The $(2,0)$ Superconformal Bootstrap

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Based on work with

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$(2,0)$ theories

Nahm's classification: superconformal algebras exist for $d \leqslant 6$.
In $d = 6$, $(\mathcal{N}, 0)$ algebras. Existence of $T_{\mu\nu}$ multiplet requires $\mathcal{N} \leqslant 2$.

$(2,0)$: maximal susy in maximal d. No marginal couplings allowed.

Interacting models inferred from string/M-theory: ADE catalogue.

Central to many recent developments in QFT.
"Mothers" of many interesting QFTs in $d < 6$.

Key properties:

- Moduli space of vacua
$$\mathcal{M}_{\mathfrak{g}} = (\mathbb{R}^5)^{r_{\mathfrak{g}}}/W_{\mathfrak{g}}, \qquad \mathfrak{g} = \{A_n, D_n, E_6, E_7, E_8\}.$$
- On $\mathbb{R}^5 \times S^1$, IR description as $5d$ MSYM with gauge algebra $\mathfrak{g}$.

At large n, A_n and D_n theories described through AdS/CFT:
M-theory on $AdS_7 \times S^4$ and $AdS_7 \times \mathbb{RP}^4$.

The $(2,0)$ theories as abstract CFTs

No intrinsic field-theoretic formulation yet.
No conventional Lagrangian (hard to imagine one from RG lore).

Working hypothesis: (at least) for correlators of local operators in $\mathbb{R}^6$, the $(2,0)$ theory is just another CFT, defined by a local operator algebra

$$\mathrm{OPE}: \quad \mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k}(x)\mathcal{O}_k(0)$$

Can symmetry and basic consistency requirements *completely determine* the spectrum and OPE coefficients?

Abstract CFT Framework

A general Conformal Field Theory hasn't much to do with "fields" (of the kind we write in Lagrangians).

We'll think more abstractly. A CFT is *defined* by its local operators,

$$\mathcal{A} \equiv \{\mathcal{O}_k(x)\} \,,$$

and their correlation functions $\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$.

$\mathcal{A}$ is an algebra. Operator Product Expansion (OPE),

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k} \left(\mathcal{O}_k(0) + \ldots \right) \,,$$

where the $\ldots$ are fixed by conformal invariance. The sum converges.

Caveat I: This definition does *not* capture non-local observables, such as conformal defects. (E.g., Wilson lines in a conformal gauge theory.).

Reduce $n$pt to $(n-1)$pt,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle = \sum_k c_{12k}(x_2)\,\langle \mathcal{O}_k(x_2)\ldots\mathcal{O}_n(x_n)\rangle\,.$$

1pt functions are trivial, $\langle \mathcal{O}_i(x)\rangle = 0$ except for $\langle \mathbf{1}\rangle \equiv 1$.

$\mathcal{O}_{\Delta,\ell,f}(x)$ labeled by conformal dimension Δ, Lorentz representation ℓ and possibly flavor quantum number f.

The CFT data $\{(\Delta_i,\ell_i,f_i)\,,c_{ijk}\}$ completely specify the theory.

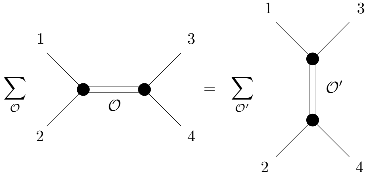
But not anything goes! Consistency conditions:

- Associativity:
$$(\mathcal{O}_1\mathcal{O}_2)\,\mathcal{O}_3 = \mathcal{O}_1\,(\mathcal{O}_2\mathcal{O}_3)\ .$$
- Unitarity (reflection positivity):
  Lower bounds on Δ for given ℓ;
  $c_{ijk}\in\mathbb{R}$

Caveat II: In non-trivial geometries, $\langle\mathcal{O}\rangle \neq 0 \to$ additional constraints.
In $d=2$, modularity. In $d>2$, harder to analyze, have been ignored so far.

The bootstrap program

Old aspiration (1970s) Ferrara Gatto Grillo, Polyakov.
Associativity $\equiv$ crossing symmetry of $4pt$ functions



Vastly over-constrained system of equations for $\{\Delta_i, c_{ijk}\}$.
Classification and construction of CFTs reduced to an algebraic problem.

• Famous success story in $d = 2$, starting from BPZZ (1984).
$2d$ conformal symmetry is infinite dimensional, $z \to f(z)$.
In some cases, *finite*-dimensional bootstrap problem (rational CFTs).
Many exact solutions, partial classification.

Bootstrapping in two steps

For $d = 6$, $\mathcal{N} = (2, 0)$ SCFTs (as well as $d = 4$, $\mathcal{N} \geqslant 2$ SCFTs) the crossing equations split into

(1) Equations that depend only on intermediate BPS operators. Captured by the $2d$ chiral algebra. "Minibootstrap"

(2) Equations that also include intermediate non-BPS operators. "Maxibootstrap"

(1) are tractable and determine an infinite amount of CFT data.

This is essential input to the full-fledged bootstrap (2), which can be studied numerically.

Beem Lemos Liendo Peelaers LR van Rees, Beem LR van Rees

Meromorphy in $(2,0)$ SCFTs

Fix a plane $\mathbb{R}^2 \subset \mathbb{R}^6$, parametrized by $(z, \bar{z})$.

Claim : $\exists$ subsector $\mathcal{A}_\chi = \{\mathcal{O}_i(z_i, \bar{z}_i)\}$ with meromorphic

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = f(z_i)\,.$$

Rationale: $\mathcal{A}_\chi \equiv$ cohomology of a nilpotent $\mathbb{Q}$,

$$\mathbb{Q} = \mathcal{Q} + \mathcal{S}\,,$$

$\mathcal{Q}$ Poincaré, $\mathcal{S}$ conformal supercharges.

$\bar{z}$ dependence is $\mathbb{Q}$-exact: cohomology classes $[\mathcal{O}(z, \bar{z})]_{\mathbb{Q}} \rightsquigarrow \mathcal{O}(z)$.

Analogous to the $d = 4$, $\mathcal{N} = 1$ chiral ring:
cohomology classes $[\mathcal{O}(x)]_{\tilde{\mathcal{Q}}_{\dot{\alpha}}}$ are x-independent.

Cohomology

At the origin of $\mathbb{R}^2$, $\mathbb{Q}$-cohomology $\mathcal{A}_\chi$ easy to describe.

$\mathcal{O}(0,0) \in \mathcal{A}_\chi \leftrightarrow \mathcal{O}$ obeys the chirality condition

$$\frac{\Delta - \ell}{2} = R$$

Δ conformal dimension, ℓ angular momentum on $\mathbb{R}^2$,
R Cartan generator of $SU(2)_R \cong SO(3)_R \subset SO(5)$ R-symmetry.

$$[\mathbb{Q}, \mathfrak{sl}(2)] = 0 \quad \text{but} \quad [\mathbb{Q}, \overline{\mathfrak{sl}(2)}] \neq 0$$

To define $\mathbb{Q}$-closed operators $\mathcal{O}(z, \bar{z})$ away from origin, we twist the right-moving generators by $SU(2)_R$,

$$\widehat{L}_{-1} = \bar{L}_{-1} + \mathcal{R}^{-}\,, \quad \widehat{L}_0 = \bar{L}_0 - \mathcal{R}\,, \quad \widehat{L}_1 = \bar{L}_1 - \mathcal{R}^{+}$$

$$\widehat{\mathfrak{sl}(2)} = \{\mathbb{Q}, \dots\}$$

$\mathbb{Q}$-closed operators are "twisted-translated"

$$\begin{aligned} \mathcal{O}(z, \bar{z}) &= e^{zL_{-1} + \bar{z}\widehat{L}_{-1}}\, \mathcal{O}^{1\dots 1}(0)\, e^{-zL_{-1} - \bar{z}\widehat{L}_{-1}} \\ &= u_{\mathcal{I}_1}(\bar{z}) \dots u_{\mathcal{I}_k}(\bar{z}) \mathcal{O}^{\mathcal{I}_1 \dots \mathcal{I}_k}(z, \bar{z}) \qquad\qquad u_{\mathcal{I}} \equiv (1, \bar{z}) \end{aligned}$$

$SU(2)_R$ orientation correlated with position on $\mathbb{R}^2$.

Example: free $(2,0)$ tensor multiplet

$$\Phi_I\,,\quad \lambda_{aA}\,,\quad \omega^+_{ab}$$

$I = SO(5)_R$ vector index.

Scalar in $SO(3)_R \subset SO(5)_R$ h.w. is only field obeying $\Delta - \ell = 2R$

$$\Phi_{h.w.} = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}\,,\quad \Delta = 2R = 2\,,\quad \ell = 0\,.$$

Cohomology class of twisted-translated field

$$\Phi(z) := \left[\Phi_{h.w.}(z,\bar z) + \bar z \Phi_3(z,\bar z) + \bar z^2 \Phi^*_{h.w.}(z,\bar z)\right]_{\mathbb{Q}}$$

$$\Phi(z)\,\Phi(0) \sim \bar z^2 \Phi^*_{h.w.}(z,\bar z)\,\Phi_{h.w.}(0) \sim \frac{\bar z^2}{z^2\,\bar z^2} = \frac{1}{z^2}\,.$$

$\Phi(z)$ is an $\mathfrak{u}(1)$ affine current, $\Phi(z) \rightsquigarrow J_{\mathfrak{u}(1)}(z)$.

χ_6 : 6d (2,0) SCFT $\longrightarrow$ 2d Chiral Algebra.

- Global $\mathfrak{sl}(2) \to$ Virasoro, indeed $T(z) := [\Phi_{(IJ)}(z, \bar{z})]_{\mathbb{Q}}$,
  with $\Phi_{(IJ)}$ the stress-tensor multiplet superprimary.

$$c_{2d} = c_{6d}$$

  in normalizations where c_{6d} (free tensor) $\equiv 1$.

- All $\frac{1}{2}$-BPS operators ($\Delta = 2R$) are in $\mathbb{Q}$ cohomology.

  Generators of the $\frac{1}{2}$-BPS ring $\to$ generators of the chiral algebra.

- Some semi-short multiplets with non-zero spin also play a role.

Chiral algebra for $(2,0)$ theory of type A_{N-1}

One $\frac{1}{2}$-BPS generator each of dimension $\Delta = 4, 6, \dots 2N$

$$\Downarrow$$

One chiral algebra generator each of dimension $h = 2, 3, \dots N$.

Most economical scenario: these are all the generators.
Check: the superconformal index computed by Kim[3] is reproduced:

$$\mathcal{I}(q,s) := \mathrm{Tr}(-1)^F q^{E-R} s^{h_2+h_3}$$

$$\mathcal{I}(q,s;n) = \prod_{k=2}^{n} \prod_{m=0}^{\infty} \frac{1}{1-q^{k+m}} = \mathrm{PE}\left[\frac{q^2+\cdots+q^n}{1-q}\right].$$

Plausibly a unique solution to crossing for this set of generators.

- The chiral algebra of the A_{N-1} theory is $\mathcal{W}_N$, with

$$c_{2d} = 4N^3 - 3N - 1\,.$$

Generalization to all ADE cases: $\mathcal{W}_{\mathfrak{g}}$ with $c_{2d} = 4d_{\mathfrak{g}}h^{\vee}_{\mathfrak{g}} + r_{\mathfrak{g}}$.

Half-BPS 3pt functions of $(2,0)$ SCFT

OPE of $\mathcal{W}_{\mathfrak{g}}$ generators $\Rightarrow$ half-BPS 3pt functions of SCFT.

Let us check the result at large N.

$W_{N\to\infty}$ with $c_{2d} \sim 4N^3 \to$ a *classical* Poisson algebra.
We can use results on universal Poisson algebra $W_\infty[\mu]$, with $\mu = N$.
(Gaberdiel Hartman, Campoleoni Fredenhagen Pfenninger)

We find

$$C(k_1,k_2,k_3) = \frac{2^{2\alpha-2}}{(\pi N)^{\frac{3}{2}}}\Gamma\left(\frac{\alpha}{2}\right)\left(\frac{\Gamma\left(\frac{k_{123}+1}{2}\right)\Gamma\left(\frac{k_{231}+1}{2}\right)\Gamma\left(\frac{k_{312}+1}{2}\right)}{\sqrt{\Gamma(2k_1-1)\Gamma(2k_2-1)\Gamma(2k_3-1)}}\right)$$

$k_{ijk} \equiv k_i + k_j - k_k$, $\alpha \equiv k_1 + k_2 + k_3$,
in precise agreement with calculation in $11d$ sugra on $AdS_7 \times S^4$!
(Corrado Florea McNees, Bastianelli Zucchini)

$1/N$ corrections in W_N OPE $\Rightarrow$ quantum M-theory corrections.

$(2,0)$ maxibootstrap Beem Lemos LR van Rees

Universal 4pt function of $\Phi_{(IJ)}$, superprimary of $T_{\mu\nu}$ multiplet.

Unique structure in superspace.

Only input: $6d$ Weyl anomaly coefficient c.

For ADE theories,

$$c = 4 d_{\mathfrak{g}} h^{\vee}_{\mathfrak{g}} + r_{\mathfrak{g}} \,,$$

but we keep it general.

Double OPE expansion

$$\langle \Phi\Phi\Phi\Phi \rangle = \sum_{\mathcal{O} \in \Phi \times \Phi} f^2_{\Phi\Phi\mathcal{O}}\, G^{\Phi}_{\mathcal{O}}$$

We impose the absence of higher-spin currents.

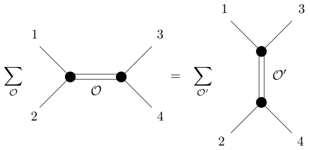
The $\mathcal{O}$s $\in \Phi \times \Phi$ are:

- Infinite set $\{\mathcal{O}_\chi\}$ of $\mathbb{Q}$-chiral BPS multiplets, fixed from χ-algebra.
- Infinite tower of BPS multiplet $\{\mathcal{D}, \mathcal{B}_1, \mathcal{B}_3, \ldots\}$, *not* in χ-algebra.
- Infinite set of non-BPS multiplets $\mathcal{L}_{\Delta,\ell}$, $\mathfrak{so}(5)_R$ singlets.
  Bose symmetry $\rightarrow$ ℓ is *even*. Unitarity bound $\Delta \geqslant \ell + 6$.

Unfixed BPS multiplets correspond to long multiplets at threshold,

$$\lim_{\Delta \to \ell+6} G^{\Phi}_{\mathcal{L}_{\Delta,\ell}} = G^{\Phi}_{\mathcal{B}_{\ell-1}} \qquad (\mathcal{D} \equiv \mathcal{B}_{-1}),$$

Bootstrap sum rule



When the dust settles, a *single* sum rule

$$\sum_{\text{long } \textit{super}\text{primaries}} f^2_{\Delta,\ell}\, \mathcal{F}_{\Delta,\ell}(z,\bar{z}) + \mathcal{F}^{\chi}(z,\bar{z};c) = 0$$

z, $\bar{z}$: conformal cross ratios;

$\mathcal{F}_{\Delta,\ell} \equiv \mathcal{G}_{\Delta,\ell} - \mathcal{G}^{\times}_{\Delta,\ell}$: *super*conformal block minus its crossing;

$\mathcal{F}^{\chi}(z,\bar{z};c)$: an explicitly known function (from minibootstrap).

The unknown CFT data to be constrained are:

- Set of (dimension, spin) $\{(\Delta_i, \ell_i)\}$ of the intermediate multiplets.
- The (squared) OPE coefficients $f^2_{\Delta_i,\ell_i}$. Non-negative by unitarity.

The numerical oracle (Rattazzi Rychkov Tonni Vichi)

$$\sum_{\Delta,\ell} f^2_{\Delta,\ell}\, \mathcal{F}_{\Delta,\ell}(z,\bar{z}) + \mathcal{F}^{\text{known}}(z,\bar{z};c) = 0$$

Use the sum rule to constrain the space of CFT data.

For example, consider a trial spectrum with $\Delta \geqslant \bar{\Delta}_\ell$ for operators of spin ℓ.
If there exists a linear functional χ such that

$$\chi \cdot \mathcal{F}_{\Delta,\ell}(z,\bar{z}) \geqslant 0 \quad \text{when } \Delta \geqslant \bar{\Delta}_\ell$$

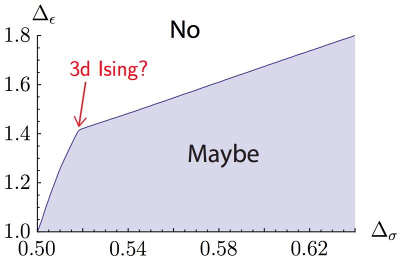
$$\chi \cdot \mathcal{F}^{\text{known}}(z,\bar{z};c) = 1$$

that trial spectrum is ruled out – oracle says NO.
If one cannot find such a χ, oracle says MAYBE.

Implemented by linear programming or semi-definite programming.
Surprisingly powerful!

Scalar bound in general $d = 3$ CFT

[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi, PRD **86**, 025022]



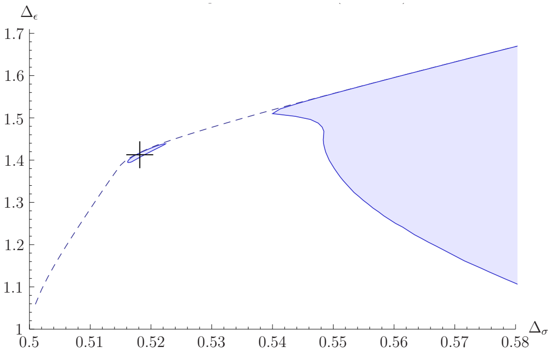
Exclusion plot in the subspace of $d = 3$ CFT data $(\Delta_\sigma, \Delta_\epsilon)$ with $\sigma \times \sigma = \mathbf{1} + \epsilon + \ldots$, from the bootstrap of a single 4pt function $\langle \sigma\sigma\sigma\sigma \rangle$.

Two real surprises:

- $3d$ Ising appears to lie on the exclusion curve (*i.e.* it saturates the bound)
- $3d$ Ising appears to sit at a special kink on the exclusion curve.

Multiple Correlators [Kos, Poland, Simmons-Duffin, '14]

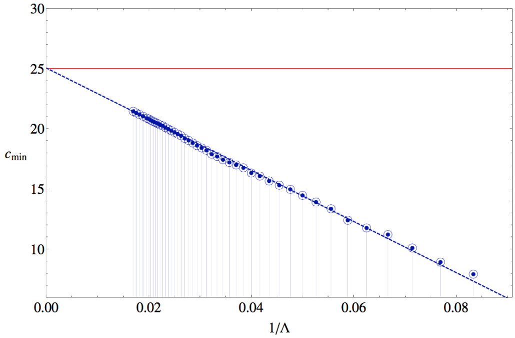
CFT$_3$ with $\mathbb{Z}_2$ symmetry. σ odd, ϵ even, $\sigma \times \sigma = \mathbf{1} + \epsilon + \ldots$
System of correlators $\langle \sigma\sigma\sigma\sigma \rangle$, $\langle \sigma\sigma\epsilon\epsilon \rangle$, $\langle \epsilon\epsilon\epsilon\epsilon \rangle$.
Allowed region assuming that only one odd scalar is relevant ($\Delta_{\sigma'} \geqslant 3$):



$3d$ Ising gets cornered! $\Delta_\sigma = 0.518151(5)$, $\Delta_\epsilon = 1.41263(5)$,
most accurate to date [Simmons-Duffin '15]

A lower bound on c

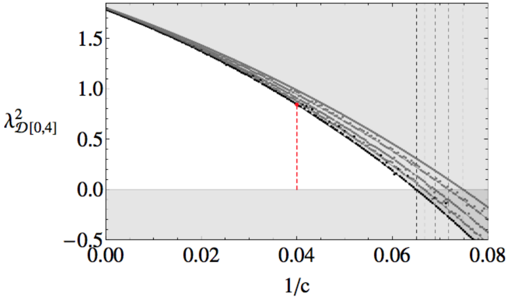
There is a minimum anomaly c_{min} compatible with crossing and unitarity. The bound c_{min} increases as we increase the search space for the functional, parametrized by a cutoff Λ.



Extrapolating, $c_{min} \to 25$, the value of the A_1 theory ($\equiv$ two $M5$s)!

(We are disallowing the free theory $(c = 1)$ by forbidding HS currents.)

For $c < c_{min}$, the oracle says NO. Why?



For $c < c_{min}$, solutions to crossing have $\lambda_D^2 < 0$, violating unitarity.

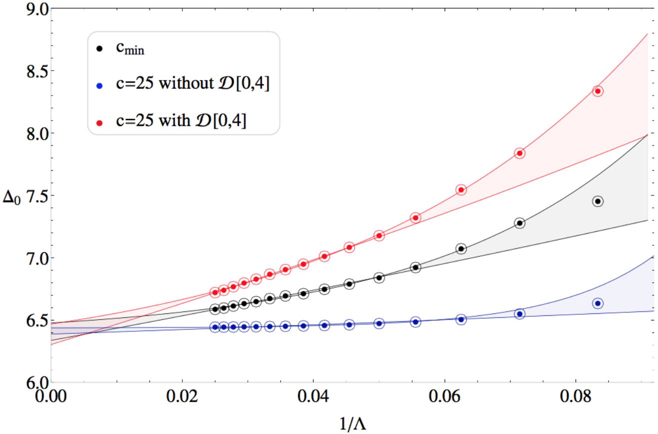
$\lambda_D^2 = 0$ precisely at $c = c_{min}$.

Agrees with conjecture of Batthacharyya and Minwalla about $\frac{1}{4}$BPS partition function of A_1 theory: $\mathcal{D}$ multiplet absent!

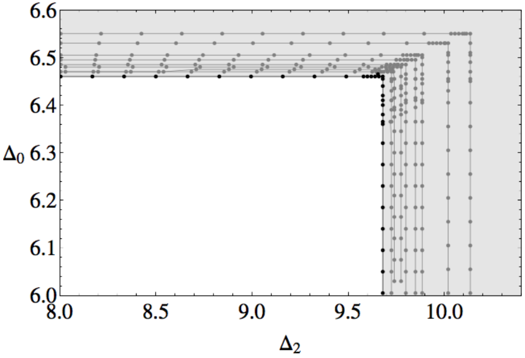
Bootstrapping the A_1 theory

For $c = c_{min} \to 25$, $\exists$ unique unitary solution to crossing.

Claim: The A_1 theory can be completely bootstrapped!



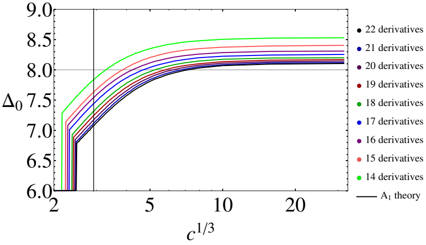
Upper bounds on the dimension of the leading-twist unprotected scalar, under different assumptions. Perfectly consistent.

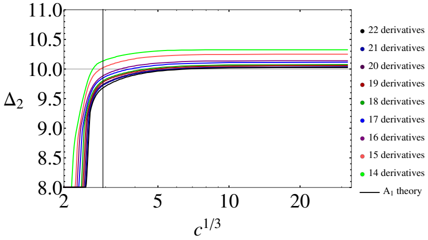


Exclusion region in (Δ_0, Δ_2) plane for $c = 25$ (A_1 value).

The corner values are conjectured to be the true leading-twist dimensions of the physical A_1 theory.

General c





Bounds for the leading-twist unprotected operators of spin $\ell = 0, 2$.

For $c \to \infty$, they appear to be saturated by $AdS_7 \times S^4$ sugra, including $1/c$ corrections.

For large c, leading-twist unprotected operators are double-traces of the form $\mathcal{O}_s = \mathcal{O}_{\mathbf{14}} \partial^s \mathcal{O}_{\mathbf{14}}$, with $\Delta_s = 8 + s - O(1/c)$.

Summary: Both at small and large c the bootstrap bounds appear to be saturated by physical $(2, 0)$ theories.

Outlook

The $(2,0)$ theories can be successfully studied by bootstrap methods.

- Exact results from the chiral algebra, *e.g.* $\frac{1}{2}$ BPS $3pt$ functions.

  Systematic $1/N$ expansion and its M-theory interpretation?

  A derivation of the AGT correspondence?
  Codimension-two defects $\Rightarrow$ Toda vertex operators?

- Numerical results for the non-protected spectrum.

  A_1 theory completely cornered by bootstrap equations.
  Beginning of a systematic algorithm to solve it.

  $A_{n>1}$ theories need input on BPS spectrum and multiple correlators.

  Precision numerics? Multiple correlators?

  Further analytic insights?