New construction method of matrix regularization using coherent states

Takaki Matsumoto (University of Tsukuba)
Work in progress with
G. Ishiki (Tsukuba), Y. Miyama (Tsukuba), H. Shimada (Okayama)

Developments in String Theory and Quantum Field Theory @ YITP November 13, 2015

1. Introduction

Matrix regularization (MR)

- It is known as a regularization of String/M theory.

  [de Wit-Hoppe-Nicolai, BFSS, IKKT]

- MR procedure is a mapping on surfaces (world sheet/volume):

$$
\begin{array}{ccc}
\text{functions} & \longrightarrow & \text{matrices} \\
\text{Poisson bracket} & \longrightarrow & \text{commutator} \\
\text{integral} & \longrightarrow & \text{trace}
\end{array}
$$

- Then, (commutative) geometry of surfaces is replaced by a matrix geometry.

$$
\begin{array}{ccc}
\text{Commutative geometry} & & \text{Matrix geometry} \\
\updownarrow & & \updownarrow \\
\text{Commutative algebra} & \xrightarrow{\textbf{MR}} & \text{Matrix algebra}
\end{array}
$$

Example of MR: Fuzzy sphere

- Unit 2-sphere embedded in $\mathbb{R}^3$.

$$\mathbb{S}^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 | x^i x^i = 1\}$$

- Fuzzy sphere is given by a mapping,

$$x^i \longrightarrow X^i \equiv \frac{2}{\sqrt{N^2-1}} L^i, \quad [L^i, L^j] = i\epsilon^{ijk} L^k$$

where L^i is N dim irrep of generators of SU(2).

- Then, $\begin{cases} (X^1)^2 + (X^2)^2 + (X^3)^2 = 1_N \\ [X^i, X^j] = \dfrac{2}{\sqrt{N^2-1}} i\epsilon^{ijk} X^k \xrightarrow[N\to\infty]{} 0 \end{cases}$

Commutative limit

[Madore]

Example of MR: Fuzzy sphere

- Using spherical harmonics functions $Y_{\ell m}$ and it's properties, we can get the following relation.

UV cut off

Functions | Matrices

$$f = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m} \longrightarrow \hat{f} = \sum_{\ell=1}^{(N-1)/2} \sum_{m=-\ell}^{\ell} c_{\ell m} \hat{Y}_{\ell m}$$

$$Y_{\ell m} = \sum_{k=0}^{\ell} t^{(\ell m)}_{A_1 \cdots A_k} x^{A_1} \cdots x^{A_k} \longrightarrow \hat{Y}_{\ell m} = \sum_{k=0}^{\ell} t^{(\ell m)}_{A_1 \cdots A_k} X^{A_1} \cdots X^{A_k}$$

Poisson bracket | Commutator

$$\{Y_{\ell m}, Y_{\ell' m'}\} \longrightarrow -iN[\hat{Y}_{\ell m}, \hat{Y}_{\ell' m'}]$$

Integral | Trace

$$\int_{\mathbb{S}^2} \frac{d\Omega}{2\pi} Y_{\ell m} = \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr} \hat{Y}_{\ell m}$$

[Madore]

Example of MR: Fuzzy sphere

Application

- Membrane theory with world volume $\mathbb{R} \times \mathbb{S}^2$. [de Wit-Hoppe-Nicolai]

$$S = \int_{\mathbb{R}\times\mathbb{S}^2} d^3\sigma \left((\dot{X}^\mu)^2 + \{X^\mu, X^\nu\}^2 \right) \qquad X : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}^{11}$$

$$\Big\downarrow \ \textbf{MR}$$

$$S = \int dt \, \mathrm{Tr} \left((\dot{\hat{X}}^\mu)^2 - [\hat{X}^\mu, \hat{X}^\nu]^2 \right) \qquad \hat{X}^\mu(t) : N \times N \, \mathrm{Matrices}$$

Quantum mechanics with a finite number of degrees of freedom

- MR is also applicable to super membrane. [BFSS, BMN]

Motivation

- MR of some surfaces is already known.

  i.e. Fuzzy sphere, Fuzzy torus, Fuzzy Cp^n etc.

  see also [Arnlind-Bordemann-Hofer-Hoppe-Shimada]

- It is not easy to construct MR of arbitrary surfaces.

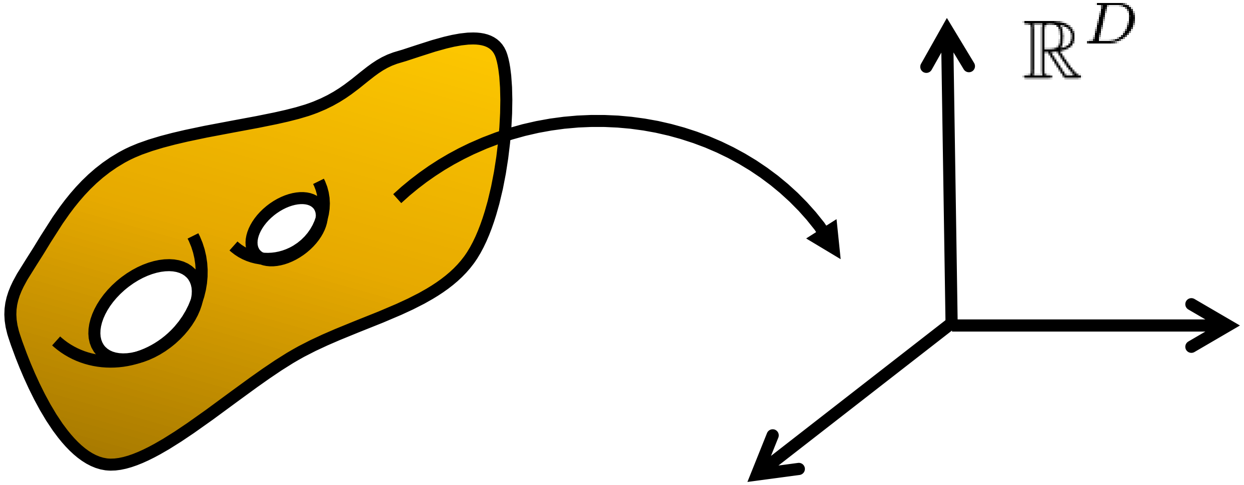
  (Formal expression of MR of Kähler manifold is already given
  by Toeplitz/geometric quantization) [Bordemann-Meinrenken-Schlichenmaier]

- We want to find an algorithmic construction method of MR.
- We want to understand the relation between geometry and matrices.

  $\longrightarrow$ Lead to better understanding of the matrix models

Our goal

- We propose a construction method of MR of arbitrary Riemann surfaces embedded in $\mathbb{R}^D$



Plan of my talk

1. Introduction
2. What we construct
3. Our strategy
4. Our result

2. What we construct

Formal definition of MR

- MR of a surface $\mathcal{M}$ is a sequence $\{T_N\}(N=1,2,\cdots)$ such that for $\forall f,h\in C^\infty(\mathcal{M})$

$$T_N : C^\infty(\mathcal{M}) \to N\times N\,\text{Hermitian matrices}$$

$$\lim_{N\to\infty} ||T_N(f)|| < \infty \qquad \cdots\cdot\ (1)$$

$$\lim_{N\to\infty} ||T_N(fh) - T_N(f)T_N(h)|| = 0 \qquad \cdots\cdot\ (2)$$

$$\lim_{N\to\infty} ||N[T_N(f),T_N(h)] - iT_N(\{f,h\})|| = 0 \qquad \cdots\cdot\ (3)$$

$$\lim_{N\to\infty} \frac{2\pi}{N}\mathrm{Tr}T_N(f) = \int_{\mathcal{M}} f\omega \qquad \cdots\cdot\ (4)$$

$||\cdot||$: norm ω : symplectic form on $\mathcal{M}$
$\{\ ,\ \}$: Poisson bracket on $\mathcal{M}$

[Arnlind-Hoppe-Huisken]

MR of surfaces embedded in $\mathbb{R}^D$

- Let us consider a linear map for embedding function $y : \mathcal{M} \to \mathbb{R}^D$ such that

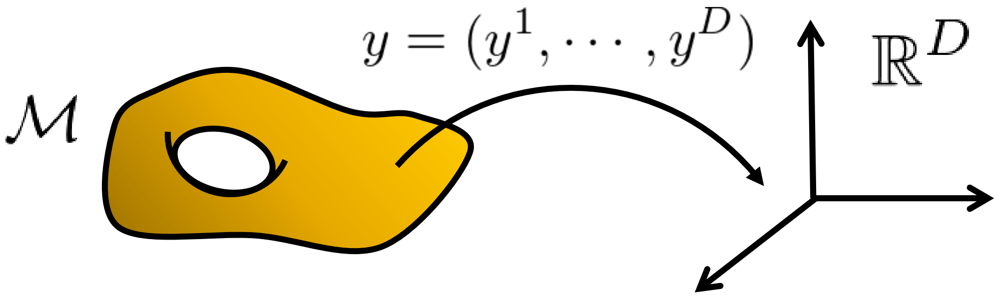
$$T_N(y^\mu) = X^\mu \quad X^\mu : N \times N \text{ Hermitian matrices}$$

- Then, a main desired condition of X^μ is

$$[X^\mu, X^\nu] \underset{N\to\infty}{\sim} \frac{i}{N} W^{\mu\nu}(X) + \mathcal{O}(N^{-2})$$

where $W^{\mu\nu}$ is induced Poisson tensor on $\mathcal{M}$.

- Using such X^μ, we can construct a linear map T_N for other functions on $\mathcal{M}$ satisfying the conditions of MR.



What we construct

- We construct $N \times N$ Hermitian matrices X^{μ} which satisfy following relation and other conditions of MR.

$$[X^{\mu}, X^{\nu}] \underset{N\to\infty}{\sim} \frac{i}{N} W^{\mu\nu}(X) + \mathcal{O}(N^{-2})$$

3. Our strategy

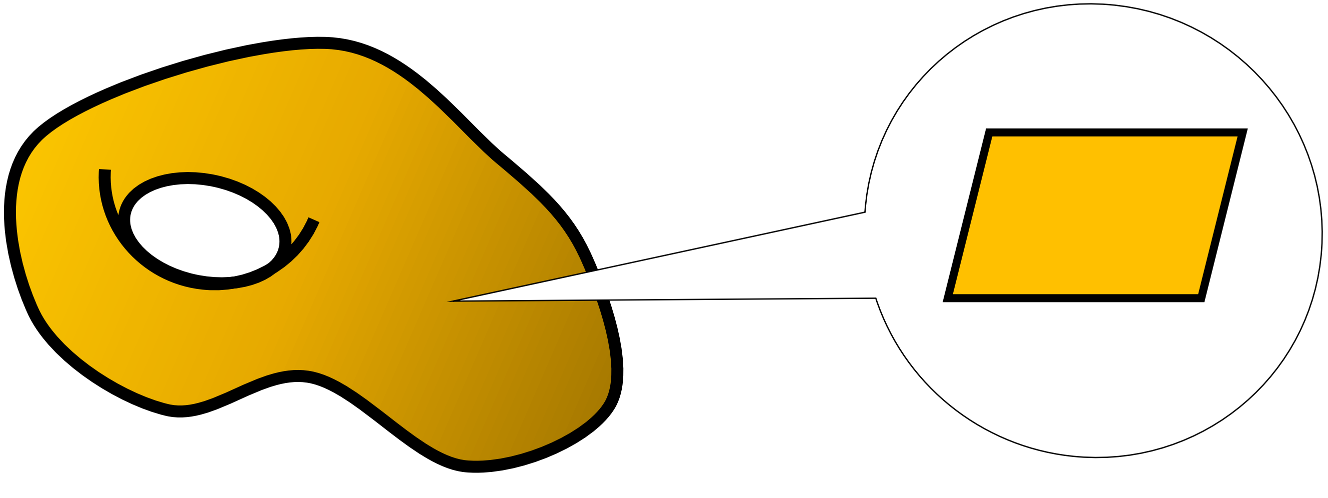
Our idea

- Any surfaces has locally the same structure as plane.

- What about matrix geometry?

but...........

What does locality mean in the matrix geometry???



The notion of locality in NC geometry

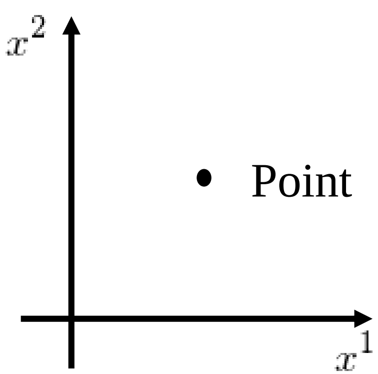
- Let us recall the NC plane (phase space in quantum mechanics).

$$[X^1, X^2] = i\hbar \quad \longleftrightarrow \quad \Delta X^1 \cdot \Delta X^2 \geq \frac{\hbar}{2}$$

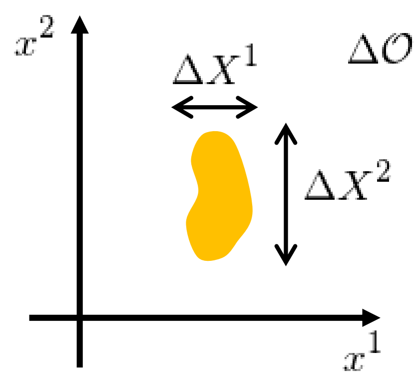
NC plane — Uncertainty relation

- Generally wave packets are nonlocal.
- We try to introduce the locality in the matrix geometry using coherent states.

Classical phase space



Quantum phase space (NC plane)



$$\Delta\mathcal{O} = \sqrt{\langle|\mathcal{O}^2|\rangle - (\langle|\mathcal{O}|\rangle)^2}$$

Canonical coherent states

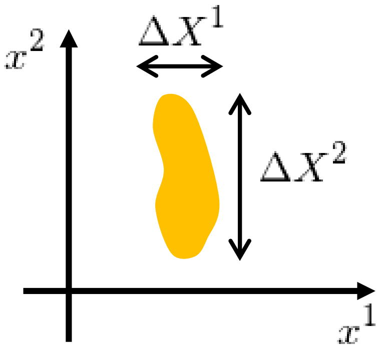
- Creation-annihilation operators : $[a, a^\dagger] = 1$
- Canonical coherent states $|y\rangle = |y^1, y^2\rangle$ in quantum mechanics (NC plane) is defined by

$$a|y\rangle = \frac{(y^1 + iy^2)}{\sqrt{2\hbar}}|y\rangle \quad y^1, y^2 \in \mathbb{R}$$

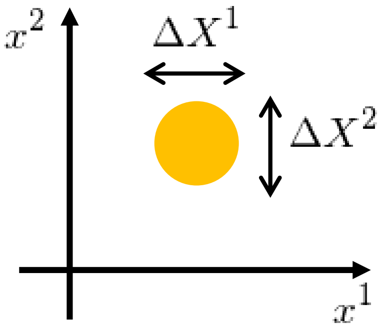
- The canonical coherent states minimize the uncertainty of the coordinate operators : $X^1 = \sqrt{\hbar/2}(a + a^\dagger),\ X^2 = i\sqrt{\hbar/2}(a^\dagger - a)$.

$$\Delta X^1 = \Delta X^2 = \sqrt{\hbar/2} \quad \longrightarrow \quad \Delta X^1 \cdot \Delta X^2 = \hbar/2$$

General states



Coherent states



wave packet is minimized !

"Local" structure of the NC plane

- Coordinate operators : $[X^1, X^2] = i\hbar$
- The coordinate operators represented on coherent states :

$$\boxed{\begin{gathered}\langle y|X^i|y'\rangle = \frac{1}{2}(y^i + y'^i - i\epsilon^{ij}(y_j - y'_j))\,\langle y|y'\rangle \\ \langle y|y'\rangle = e^{-\frac{1}{4\hbar}(y-y')^2 + \frac{i}{2\hbar}(y_1 y'_2 - y_2 y'_1)}\end{gathered}}$$

- This expression is quickly vanishing for $|y - y'| \to \infty$ (non-zero only for $|y - y'| \sim \sqrt{\hbar}$) $\longrightarrow$ local structure of X^i !!!

This local structure must be common in any matrix geometry!!

Our strategy

- The local structure has the following geometric meanings. We glue together the local structures to construct X^μ for general surface.

Poisson tensor on plane

$$\langle y|X^i|y'\rangle = \frac{1}{2}(y^i + y'^i - i\underline{\epsilon^{ij}}(y_j - y'_j))\,\langle y|y'\rangle$$

$$\langle y|y'\rangle = e^{-\frac{1}{4\hbar}\underline{(y-y')^2}+\frac{i}{2\hbar}\underline{(y_1y'_2-y_2y'_1)}}$$

Geodesic distance on plane

Integral of symplectic potential on plane

4. Our result

Our setting

$\mathcal{M}$: Riemann surface embedded in $\mathbb{R}^D$

$(z,\bar{z})$: complex local coordinate

y^μ : embedding functions

$\delta_{\mu\nu}$: flat metric on target space

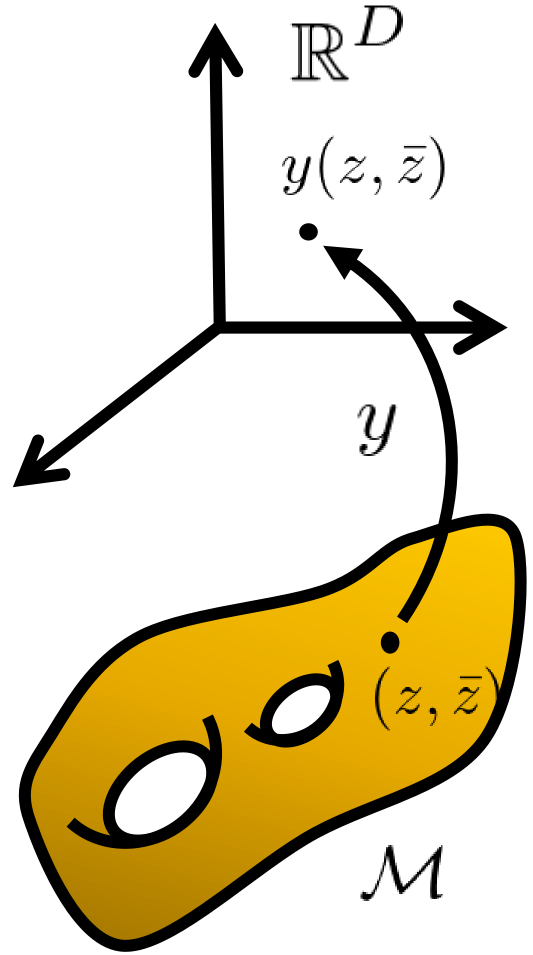
g_{ab} : induced metric on $\mathcal{M}$

ω_{ab} : symplectic form ($\int_{\mathcal{M}} \omega = 2\pi$) on $\mathcal{M}$

$\mathcal{A}_a$: symplectic potential ($\omega = d\mathcal{A}$)

w^{ab} : Poisson tensor

$W^{\mu\nu}$: induced Poisson tensor



Holomorphic section

- Let $L \to \mathcal{M}$ be a holomorphic line bundle and $\Psi_i : \mathcal{M} \to L$ be orthonormal basis of normalized holomorphic section such that

$$\int_{\mathcal{M}} \omega \Psi_i \bar{\Psi}_j = \delta_{ij}, \quad (\partial_{\bar{z}} - i(N+g-1)\mathcal{A}_{\bar{z}})\Psi_i = 0$$

where g is genus of $\mathcal{M}$.

- Then, number of such basis is "N" by index theorem.
- Holomorphic section correspond to a transformation of basis.

Section　　　　　　　　Orthonormal basis

$$\Psi_i(z, \bar{z}) \quad \longleftrightarrow \quad \overline{\langle i} \underline{| z, \bar{z} \rangle}$$

Coherent states

Our result

- We construct the Hermitian matrices $X^\mu_{ij}\ (i,j=1,\cdots,N)$ as follows.

$$X^\mu_{ij} = \int_{\mathcal{M}} \omega(z) \int_{\mathcal{M}} \omega(z')\, \Psi_i(z) X^\mu_{zz'} \bar{\Psi}_j(z')$$

$$X^\mu_{zz'} = \frac{1}{2}\left(y^\mu(z) + y^\mu(z') - \frac{i}{2}\left(W^{\mu\nu}(z) + W^{\mu\nu}(z')\right)\left(y_\nu(z) - y_\nu(z')\right)\right) P_{zz'}$$

$$P_{zz'} = e^{-\frac{N}{4} d^2_{zz'} + iN\int_z^{z'} \mathcal{A}}$$

$d^2_{zz'}$: geodesic distance of two points

- Then, we can check that all required conditions of MR is satisfied using Bergman kernel expansion.

Bergman kernel: $\displaystyle \sum_i \bar{\Psi}_i(z)\Psi_i(z') \underset{N\to\infty}{\sim} N P_{zz'}(1+\mathcal{O}(N^{-1}))$

[cf. Bordemann-Meinrenken-Schlichenmaier, Asakawa-Sugimoto-Terashima]

Summary

- We introduce the locality in the matrix geometry using coherent states.
- We proposed a construction method of MR of arbitrary Riemann surfaces embedded in $\mathbb{R}^D$.

Future work

- Our construction method gives an algorithm to numerical generate matrix elements of X^μ.
- Application to matrix models.

Thank you for your attention !!