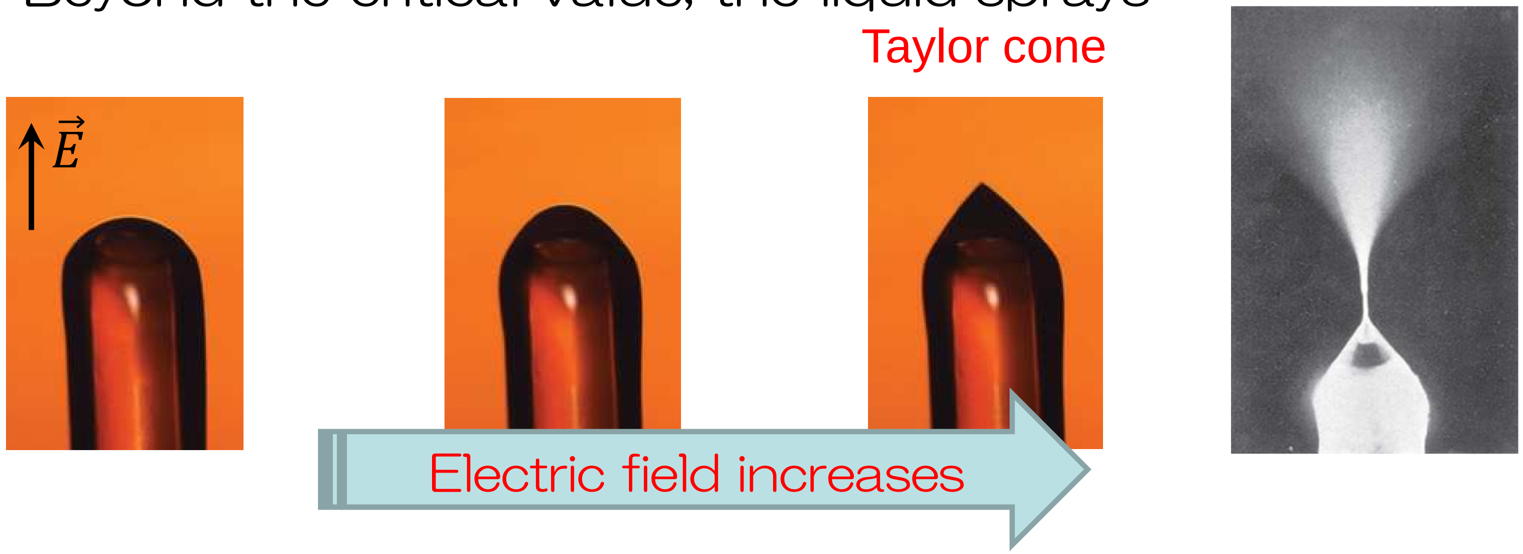
Critical embedding

Black hole embedding

Electric field increases

# Taylor cone

- A hydrodynamic phenomena, which are used in electrospray in material/industrial science
- As an electric field increases the surface of a conductive liquid is sharpening, and at a critical electric field a cone is formed
  - Beyond the critical value, the liquid sprays



Ref. R.Krpoun "Micromachined Electrospray Thrusters for Spacecraft Propulsion" (2009)

- The first theoretical model of this phenomena is given by Taylor (1964)

G.Taylor Proc. R. Soc. Lond. A 280, 383 (1964)

  - He assumed the liquid was a perfect conductor and the cone was formed when the surface tension and the electrostatic stress equilibrated on the liquid surface
  - Repulsive forces between the induced charges cancel surface tension forces
- A half-cone angle 49.29° predicted by Taylor is very close to experimental results
  - This angle is determined by a zero of the Legendre polynomial

Can we find something like universal properties for conic D-branes?

# RR flux background

- D2-brane in a constant Ramond-Ramond (RR) flux background in flat spacetime
  - The $d$-dim. bulk spacetime

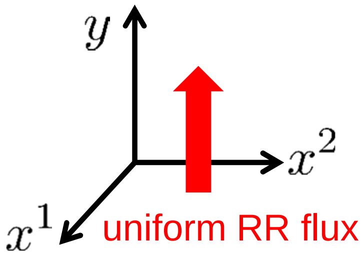
$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + (dx^1)^2 + (dx^2)^2 + d\vec{y}^2_{d-3}$$

  - Embedding function

$$y = \phi(\rho) \qquad \rho = \sqrt{(x^1)^2 + (x^2)^2}$$

  - RR field

$$C_\mu dx^\mu = C_0 dt = cy\,dt \quad (c = \text{const.})$$

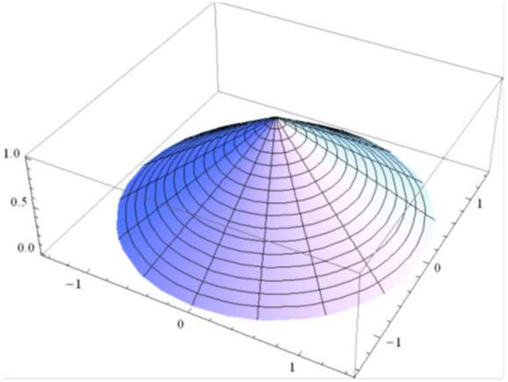


- The action is a DBI action with a coupling to the RR field

$$S = -\mathcal{T}_2 \int dx^0 dx^1 dx^2 \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab} + \partial_a\phi\partial_b\phi)}$$
$$- \frac{\mathcal{T}_2}{2} \int dx^0 dx^1 dx^2\ 2\pi\alpha' F_{ab} C_c \epsilon^{abc}$$

# Conic solution

- The equations of motion are



$$\partial_i \left[ \frac{\partial_i \phi}{\sqrt{1+(2\pi\alpha' F_{12})^2+(\partial_1\phi)^2+(\partial_2\phi)^2}} \right] + 2\pi\alpha' \frac{dC_0[\phi]}{d\phi} F_{12} = 0 \,,$$

$$\partial_i \left[ \frac{2\pi\alpha' F_{12}}{\sqrt{1+(2\pi\alpha' F_{12})^2+(\partial_1\phi)^2+(\partial_2\phi)^2}} \right] - \partial_i C_0[\phi] = 0$$

The second equation can be integrated $2\pi\alpha' F_{12} = C_0[\phi]\sqrt{\frac{1+(\partial_i\phi)^2}{1-(C_0[\phi])^2}} \,.$

$$\partial_i \left[ \frac{\partial_i \phi}{\sqrt{1+(\partial_j\phi)^2}} \sqrt{1-(C_0[\phi])^2} \right] + \frac{dC_0[\phi]}{d\phi} C_0[\phi] \sqrt{\frac{1+(\partial_j\phi)^2}{1-(C_0[\phi])^2}} = 0 \,.$$

This equation is singular when $C_0[\phi] = c\phi(\rho) = \pm 1$

- If we expand $\phi(\rho)$ around this point, we have a critical (conical) solution

$$\phi(\rho) = \frac{1}{c} - \frac{1}{\sqrt{2}}\rho + \cdots$$

Half-cone angle

$$\boxed{\theta_{\mathrm{cone}} = \arctan\sqrt{2}}$$

The apex of the cone is located at $\rho = 0$

- This analysis is local
    - The global structure of the D-brane depends on asymptotic boundary conditions.
    - In general, the RR flux does not need to be constant and uniform globally.

- If a cone has been formed at a part of the D2-brane, the cone should be locally identified with our solution at that critical point.
    - At the critical point, the apex angle can be uniquely determined.

# Other examples

- NSNS flux background
  - D$p$-brane in a constant NSNS flux

topology of the cone: $\mathbf{R}_+ \times S^{p-2}$

$$\boxed{\theta_{\rm cone} = \arctan\sqrt{2(p-2)}}$$

- D3/D7
  - Probe D7-brane with worldvolume gauge fields in $\mathrm{AdS}_5$−Schwarzschild $\times S^5$

topology of the cone: $\mathbf{R}_+ \times S^3$

$$\boxed{\theta_{\rm cone} = \arctan\sqrt{6}}$$

The cone angle is unique independent of three parameters $(E_i, B_i, r_{\rm h})$

# Universal formula?

- We have three conical D-brane solutions for different external forces and couplings
  - RR flux, NSNS flux, gravitational field (AdS curvature)
- It is expected that the half-cone angle is determined as

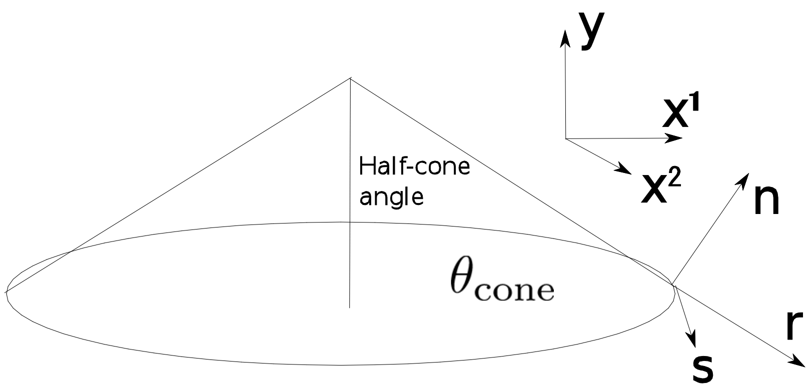
$$\boxed{\theta_{\mathrm{cone}} = \arctan \sqrt{2(d_{\mathrm{cone}} - 1)}}$$

topology of the cone: $\mathbf{R}_+ \times S^{d_{\mathrm{cone}}-1}$

- What mechanism determines the angle of conic D-branes?
- Where is the factor of 2 in the square root coming from?

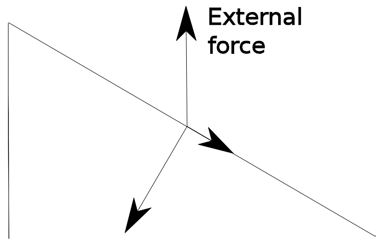
# Force balance in Newtonian mechanics

- We have two force balance conditions:
  - Normal direction (extrinsic dynamics)
  - Radial direction (intrinsic dynamics)



$$\boxed{|F_n| = -T_{ss}\frac{1}{r\tan\theta_{\rm cone}}}$$

Young-Laplace eq.



$$|F_n| = |F_r|\tan\theta_{\rm cone}$$

$$\boxed{|F_r| = -\partial_r T_{rr}}$$

Hydrodynamic (elastic) equilibrium

$$T_{ss} = (\tan\theta_{\rm cone})^2 r\frac{d}{dr}T_{rr}$$

Assuming that the tension is isotropic and its distribution behaves as

$$T_{ss} = T_{rr} \simeq Ar^{\alpha} \quad (r\sim 0)$$

$$\boxed{\theta_{\rm cone} = \arctan\sqrt{\frac{1}{\alpha}}}$$

# Equations of motion for generic membranes

- Extrinsic and intrinsic dynamics

$$T^{ab}K^{\mu}{}_{ab} = -\mathcal{F}^{\mu}_{\mathrm{n}},$$

$$D_a T^{ab} = \mathcal{F}^{b}_{\mathrm{t}},$$

External force

$$\mathcal{F}^{\mu} = \mathcal{F}^{\mu}_{\mathrm{n}} + \mathcal{F}^{a}_{\mathrm{t}} h_a{}^{\mu}$$

Induced metric: $h_{ab} \equiv g_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu}$

Extrinsic curvature: $-K^{\mu}{}_{ab} \equiv (g^{\mu}{}_{\lambda} - h^{\mu}{}_{\lambda}) h_a{}^{\nu}\nabla_{\nu} h_b{}^{\lambda}$

$$= D_a D_b X^{\mu} + \Gamma^{\mu}{}_{\alpha\beta} D_a X^{\alpha} D_b X^{\beta},$$

Embedding functions: $x^{\mu} = X^{\mu}(y^a)$

Nambu-Goto brane

$T^{ab} = -\sigma h^{ab} \quad (\sigma = \mathrm{const.})$

$\mathrm{Tr}K^{\mu} = 0$ Extremal surface

In general, the energy density is not equal to the tension (negative pressure).

# Force balance in curved spacetimes

- A membrane in an “axisymmetric” spacetime

Bulk metric: $g_{\mu\nu}dx^{\mu}dx^{\nu} = \mathsf{A}_{ij}(\rho,\zeta)dy^i dy^j + \mathsf{B}(\rho,\zeta)(d\rho^2 + d\zeta^2) + \mathsf{C}(\rho,\zeta)d\Omega^2_{d-1}$

$$+ \mathsf{D}_{kl}(\rho,\zeta)dw^k dw^l$$

Induced metric on the memebrane:

Embedding functions

$$\zeta = \phi(\rho),\ w^k = \text{const.}$$

$$h_{ab}dy^a dy^b = \mathsf{A}_{ij}(\rho,\phi(\rho))dy^i dy^j$$

$$+ [1 + \phi'(\rho)^2]\mathsf{B}(\rho,\phi(\rho))d\rho^2 + \mathsf{C}(\rho,\phi(\rho))d\Omega^2_{d-1}$$

Topology of the cone $\mathbf{R}_+ \times S^{d-1}$

We assume the membrane has an isotropic tension on the cone

stress-energy tensor $T_{ab} = \tau_{ab} - \sigma(r_a r_b + s_{ab})$

tension

induced metric on the cone

$$\boxed{\begin{aligned} T^{ab}K^{\mu}{}_{ab} &= -\mathcal{F}^{\mu}_{\rm n}, \\ D_a T^{ab} &= \mathcal{F}^b_{\rm t}, \end{aligned}} \Rightarrow \frac{\sin\theta}{\sqrt{-\mathsf{AB}}}\frac{d}{d\rho}(\sqrt{-\mathsf{A}}\sigma\sin\theta) - \sigma\cos\theta n^{\mu}\partial_{\mu}\log(\sqrt{\mathsf{B}}\mathsf{C}^{(d-1)/2}) + \frac{1}{2\sqrt{\mathsf{B}}}\tau^{ij}\partial_{\rho}\mathsf{A}_{ij} = 0$$

If the external force is along the axis of the cone, we can combine two equations.

- If we assume that the bulk spacetime is regular at the membrane (the membrane does not touch event horizons or some singularities) and the tension $\sigma$ plays a dominant role, then we have

$$(\sin\theta_{\rm cone})^2 \frac{d\sigma}{d\rho} \simeq (d-1)(\cos\theta_{\rm cone})^2 \frac{\sigma}{\rho}$$

- If the tension behaves as $\sigma \sim \rho^\alpha$ near the apex of the cone $\rho \sim 0$, the angle of the cone becomes

$$\boxed{\theta_{\rm cone} = \arctan\sqrt{\frac{d-1}{\alpha}}}$$

- The dimension of the spherical part of the cone
- The power of the stress distribution

# Stress-energy tensor of the various D-branes

- D2-brane in the RR flux

isotropic tension

$$T^0{}_0 = -\frac{1}{\sqrt{1-(C_0[\phi])^2}}, \quad \boxed{T^\theta{}_\theta = T^r{}_r = -\sqrt{1-(C_0[\phi])^2}} \qquad \mathbf{R}_+ \times S^1$$

- D$p$-brane in the NSNS flux

isotropic tension

$$T^0{}_0 = T^1{}_1 = -\frac{1}{\sqrt{1-c^2\phi^2}}, \quad \boxed{T^r{}_r = -\sqrt{1-c^2\phi^2}, \quad T^m{}_n = -\sqrt{1-c^2\phi^2}\delta^m{}_n}$$

$$\mathbf{R}_+ \times S^{p-2}$$

- D7-brane in $\mathrm{AdS}_5 \times S^5$

$$T^0{}_0 = -\frac{1}{\sigma}(1+\frac{\mathbf{B}^2}{g^2}), \quad T^0{}_i = \frac{1}{\sigma}\frac{1}{g^2 h}\epsilon_{ijk}E^j B^k, \quad T^i{}_0 = -\frac{1}{\sigma}\frac{1}{g^2}\epsilon^{ijk}E_j B_k,$$

$$T^i{}_j = -\frac{1}{\sigma}\left[(1-\frac{\mathbf{E}^2}{g^2 h})\delta^i{}_j + \frac{1}{g^2}(h^{-1}E^i E_j + B^i B_j)\right],$$

$$\boxed{T^r{}_r = -\sigma, \quad T^m{}_n = -\sigma\delta^m{}_n}$$ isotropic tension $\mathbf{R}_+ \times S^3$

$$\left(\sigma^2 \equiv 1 - \frac{1}{g^2}(h^{-1}\mathbf{E}^2 - \mathbf{B}^2) - \frac{1}{g^4 h}(\mathbf{E}\cdot\mathbf{B})^2, \quad h(u) \equiv \left(\frac{u^4 - r_\mathrm{h}^4/4}{u^4 + r_\mathrm{h}^4/4}\right)^2, \quad g(u) \equiv \frac{1}{2\pi\alpha' R^2}\frac{u^4 + r_\mathrm{h}^4/4}{u^2}\right)$$

# Mechanism for the conic D-branes

- When the isotropic tension vanishes, a cone is formed.
- The angle of the cone is universally determined by the dimension of the cone and the power of the distribution of the tension.
  - For the conic D-branes, the power is $\frac{1}{2}$ independent of the background fields, which comes from the square root of the DBI action

$$\boxed{\theta_{\text{cone}} = \arctan\sqrt{2(d_{\text{cone}} - 1)}}$$

topology of the cone: $\mathbf{R}_+ \times S^{d_{\text{cone}}-1}$

# Summary

- We found various conic D-brane solutions, whose cone angles obey an universal formula
  - The cone is formed at a critical point where the brane tension is canceled
- In general, the cone angles are determined by simply the local force balance
  - It is expected that many conic D-branes other than our limited examples exist and our formula is valid

- Beyond the critical value, what happens?
  - Spray solution? Funnel solution?

# APPENDIX

# NSNS flux background

- D$p$-brane in a constant NSNS flux in flat spacetime
  - Bulk spacetime

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\vec{x}_p^2 + d\vec{y}_{d-p}^2$$
$$= -dt^2 + dx_1^2 + d\rho^2 + \rho^2 d\Omega_{p-2}^2 + d\vec{y}_{d-p}^2$$

  - NSNS field

$$B_{\mu\nu}dx^\mu \wedge dx^\nu = 2cy\, dt \wedge dx_1, \quad H_{01y} = c\,(\mathrm{const.})$$

  - Embedding function $y = \phi(\rho)$
- The action is given by DBI action

$$S = -\mathcal{T}_p \int dx^{p+1}x\sqrt{-\det(\eta + B + \partial\phi\partial\phi)}$$
$$= -\mathcal{T}_p \int dx^0 dx^1 d\rho\, V_{p-2}\rho^{p-2}\sqrt{(1 + (d\phi/d\rho)^2)(1 - B_{01}^2)}$$

In contrast to the case of RR flux, no additional coupling term exists.

# Conic solution

- In this case there is a critical point at which the Lagrangian density vanishes.

$$\begin{aligned}\mathcal{L}[\phi(\rho)] =& \rho^{p-2}\sqrt{(1+(d\phi/d\rho)^2)(1-B_{01}^2)} \\ =& \rho^{p-2}\sqrt{(1+(d\phi/d\rho)^2)(1-c^2\phi^2)}\end{aligned}$$

The equation of motion becomes singular when $\phi(\rho) = 1/c$

- Near this critical point, we can obtain conic solution in a similar manner

$$\phi(\rho) = \frac{1}{c} - \frac{1}{\sqrt{2(p-2)}}\rho + \cdots$$

Half-cone angle

$$\boxed{\theta_{\text{cone}} = \arctan\sqrt{2(p-2)}}$$

Induced metric: $h_{ab}dx^a dx^b = -dt^2 + dx_1^2 + (1+\phi'^2)d\rho^2 + \rho^2 d\Omega_{p-2}^2$

Topology of the cone: $\mathbf{R}_+ \times S^{p-2}$

# D3/D7 system

- Probe D7-brane with worldvolume gauge field in the $\mathrm{AdS}_5-\mathrm{Schwarzschild}\times S^5$ geometry
  - Bulk metric:

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{R^2}\frac{u^4+r_\mathrm{h}^4/4}{u^2}[-h(u)dt^2+d\vec{x}_3^2]+\frac{R^2}{u^2}(du^2+u^2d\Omega_5^2)$$

$$h(u)\equiv\left(\frac{u^4-r_\mathrm{h}^4/4}{u^4+r_\mathrm{h}^4/4}\right)^2 \qquad du^2+u^2d\Omega_5^2=d\rho^2+dw^2+\rho^2d\Omega_3^2+w^2d\psi^2$$

$r_\mathrm{h}$ is a horizon radius in the usual Schwarzschild coordinates

$$(u^2=w^2+\rho^2)$$

  - Embedding functions: $w=L(\rho),\ \psi=\mathrm{const.}$
  - Worldvolume gauge field: $F_{0i}=E_i, \quad F_{ij}=e_{ijk}B_k$
- DBI action

$$S=-\mathcal{T}_7\int d^8x\sqrt{-\det(g_{ab}+2\pi\alpha' F_{ab})}\propto\int d^4x\rho^3\sqrt{\xi(1+(L'(\rho))^2)}$$

$$\xi\equiv\left(\frac{u^4+r_\mathrm{h}^4/4}{u^4}\right)^4h(u)-\left(\frac{2\pi\alpha' R^2(u^4+r_\mathrm{h}^4/4)}{u^6}\right)^2(\mathbf{E}^2-h(u)\mathbf{B}^2)-\left(\frac{2\pi\alpha' R^2}{u^2}\right)^4(\mathbf{E}\cdot\mathbf{B})^2$$

# Conic solution

- When the electric field increases, there exists a critical electric field at which the Lagrangian density vanishes
  - Critical embedding, which is the phase boundary between two series of solutions: BH and Minkowski embeddings

$$L(\rho) = L_{\rm c} - \frac{1}{\sqrt{6}}\rho + \cdots$$

Half-cone angle

$$\boxed{\theta_{\rm cone} = \arctan\sqrt{6}}$$

$$h_{ab}dy^a dy^b = \frac{1}{R^2}\frac{(\rho^2+L^2)^2+r_{\rm h}^4/4}{\rho^2+L^2}[-h(u)dt^2+dx_i^2] + \frac{R^2}{\rho^2+L^2}[(1+L'^2)d\rho^2+\rho^2 d\Omega_3^2]\,.$$

Topology of the cone: $\mathbf{R}_+ \times S^3$

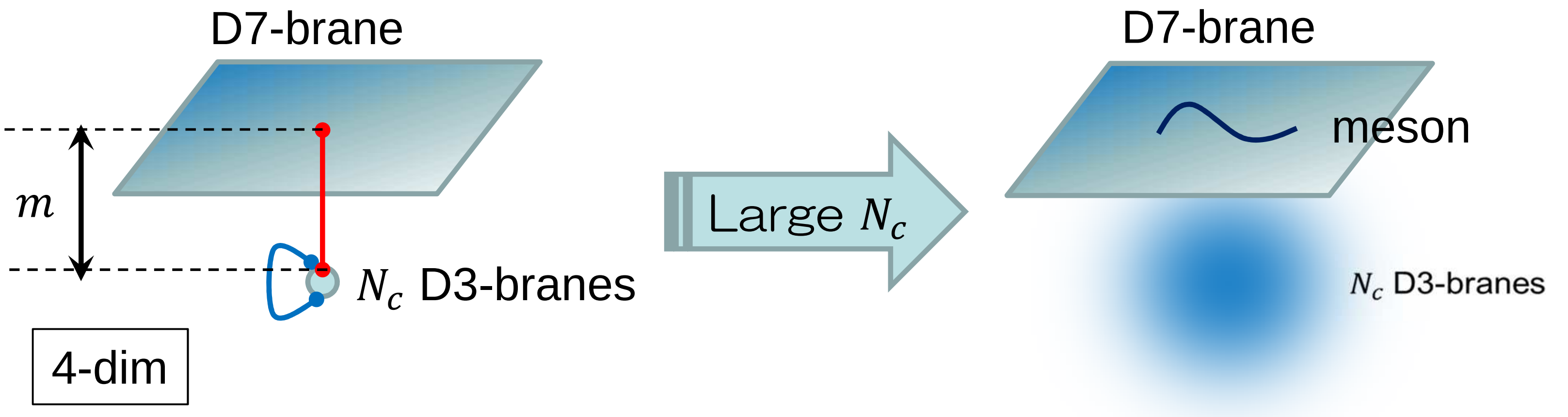
The cone angle is unique independent of three parameters $(E_i, B_i, r_{\rm h})$

# Holographic QCD constructed by D3/D7

Karch, Katz (2002), Grana, Polchinski (2002), Bertolini *et al*. (2002)

10-dim. flat spacetime

10-dim. $AdS_5 \times S^5$



N=4 super Yang-Mills + N=2 quark multiplet

A probe D7-brane is embedded in $AdS_5 \times S^5$

$$ds^2 = r^2[-dt^2 + d\vec{x}_3^2] + \frac{1}{r^2}[d\rho^2 + \rho^2 d\Omega_3^2 + dw^2 + d\bar{w}^2]$$

DBI action: $S_{\mathrm{D7}} = -\mu_7 \int d^8y \sqrt{-\det\left(g_{ab} + 2\pi\alpha' F_{ab}\right)}$

Embedding function
$w(\rho) = m\,(\text{const.})$

quark mass

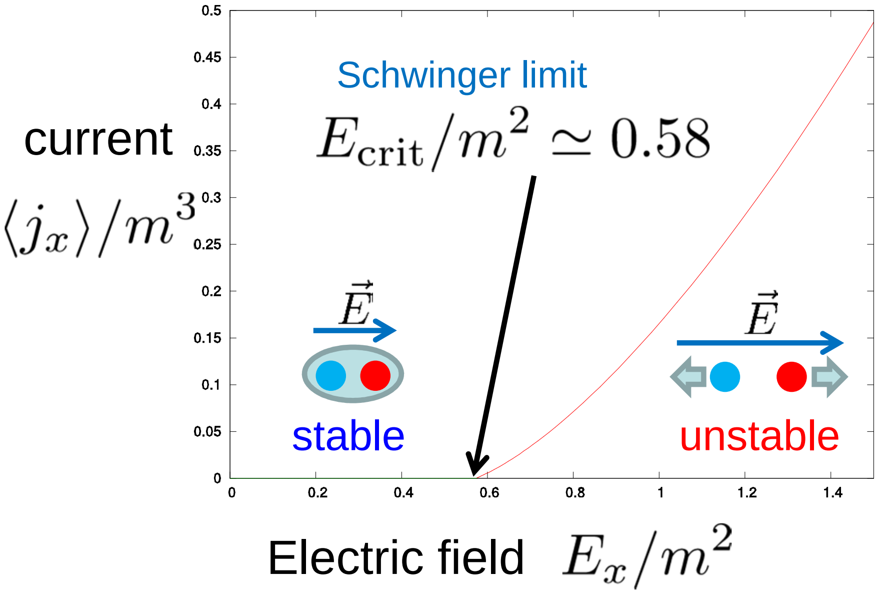
string between D3-branes and D7-brane ⇔ “quark”
fluctuations of D7-brane ⇔ “meson”

# Phase transition by applying electric fields

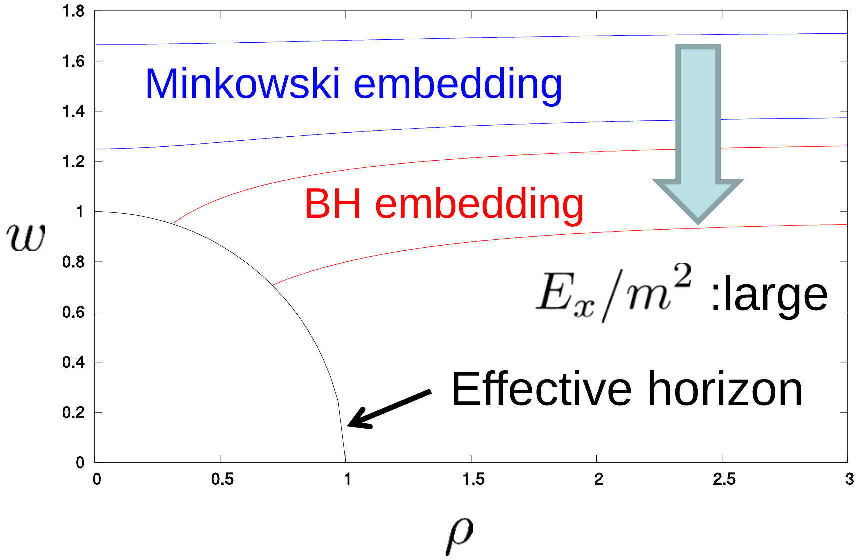
Karch, O'Bannon (2007)
Erdmenger, Meyer, Shock (2007)
Albash, Filev, Johnson, Kundu (2007)

- Schwinger effect

Electric field vs current in the boundary



Profile of the brane in the bulk



$$\mathrm{d}s^2 = r^2[-\mathrm{d}t^2 + \mathrm{d}\vec{x}_3^2] + \frac{1}{r^2}[\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_3^2 + \mathrm{d}w^2 + \mathrm{d}\bar{w}^2]$$

Beyond the critical electric field, an effective horizon emerges on the brane
The electric current becomes non-zero value = Schwinger effect