Recurrence Relations for Finite-Temperature Correlators via AdS_2/CFT_1

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Based on:
SO, `arXiv:1309.2939`

Introduction

Introduction

- Conformal symmetry is powerful enough to constrain possible forms of correlation functions.
- Indeed, up to overall normalization factors, two- and three-point functions are completely fixed by $SO(2,d)$ conformal symmetry in any spacetime dimension $d \geq 1$ **[Polyakov '70]**:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \delta_{\Delta_1\Delta_2}\frac{C_{\Delta_1\Delta_2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{\Delta_1\Delta_2\Delta_3}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}$$

- Conformal constraints work well in coordinate space.
- Then, what about conformal constraints in momentum space?

Introduction

- Correlation functions in momentum space are directly related to physical observables.
  - Example: imaginary part of retarded two-point function = spectral density
- So it would be desirable to understand how conformal symmetry constrains the possible forms of momentum-space correlators.
- In principle, momentum-space correlators are just obtained by Fourier transforms of position-space correlators.
- However, Fourier transforms of position-space correlators are generally hard.
- Indeed, in spite of its simplicity in coordinate space, three-point functions in momentum space are known to be very complicated.
  - In fact, Fourier transform of three-point functions in finite-temperature CFT_2 was first computed in 2014! **[Becker-Cabrera-Su '14]**
  - The study of conformal constraints in momentum space is still ongoing **[Corianò-Delle Rose-Mottola-Serino '13] [Bzowski-McFadden-Skenderis '13]**.

Introduction

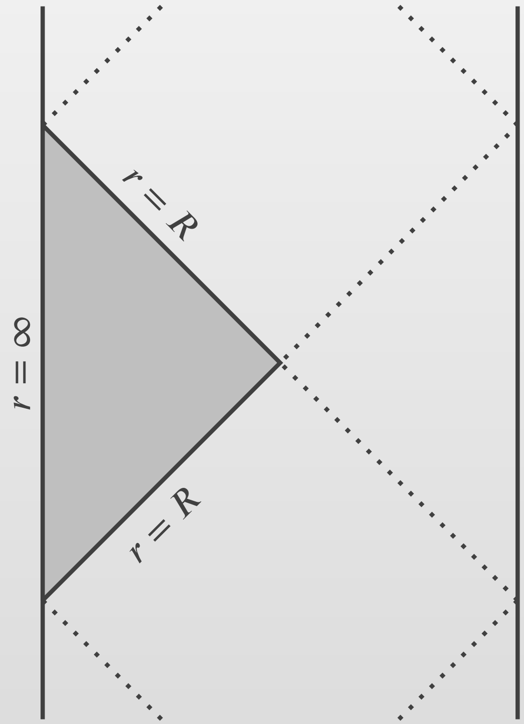
- Today I will present a simple algebraic approach to compute **finite-temperature CFT two-point functions in momentum space**.
- For the sake of simplicity I shall focus on finite-temperature CFT_1.
- The keys to my approach are:
  - 1d conformal algebra $\mathfrak{so}(2,1)$ in the basis in which the $SO(1,1)$ generator becomes diagonal; and
  - Killing vectors of AdS_2 black hole.

AdS_2 Black Hole

AdS_2 black hole

- The AdS_2 black hole is a portion of AdS_2; it is just a single **Rindler wedge** of AdS_2 and described by the following metric:

$$ds^2_{AdS_2} = -\left(\frac{r^2}{R^2} - 1\right) dt^2 + \frac{dr^2}{r^2/R^2 - 1}, \quad r \in (R, \infty)$$



- ∘ AdS_2 is topologically an infinite strip.
- ∘ The AdS_2 black hole covers only a part of the whole AdS_2.
  - $r = R$: Rindler horizon
  - $r = \infty$: AdS_2 boundary

AdS_2 black hole

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- For the following discussions it is convenient to introduce a new coordinate system (t, x) via

$$r = R\coth(x/R), \quad x \in (0, \infty)$$

in which the metric becomes conformally flat:

$$ds^2_{AdS_2} = \frac{-dt^2 + dx^2}{\sinh^2(x/R)}$$

- **Below I will work in the units $R = 1$.**

AdS_2 and $SO(2,1)$

- The one-dim'l conformal group $SO(2,1)$, which is the isometry of AdS_2, contains three distinct one-parameter subgroups:
  - compact rotation group $SO(2)$
  - noncompact Euclidean group $E(1)$
  - noncompact Lorentz group $SO(1,1)$

- Correspondingly, there exist three distinct classes of static AdS_2 coordinate patches in which time-translation Killing vectors generate these one-parameter subgroups $SO(2)$, $E(1)$ and $SO(1,1)$.

- In Lorentzian signature, these coordinate patches are given by the global, Poincaré and Rindler coordinates, respectively.

coordinate patch	time-translation group		frequency spectrum	
	Lorentzian	Euclidean	Lorentzian	Euclidean
global	$SO(2)$	$SO(1,1)$	discrete	continuous
Poincaré	$E(1)$	$E(1)$	continuous	continuous
Rindler	$SO(1,1)$	$SO(2)$	continuous	discrete (Matsubara frequency)

Correlator Recurrence Relations

1d conformal algebra: $SO(2)$ diagonal basis

- The one-dim'l conformal algebra $\mathfrak{so}(2,1)$ is spanned by the three generators $\{J_1, J_2, J_3\}$ that satisfy the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = -iJ_2$$

- In the Cartan-Weyl basis $\{J_3, J_\pm := -J_1 \pm iJ_2\}$ the commutation relations become

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_3$$

- The quadratic Casimir of the Lie algebra $\mathfrak{so}(2,1)$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = J_3(J_3 \pm 1) - J_\mp J_\pm$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of C and J_3 that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad J_3|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

Then the state $J_\pm|\Delta, \omega\rangle$ satisfies $J_3 J_\pm|\Delta, \omega\rangle = (\omega \pm 1)J_\pm|\Delta, \omega\rangle$, which implies the ladder equations

$$\boxed{J_\pm|\Delta, \omega\rangle \propto |\Delta, \omega \pm 1\rangle}$$

1d conformal algebra: $SO(1,1)$ diagonal basis

- Let us next consider the following hermitian linear combinations

$$A_1 = J_1, \quad A_\pm = J_2 \pm J_3$$

which satisfy the commutation relations

$$[A_1, A_\pm] = \pm i A_\pm, \quad [A_+, A_-] = 2i A_1$$

- The quadratic Casimir of the Lie algebra $\mathfrak{so}(2,1)$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = -A_1(A_1 \pm i) - A_\mp A_\pm$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of C and A_1 that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad A_1|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

Then the state $A_\pm|\Delta, \omega\rangle$ satisfies $A_1 A_\pm|\Delta, \omega\rangle = (\omega \pm i)A_\pm|\Delta, \omega\rangle$, which implies the ladder equations

$$\boxed{A_\pm|\Delta, \omega\rangle \propto |\Delta, \omega \pm i\rangle}$$

1d conformal algebra: $SO(1,1)$ diagonal basis

- In the AdS_2 black hole problem, the $SO(2,1)$ generators (**Killing vectors**) are given by the following first-order differential operators:

$$\begin{aligned} A_1 &= i\partial_t \\ A_\pm &= \mathrm{e}^{\pm t}\left[\sinh x(i\partial_x) \pm \cosh x(i\partial_t)\right] \end{aligned}$$

- The quadratic Casimir gives the d'Alembertian on the AdS_2 black hole:

$$C = A_1(A_1 \pm i) - A_\mp A_\pm = \sinh^2 x\left(-\partial_t^2 + \partial_x^2\right)$$

- The eigenvalue equations reduce to the Schrödinger equation:

$$\begin{aligned} A_1|\Delta,\omega\rangle = \omega|\Delta,\omega\rangle \quad &\Leftrightarrow \quad i\partial_t\Phi_{\Delta,\omega}(t,x) = \omega\Phi_{\Delta,\omega}(t,x) \\ C|\Delta,\omega\rangle = \Delta(\Delta-1)|\Delta,\omega\rangle \quad &\Leftrightarrow \quad \left(-\partial_x^2 + \frac{\Delta(\Delta-1)}{\sinh^2 x}\right)\Phi_{\Delta,\omega}(t,x) = \omega^2\Phi_{\Delta,\omega}(t,x) \end{aligned}$$

- The ladder equations are

$$\boxed{\boldsymbol{A_\pm\Phi_{\Delta,\omega} \propto \Phi_{\Delta,\omega\pm i}}}$$

1d conformal algebra: $SO(1,1)$ diagonal basis

- Finite-temperature CFT_1 lives on the boundary $x = 0$. To analyze this, let us consider the asymptotic near-boundary limit $x \to 0$ of the Killing vectors

$$A_1^0 := \lim_{x\to 0} A_1 = i\partial_t$$

$$A_\pm^0 := \lim_{x\to 0} A_\pm = \mathrm{e}^{\pm t}\left(ix\partial_x \pm i\partial_t\right)$$

- The quadratic Casimir near the boundary is

$$C^0 = A_1^0(A_1^0 \pm i) - A_\mp^0 A_\pm^0 = x^2\partial_x^2$$

- The eigenvalue equations are

$$i\partial_t \Phi^0_{\Delta,\omega}(t,x) = \omega \Phi^0_{\Delta,\omega}(t,x)$$

$$\left(-\partial_x^2 + \frac{\Delta(\Delta-1)}{x^2}\right)\Phi^0_{\Delta,\omega}(t,x) = 0$$

which are easily solved with the result

$$\Phi^0_{\Delta,\omega}(t,x) = A_\Delta(\omega)x^\Delta \mathrm{e}^{-i\omega t} + B_\Delta(\omega)x^{1-\Delta}\mathrm{e}^{-i\omega t}$$

where $A_\Delta(\omega)$ and $B_\Delta(\omega)$ are integration constants which may depend on Δ and ω.

Correlator recurrence relations

- The ladder equations $A^0_\pm \Phi^0_{\Delta,\omega} \propto \Phi^0_{\Delta,\omega\pm i}$ become

$$
\begin{aligned}
&(i\Delta \pm \omega)A_\Delta(\omega)x^\Delta \mathrm{e}^{-i(\omega\pm i)t} + (i(1-\Delta)\pm\omega)B_\Delta(\omega)x^{1-\Delta}\mathrm{e}^{-i(\omega\pm i)t} \\
\propto\ &A_\Delta(\omega\pm i)x^\Delta \mathrm{e}^{-i(\omega\pm i)t} + B_\Delta(\omega\pm i)x^{1-\Delta}\mathrm{e}^{-i(\omega\pm i)t}
\end{aligned}
$$

  from which we get

$$
\begin{aligned}
(i\Delta \pm \omega)A_\Delta(\omega) &\propto A_\Delta(\omega\pm i) \\
(i(1-\Delta)\pm\omega)B_\Delta(\omega) &\propto B_\Delta(\omega\pm i)
\end{aligned}
$$

- According to the real-time prescription of AdS/CFT correspondence, two-point functions of dual CFT_1 are given by the ratio **[Iqbal-Liu '09]**

$$G_\Delta(\omega) = (2\Delta - 1)\frac{A_\Delta(\omega)}{B_\Delta(\omega)}$$

  which satisfies the recurrence relations

$$\boxed{G_\Delta(\omega) = \frac{-1+\Delta\pm i\omega}{-\Delta\pm i\omega}G_\Delta(\omega\pm i)}$$

Correlator recurrence relations

- The recurrence relations

$$G_\Delta(\omega) = \frac{-1+\Delta \pm i\omega}{-\Delta \pm i\omega} G_\Delta(\omega \pm i)$$

are easily solved by iteration. Minimal solutions are

$$G_\Delta^{A/R}(\omega) = \frac{\Gamma(\Delta \pm i\omega)}{\Gamma(1-\Delta \pm i\omega)} g^{A/R}(\Delta)$$

where $g^{A/R}(\Delta)$ are ω-independent normalization factors.

- Restoring R via $\omega \to \omega R$, we get the advanced/retarded two-point functions for a scalar primary operator of scaling dimension Δ:

$$G_\Delta^{A/R}(\omega) = \frac{\Gamma(\Delta \pm \frac{i\omega}{2\pi T})}{\Gamma(1-\Delta \pm \frac{i\omega}{2\pi T})} g^{A/R}(\Delta)$$

where T is the Hawking temperature given by

$$T = \frac{1}{2\pi R}$$

Summary & Outlook

Summary & outlook

Summary

- $SO(2,1)$ isometry of the AdS_2 black hole induces the recurrence relations for finite-temperature CFT_1 two-point functions:

$$G_\Delta(\omega) = \frac{-1+\Delta \pm i\omega}{-\Delta \pm i\omega} G_\Delta(\omega \pm i)$$

- The minimal solutions to the recurrence relations give the advanced/retarded two-point functions in frequency space.

Outlook

- Generalizations to finite-temperature CFT_d. The simplest approach would be to consider the Rindler-AdS_{d+1} described by the metric

$$ds^2_{\mathrm{AdS}_{d+1}} = -\left(\frac{r^2}{R^2} - 1\right) dt^2 + \frac{dr^2}{r^2/R^2 - 1} + r^2 dH^2_{d-1}$$

where dH_{d-1} stands for the line element of $(d-1)$-dim'l hyperbolic space $\mathbb{H}^{d-1}$.
(The case $d = 2$ has been done in the previous work `arXiv:1312.7348`.)