Mathematical proof for “physicist-friendly” reformulation of Atiyah-Patodi-Singer index

OSAKA UNIVERSITY
Live Locally, Grow Globally

Hidenori Fukaya (Osaka U.)

HF,. T Onogi, S. Yamaguchi PRD96(2017) no. 12, 125004 [arXiv:1710.03379]

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[arXiv: 19xx.xxxxx]

My talk today

In 2017 we proposed

"A physicist-friendly reformulation

of the Atiyah-Patodi-Singer(APS)

index theorem."

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Recently, we invited 3 mathematicians and succeeded in a mathematical proof.

F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, in progress

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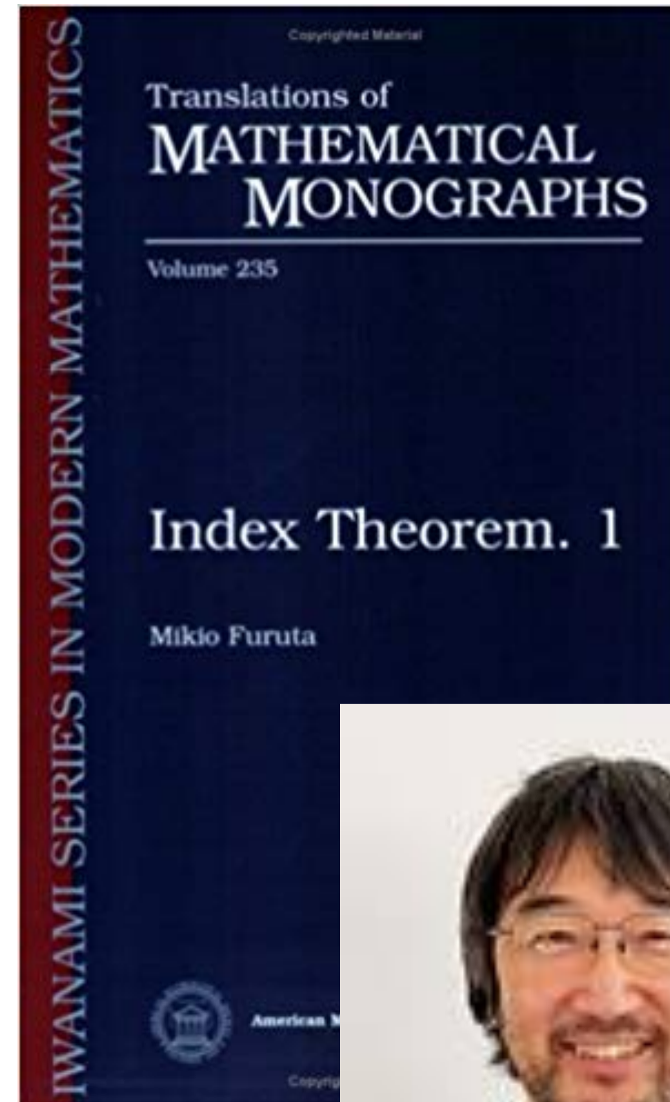


Recently, we invited 3 mathematicians and succeeded in a mathematical proof.

F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, in progress

Atiyah-Patodi-Singer index theorem

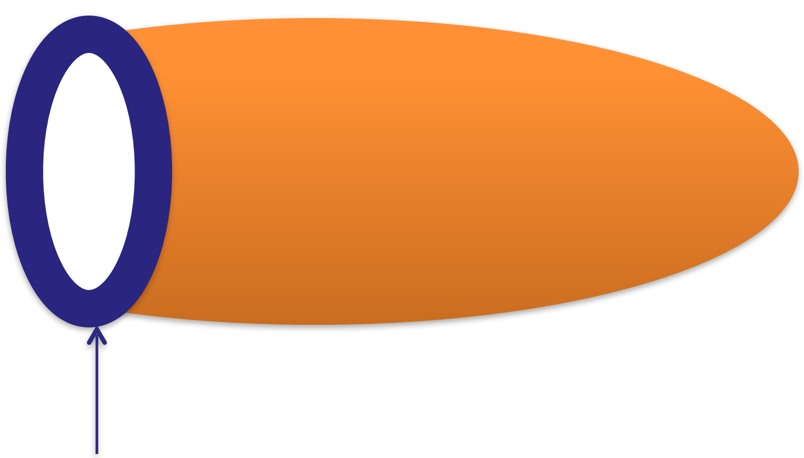
index on a manifold with boundary,

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 e^{D^2_{4\mathrm{D}}/\Lambda^2} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

integer

non-integer

non-integer



$x_4 = 0$

$$\eta(iD^{3\mathrm{D}}) = \sum_{\lambda\geq 0}{}^{reg} - \sum_{\lambda<0}{}^{reg} = \sum_{\lambda}{}^{reg}\mathrm{sgn}\lambda$$

[Atiyah-Patodi-Singer 1975]

APS index in topological insulator

Witten 2015 : APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator:

fermion path integrals

$$Z_{\rm edge} \propto \exp(-i\pi\eta(iD^{\rm 3D})/2)$$ T-anomalous

$$Z_{\rm bulk} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$ T-anomalous

$$Z_{\rm edge}Z_{\rm bulk} \propto (-1)^{\mathfrak{I}} = (-1)^{-\mathfrak{I}}$$ ➡ T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16&18, Freed-Hopkins 16, Witten 16, Yonekura 16&19 …]

What puzzled us

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   [except for Alvarez-Gaume et al. 1985 but boundary condition is obscure.]

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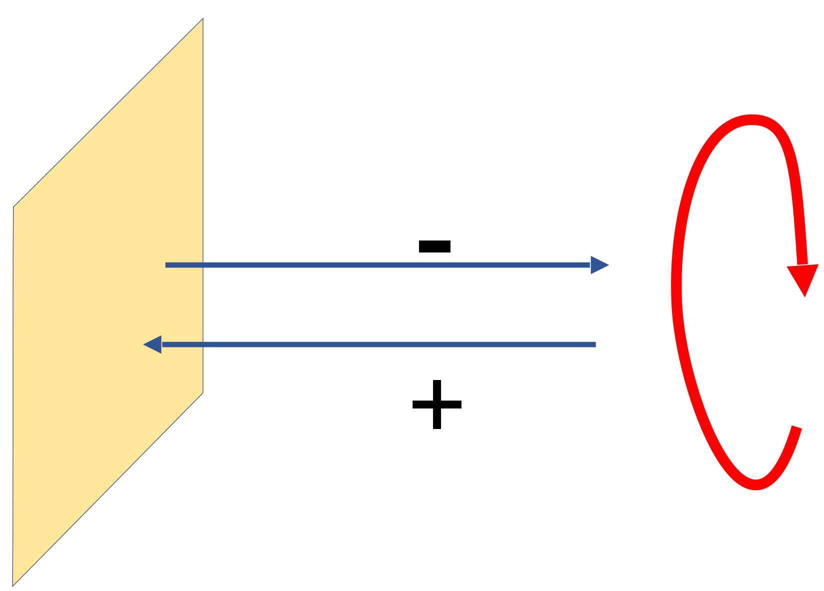
→ We launched a study group reading original APS paper and it took 3 months to translate it into “physics language”, and we reached an alternative expression.

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



angular momentum is conserved

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition

[Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry but keeps the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$\Rightarrow$

$$\mathrm{index} = n_+ - n_-$$

Beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.

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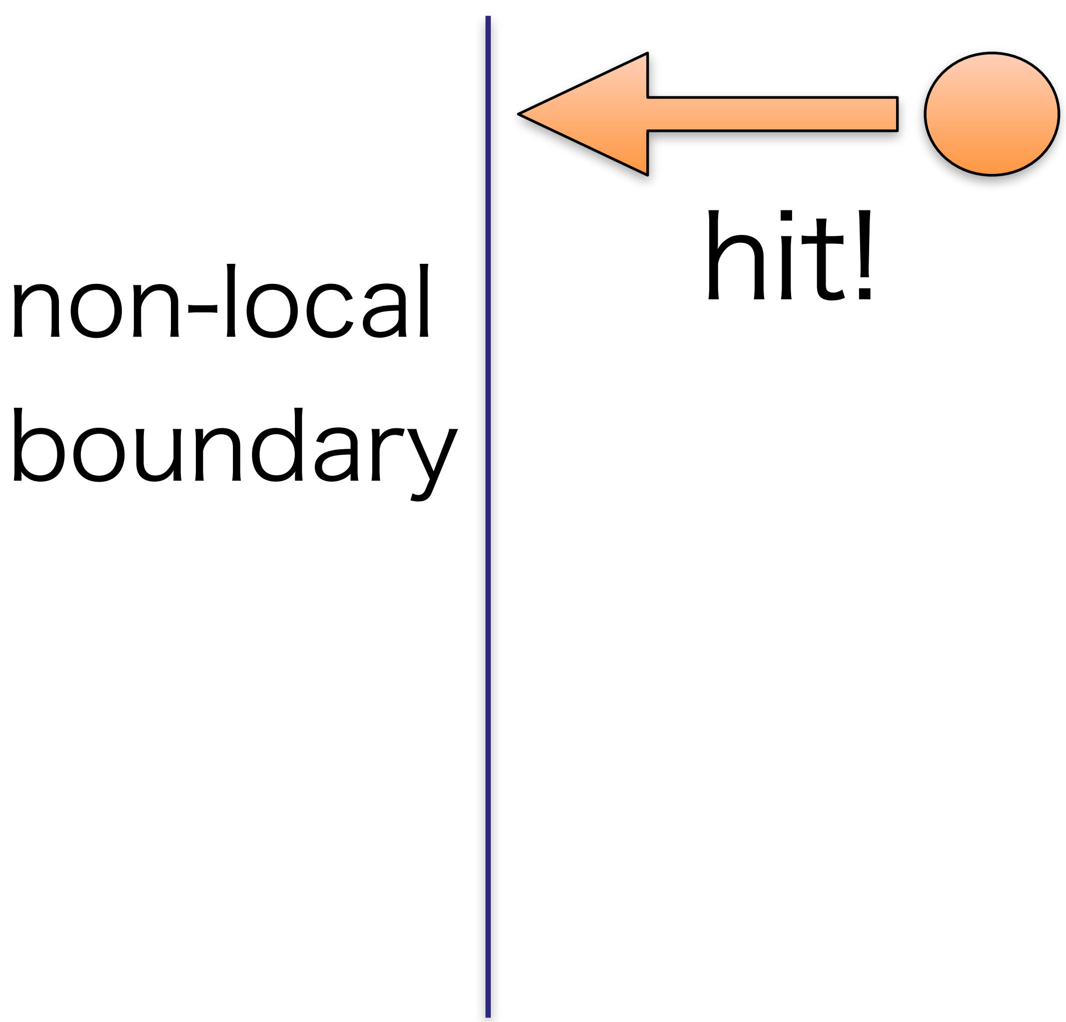

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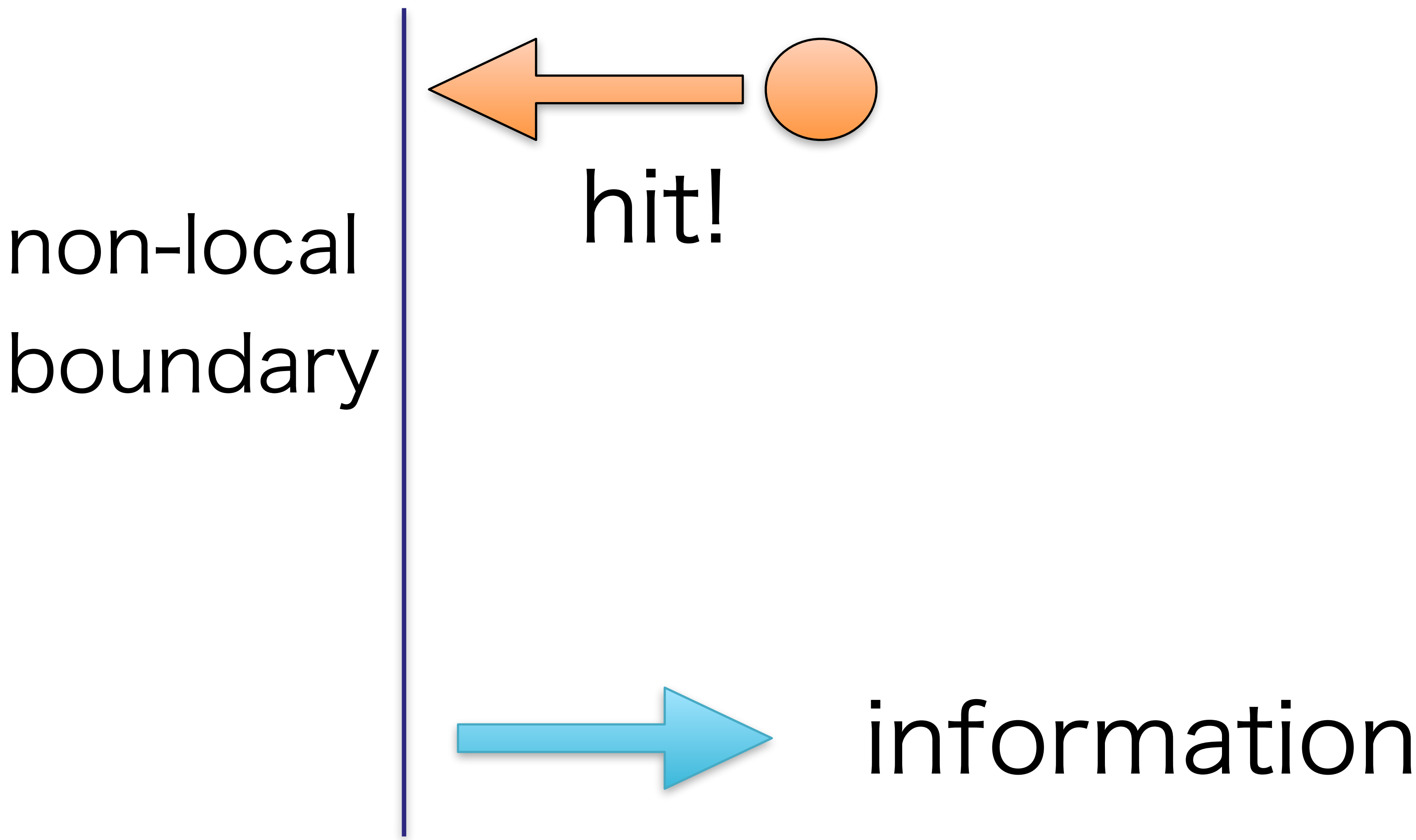

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information propagates faster than speed of light.

Locality >> chirality for physicists

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We cannot accept APS condition even if it is beautiful.

$\rightarrow$ need to give up chirality and consider L/R mixing

(massive case)

$$\cancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is "Yes, we can".

Contents

Atiyah-Singer(AS) index from massive Dirac operator

$$H = \gamma_5(D + M)$$

Zero-modes of D = still eigenstates of H:

$$H\phi_0 = \gamma_5 M\phi_0 = \pm M\phi_0.$$

Non-zero modes make ± pairs

$$H\phi_i = \lambda_i\phi_i \quad HD\phi_i = -DH\phi_i = -\lambda_i D\phi_i$$

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i$$

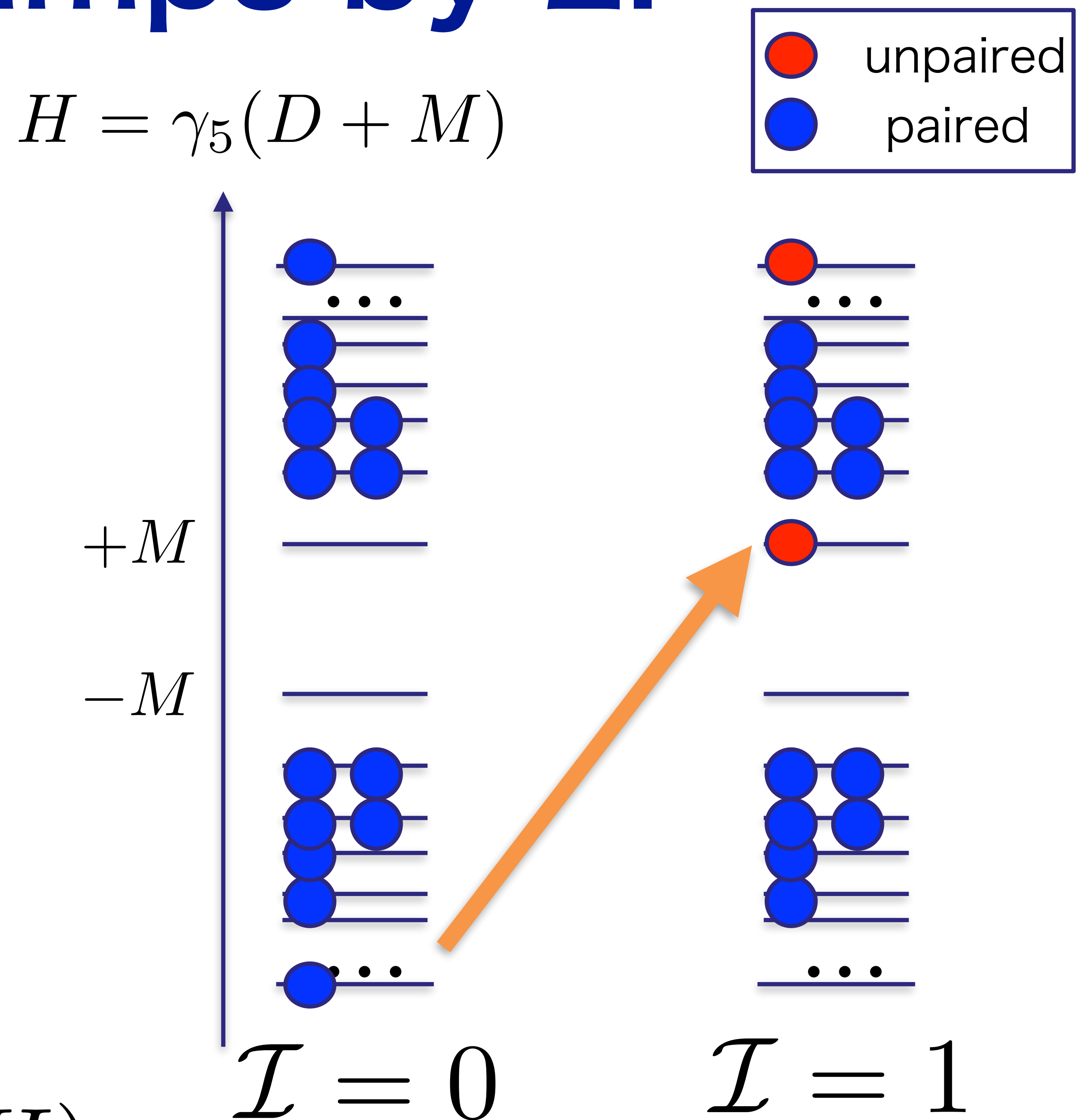
$$= \# \text{ of } +M - \# \text{ of } -M$$ = AS index!

$\eta(H)$ always jumps by 2.

To increase + modes,
we have to borrow
one from - (UV) modes.

Good regularizations
(e.g. Pauli-Villars, lattice)
respect this fact.

$$\Rightarrow \quad \mathrm{Index}(D) = \frac{1}{2}\eta(H).$$



Perturbative "proof" (in physics sense)

using Pauli-Villars regulator

$$\frac{1}{2}\eta(H)^{reg} = \frac{1}{2}\left[\eta(H) - \eta(H_{PV})\right].$$

$$H = \gamma_5(D+M)$$

$$H_{PV} = \gamma_5(D+\Lambda), \quad \Lambda \gg M$$

$$\eta(H) = \lim_{s\to 0} \mathrm{Tr}\frac{H}{(\sqrt{H^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr} H e^{-tH^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(M + \frac{D}{M}\right)e^{-t'D^\dagger D/M^2}e^{-t'},$$

$(t' = M^2 t)$

Fujikawa-method

does not contribute.

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathcal{O}(1/M^2).$$

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More physical set-up?

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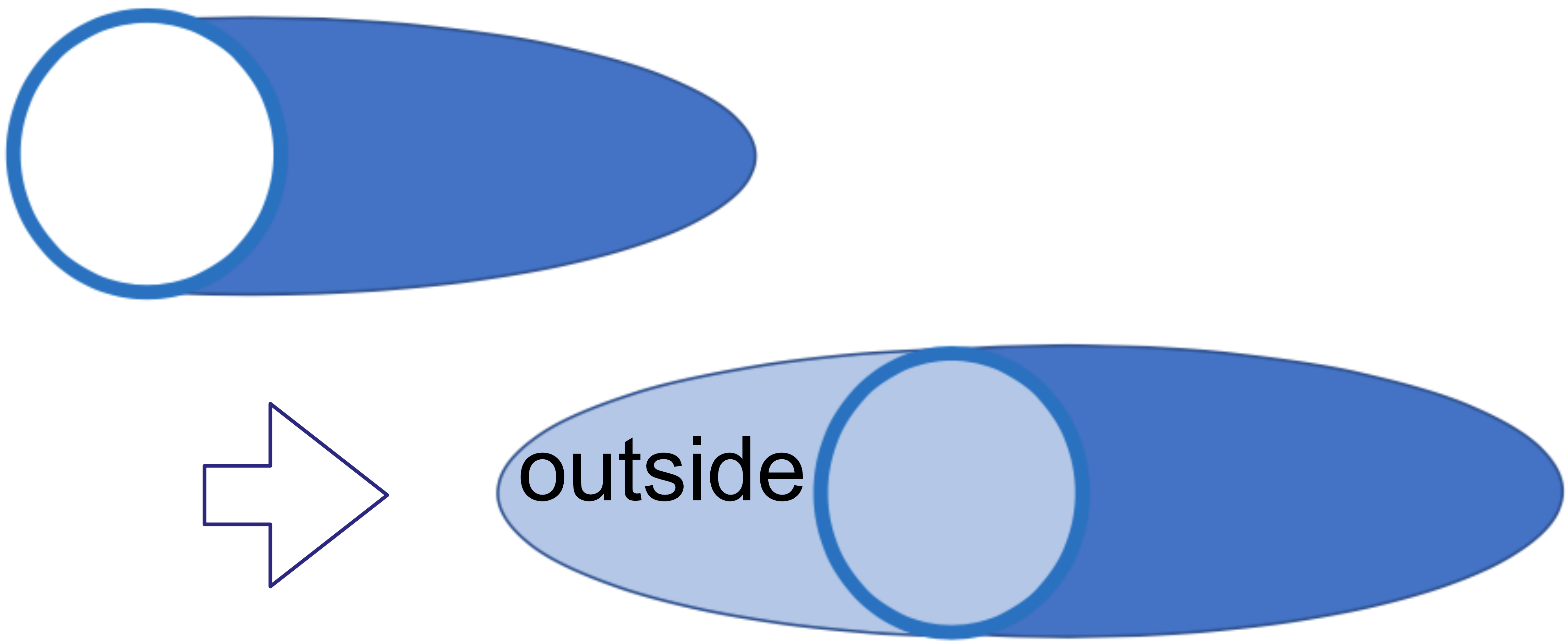
In physics,

1. Any boundary has "outside": manifold + boundary → domain-wall.



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In physics,

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3. Boundary condition should not be put by hand → but automatically chosen.
4. Edge-localized modes play the key role.

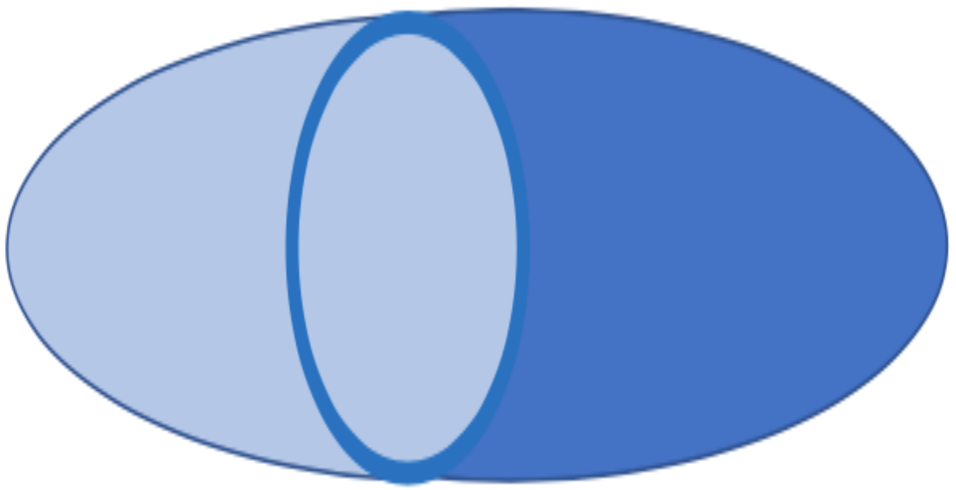
Domain-wall Dirac operator

Let us consider

$$D_{4\mathrm{D}} + M\epsilon(x_4), \quad \epsilon(x_4) = \mathrm{sgn} x_4$$

[Jackiw-Rebbi 1976,
Callan-Harvery 1985,
Kaplan 1992]

on a closed manifold
with sign flipping mass,
without assuming any
boundary condition
(we expect it dynamically given.).

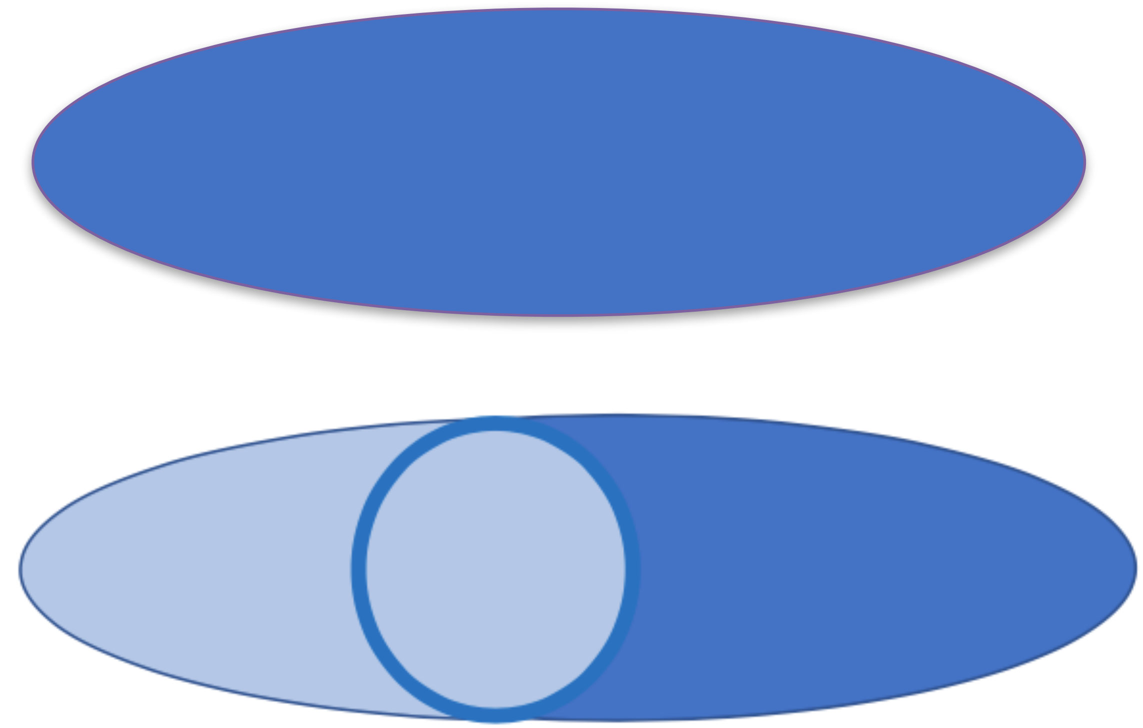


"new" APS index [F-Onogi-Yamaguchi 2017]

$$\frac{1}{2}\eta(\gamma_5(D+M))^{reg}$$ = AS index

⬇

$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$



$$=\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]-\frac{\eta(iD^{\mathrm{3D}})}{2}$$

which can be shown by Fujikawa-method.

See our paper or my talk slide in 2007.

Complete set in the free case

Solutions to

$$\{\gamma_5(D^{\rm free}+M\varepsilon(x_4))\}^2\phi=\left[-\partial_\mu^2+M^2-2M\gamma_4\delta(x_4)\right]\phi=\lambda^2\phi$$

are $\varphi(x_4)\otimes e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ where

$$\varphi^{\omega}_{\pm,o}(x_4)=\frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4}-e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4)=\frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega\mp M)e^{i\omega|x_4|}+(i\omega\pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\rm edge}_{+,e}(x_4)=\sqrt{M}e^{-M|x_4|},$$

→ Edge mode appears !

Here, $\omega=\sqrt{\boldsymbol{p}^2+M^2-\lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,{\rm edge}}_{\pm,e/o}=\pm\varphi^{\omega,{\rm edge}}_{\pm,e/o}$

“Automatic” boundary condition

We didn’t put any boundary condition by hand.
But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\bigg|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the domain-wall. This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

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Overview

$$\mathfrak{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

|| ||

CONJECTURE from perturbation theory in 4D flat space

$Ind(D_{\mathrm{APS}})$ with physicist-unfriendly boundary condition [APS 1975]	=	$\frac{1}{2}\eta(H_{DW})$ with physicist-friendly set-up (topological insulator) [FOY 2017]

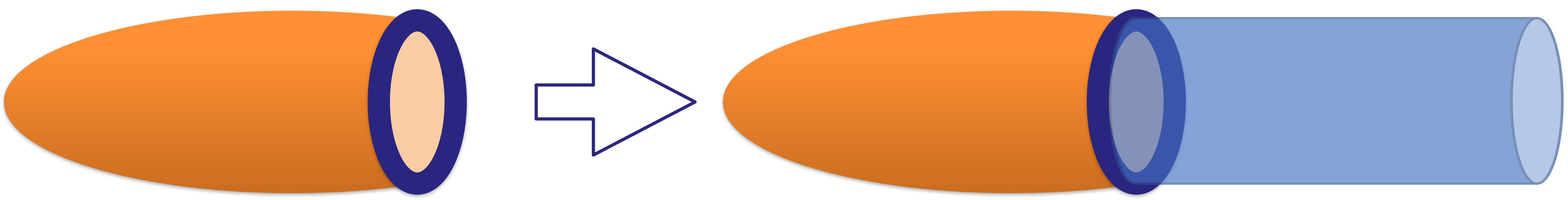
This work = THEOREM

(on any even-dim. curved manifold)

[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

Theorem 1:
APS index = index with infinite cylinder

In original APS paper, they showed

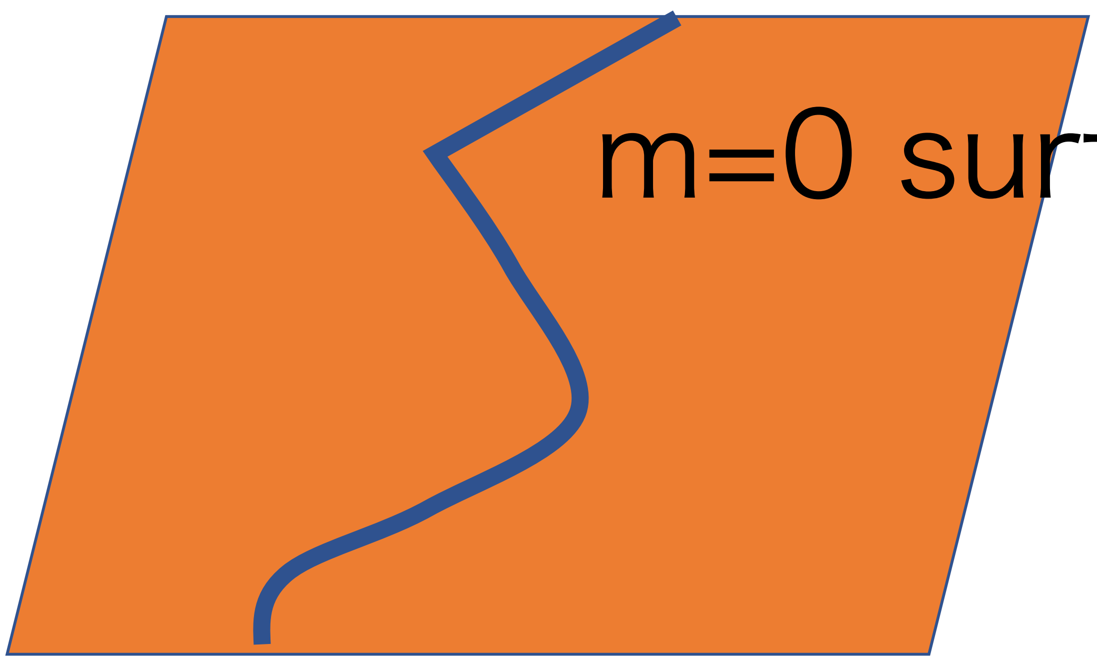


Index w/ APS b.c. = Index with infinite cylinder attached to the original boundary (w.r.t. square integrable modes).

* On cylinder, gauge fields are constant in the extra-direction.

Theorem 2:
Localization (& product formula)

By giving position-dependent "mass", we can localize the zero modes to "massless" lower-dimensional surface and the index is given by the product:



$$Ind(\gamma_s(D^d + \partial_s + i\gamma_s M(s))) =$$
$$Ind(D^d) \times Ind(\gamma_s \partial_s + M(s))$$

= generalization of domain-wall fermion

Theorem 3:
In odd-dim, APS index = boundary eta-invariant

$$\int F \wedge F \wedge \cdots$$

exists only in even-dim.

$$Ind(D^{odd-dim}_{\mathrm{APS}}) = \frac{1}{2}\left[\eta(D^{\mathrm{boundary1}}) - \eta(D^{\mathrm{boundary2}})\right]$$

5-dimensional Dirac operator

we consider

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5 (D^{4\mathrm{D}} + m(x_4, x_5)) \\ -\partial_5 + \gamma_5 (D^{4\mathrm{D}} + m(x_4, x_5)) & 0 \end{pmatrix}$$

where

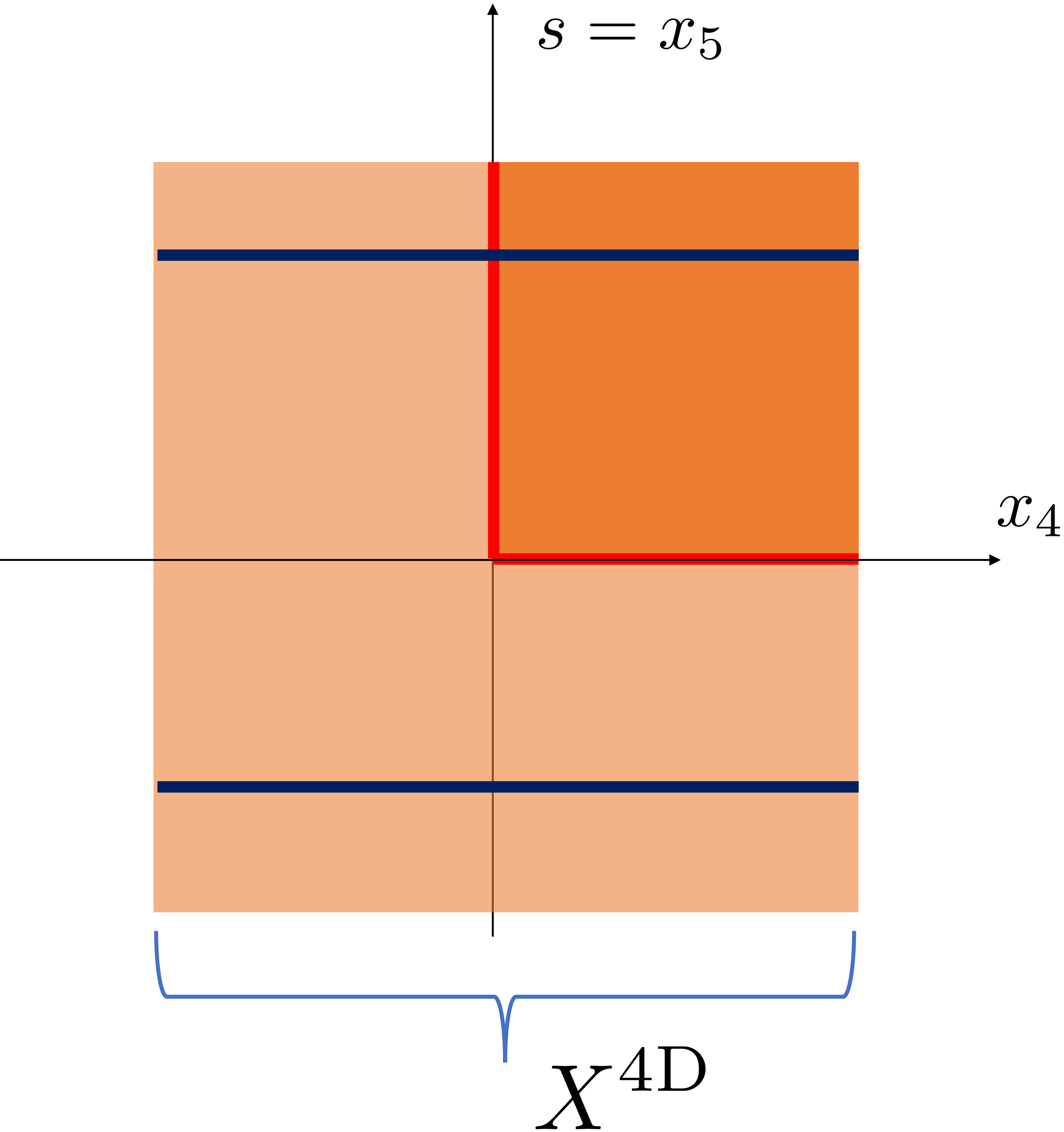
$$m(x_4, x_5) = \begin{cases} M & \text{for } x_4 > 0 \ \& \ x_5 > 0 \\ 0 & \text{for } x_4 = 0 \ \& \ x_5 = 0 \\ -M_2 & \text{otherwise} \end{cases}$$

and A_μ is

independent of x_5 .

* The following proof is valid for any 2n+1 dimension.

On X^{4D} x R,



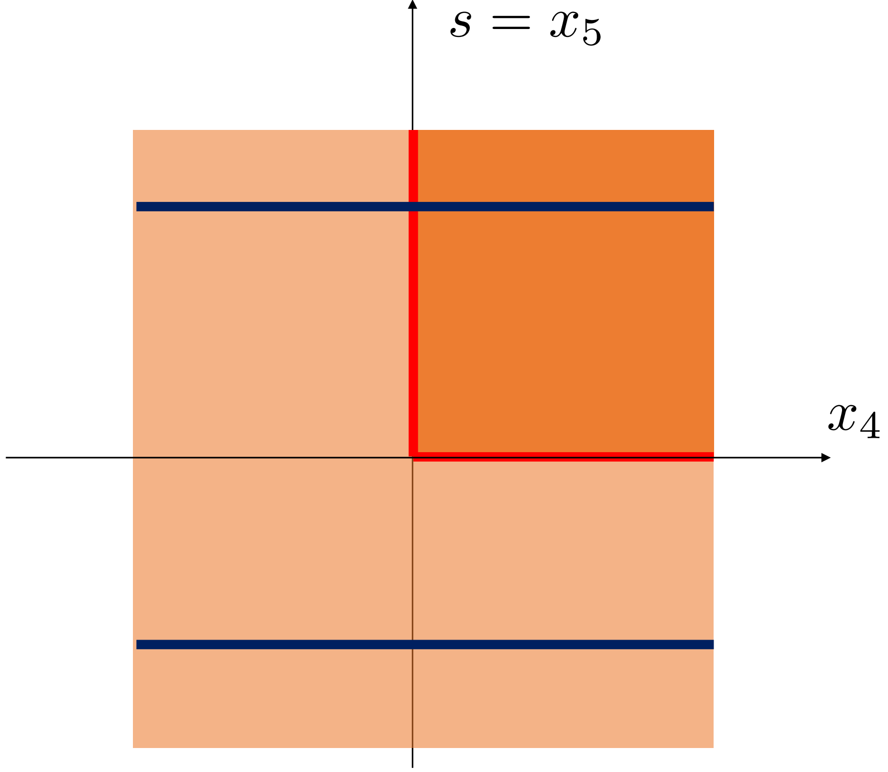
we compute

$$Ind(D^{5\mathrm{D}})$$

in two different ways:

1. localization
2. eta-inv. at $x_5 = \pm 1.$

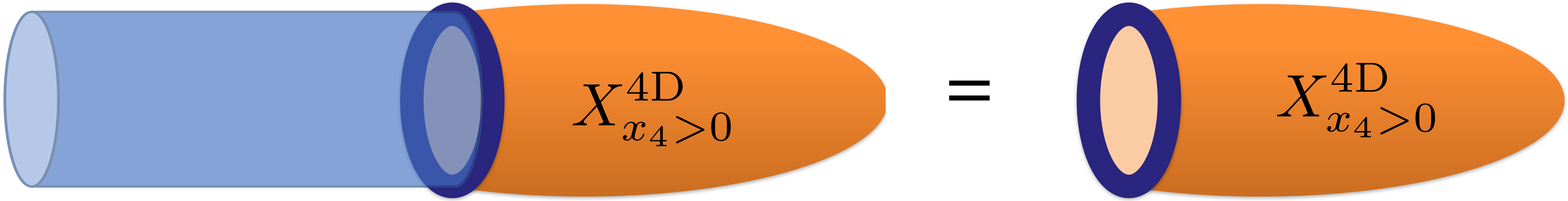
Localization



Theorem 2 tells us

$$Ind(D^{5D})|_{M,M_2\to\infty} = Ind(D^{4D}_{m=0\text{surface}}) \times \underbrace{IndD^{1D}_{normal}}_{=1}$$

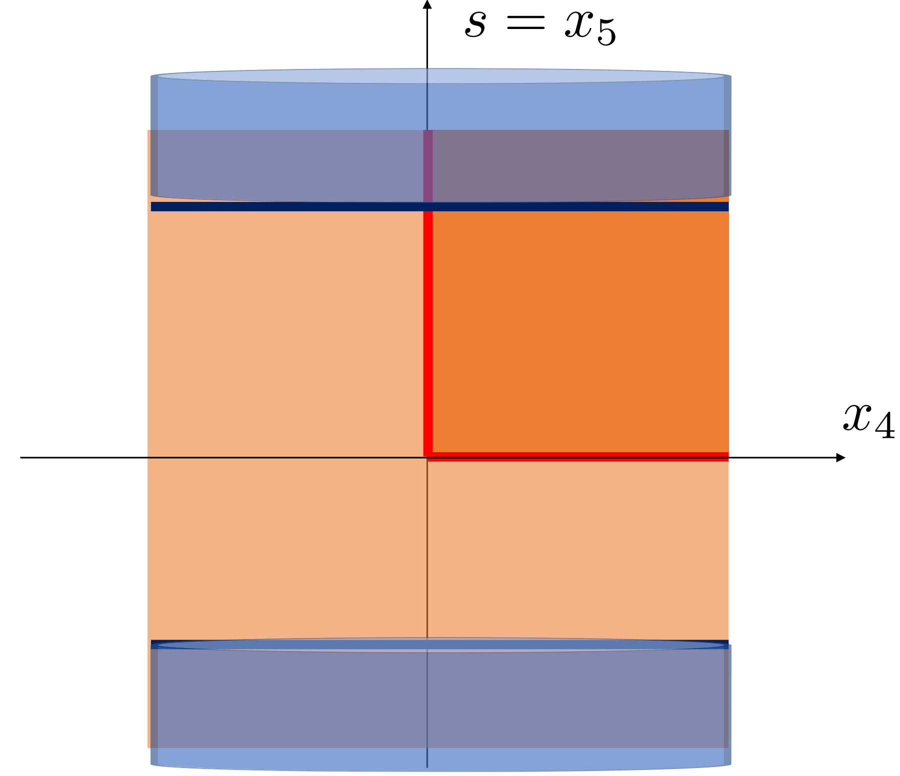
and on the massless surface



theorem 1 indicates

$$Ind(D^{4D}_{m=0\text{surface}}) = Ind(D^{X^{4D}_{x_4>0}}_{\text{APS}})$$

Boundary eta invariants



Theorem 1 tells us

$$Ind(D^{\rm 5D}) = Ind(D^{\rm 5D}_{\rm APS\ b.c.}{}_{ats=\pm 1})$$

and from theorem 3, we obtain

$$Ind(D^{\rm 5D}_{\rm APS\ b.c.}{}_{ats=\pm 1}) = \frac{1}{2}\left[\eta(D^{\rm 4D}_{s=1}) - \eta(D^{\rm 4D}_{s=-1})\right]$$

$$= \frac{1}{2}\left[\eta(\gamma_5(D^{\rm 4D} + M\epsilon(x_4)) - \eta(\gamma_5(D^{\rm 4D} - M_2)\right] = \frac{1}{2}\eta^{PV reg.}(\gamma_5(D^{\rm 4D} + M\epsilon(x_4))$$

therefore,

$$Ind(D^{5D}) = Ind(D_{\rm APS}) = \frac{1}{2}\eta(H_{DW})$$ Q.E.D.

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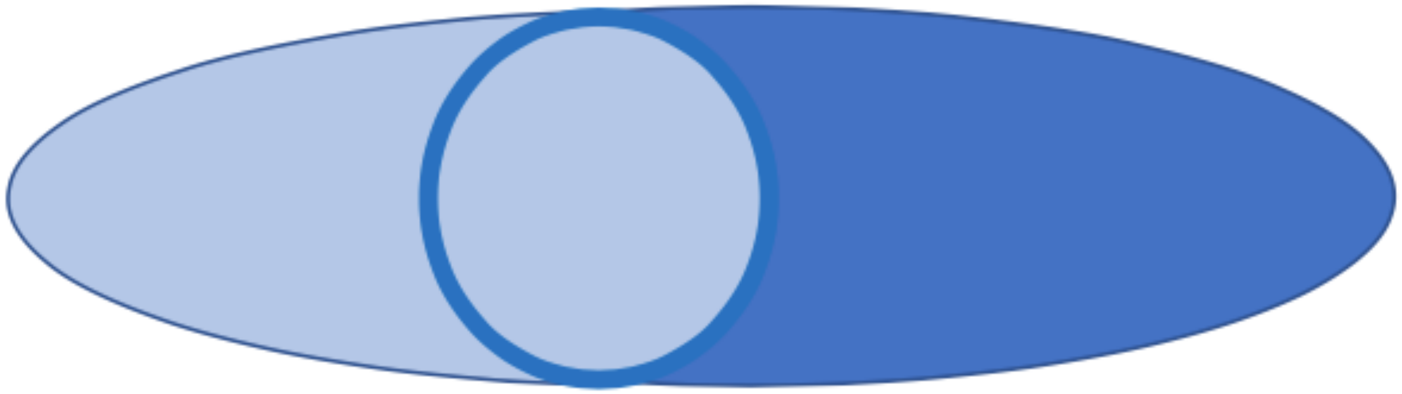
✔ 4. Mathematical proof

$$Ind(D^{5\text{D}}) = Ind(D_{\text{APS}}) = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$$

5. Discussion

6. Summary

We don't need any boundary condition by hand.



The kink structure automatically chooses a local and rotationally symmetric boundary condition,

and extension from AS index is simple:

$$\frac{1}{2}\eta(\gamma_5(D+M)) \to \frac{1}{2}\eta(\gamma_5(D+M\epsilon(x)))$$

massive fermion = chiral symmetry is NOT important.

The lattice fermion "knew" this fact:

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \quad D_{ov} = \frac{1}{a}\left(1 + \gamma_5\frac{H_W}{\sqrt{H_W^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

If the original AS index were given by

$$-\frac{1}{2}\eta(\gamma_5(D - M))$$

we should have known the lattice index theorem much before Hasenfratz or Neuberger 1998.

Massless vs. massive

index theorem with massless Dirac op.

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1-aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

index theorem with massive Dirac op.

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
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APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))?$

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↓ Next talk by Kawai

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$$Ind(D^{5\mathrm{D}}) = Ind(D_{\mathrm{APS}}) = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$$

✔ 5. Discussion

For index theorems, chiral sym. is NOT important.

6. Summary

Summary

$$Ind(D_{\rm APS}) = \frac{1}{2}\eta(H_{DW})$$

1. APS index describes bulk-edge correspondence of topological insulators.
2. APS (as well as AS) index can be reformulated by the eta-inv. of massive domain-wall operator.
3. We have given a mathematical proof for general cases through the 5D index.
4. eta-invariant of massive operator unifies the index theorems (including their lattice version).

Backup slides

Example : 1+1d bulk + 0+1d edge Majorana fermion coupled to gravity

APS index tells

$$Z \propto \exp\left(2\pi i \frac{n}{8}\right)$$

consistent with Z_8 classification of Kitaev's interacting Majorana chain.

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{\rm 3D})}{2} = \frac{CS}{2\pi} + \text{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x \, \mathrm{tr}_c \left[\epsilon_{\nu\rho\sigma}\left(A^\nu \partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$