ABJM Matrix Model and two-dimensional Toda Lattice Hierarchy

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JHEP **1903**, 197 (2019), [arXiv:1901.00541 [hep-th]]

Purpose

We clarify the relation between ABJM matrix model and integrable system.

ABJM matrix model: 1-point function, 2-point function ⇄ Integrable system: mKP hierarchy, 2D Toda Lattice hierarchy

ABJM Theory

(k:Chern-Simons level)

3D $\mathcal{N}=6$ Superconformal Chern-Simons theory with gauge group $SU(N)_k \times SU(N+M)_{-k}$

[Aharony, Bergman, Jafferis, Maldacena 2008]
[Aharony, Bergman, Jafferis 2008]
[Hosomichi, Lee, Lee, Lee, Park 2008]

Localization

(saddle point method)

$$Z=\int [dX]\, e^{-S[X]}$$

$$Z=\int dX_0 e^{-S[X_0]} Z_{1-loop}[X_0]$$

exact result

ABJM Matrix Model

$$Z_k=\frac{i^{\frac{1}{2}M(M+2N)}}{N!(N+M)!}\int\frac{d^Nx}{(2\pi)^N}\frac{d^{N+M}y}{(2\pi)^{N+M}}e^{\frac{ik}{4\pi}\left(\sum_{m=1}^N x_m^2-\sum_{n=1}^{N+M}y_n^2\right)}$$
$$\times\frac{\prod_{m<m'}^N\left(2\sinh\frac{x_m-x_{m'}}{2}\right)^2\prod_{n<n'}^{N+M}\left(2\sinh\frac{y_n-y_{n'}}{2}\right)^2}{\prod_{m=1}^N\prod_{n=1}^{N+M}\left(2\cosh\frac{x_m-y_n}{2}\right)^2}$$

[Kapustin, Willett, Yaakov 2010]

canonical 1-point function of 1/2-BPS Wilson loop $\langle W_\lambda\rangle_k(N,N+M)$ (λ : Young diagram)

In ABJM theory, Wilson loop → Localization → In ABJM matrix model, Super Suchur function

$$W_\lambda=W_\lambda(e^x|e^y)$$

grand canonical 1-point function of 1/2-BPS Wilson loop

$$\langle W_\lambda\rangle_{k,M}^{GC}=\sum_{N=0}^{\infty}z^N\langle W_\lambda\rangle_k(N,N+M)$$

(z : fugacity)

Generalize to 2-point function (**not** obtained by Localization)

[Kubo, Moriyama 2018]

$$\langle W_\lambda\overline{W}_\mu\rangle_{k,M}^{GC}=\sum_{N=0}^{\infty}z^N\langle W_\lambda\overline{W}_\mu\rangle_k$$

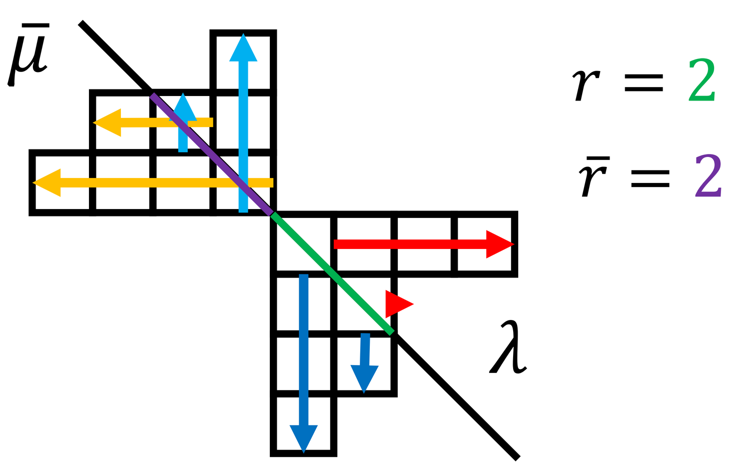
$$W_\lambda\overline{W}_\mu=W_\lambda(e^x|e^y)W_\mu(e^{-x}|e^{-y})$$

Young Diagram

- Convenient to use **two** Young diagrams.
- Represent to rank difference M also.

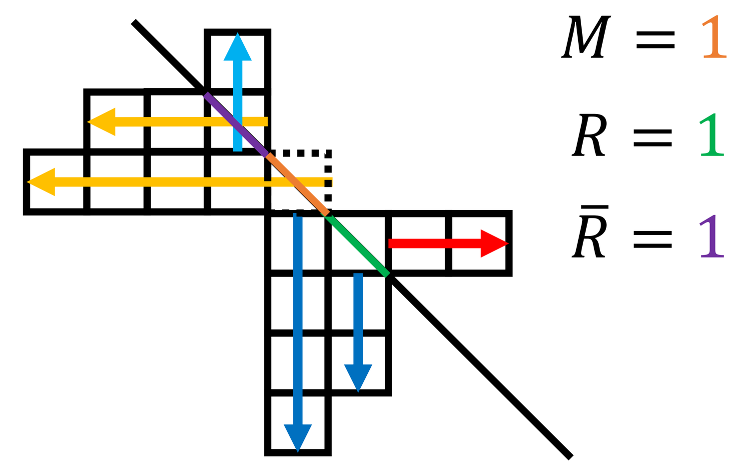
Frobenius Notation

$\lambda\bar{\mu}=(3,0,-2,-4|3,1,-1,-3)$



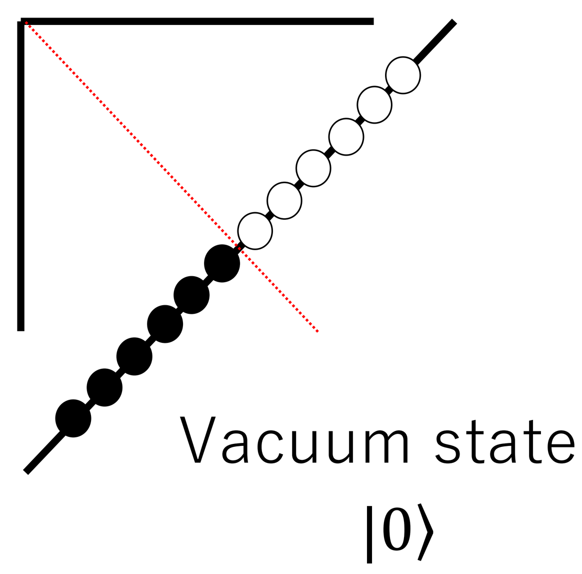
Shifted Frobenius Notation

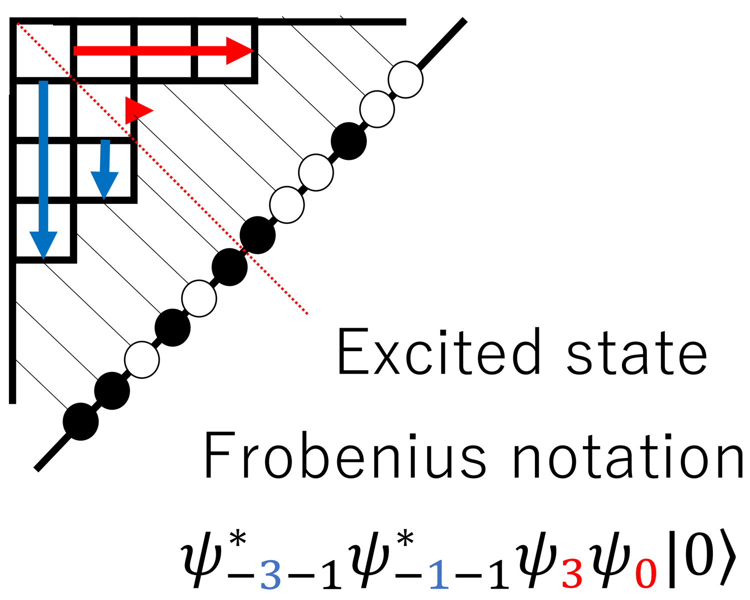
$\lambda\bar{\mu}=(2,-3,-5|4,2,-2)$



(M : shift parameter)

Maya Diagram → Fermion Formula

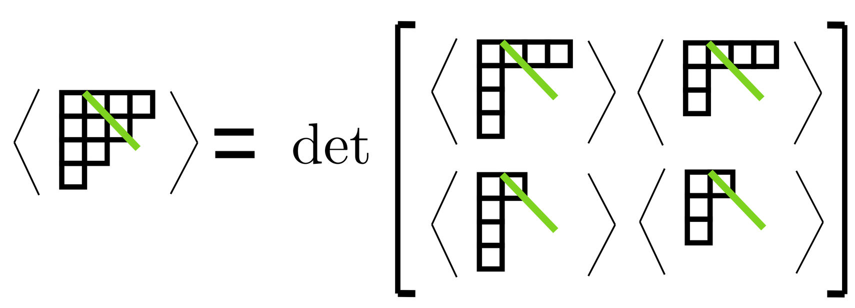




Result
We find Fermion notation for shifted Frobenius notation.

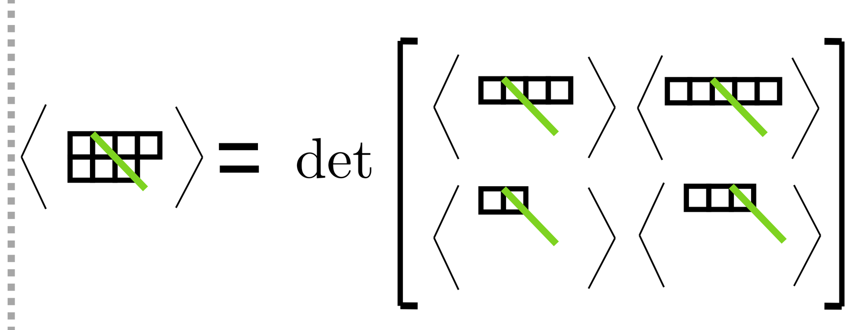
For normalized 1-point functions $S_\lambda^M=\langle W_\lambda\rangle_{k,M}^{GC}/\langle 1\rangle_{k,0}^{GC}$

Giambelli Identity



[S. Matsuno, S. Moriyama 2016]

Jacobi-Trudi Identity



[T. Furukawa, S. Moriyama 2017]

These identities also appear in integrable system.

Why do we consider 2-point function?

For 1-point functions, shifted Giambelli identity

$$\begin{cases}\dfrac{\langle W_\lambda\rangle_{k,M}^{GC}}{\langle 1\rangle_{k,0}^{GC}}=det\begin{pmatrix}\mathcal{H}_{(\bar{A}|B)}\\ \mathcal{H}_{(A|B)}\end{pmatrix} & \text{for M} \geq 0\\ \dfrac{\langle W_\lambda\rangle_{k,M}^{GC}}{\langle 1\rangle_{k,0}^{GC}}=det\begin{pmatrix}\mathcal{H}_{(A|\bar{B})} & \mathcal{H}_{(A|B)}\end{pmatrix} & \text{for M} \leq 0\end{cases}$$

- Component $\mathcal{H}_{(-|-)}$ does **not** appear.
- We can **not** write down components $\mathcal{H}$ with 1-point functions.
- We can **not** represent shifted Giambelli identity as one formula.

We can solve these unsatisfactory points by generalization to 2-point function.

For normalized 2-point functions $S_{\lambda\bar{\mu}}^M=\langle W_\lambda\overline{W}_\mu\rangle_{k,M}^{GC}/\langle 1\rangle_{k,0}^{GC}$

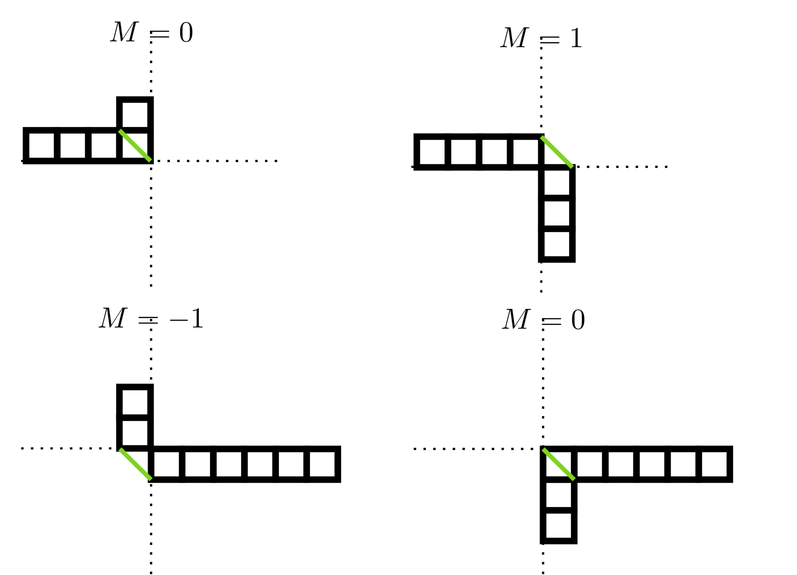
Shifted Giambelli Identity

$$S_{\lambda\bar{\mu}}^{M=1}=det\begin{pmatrix}S^{M=0}_{(-3|-2)} & S^{M=1}_{(-3|4)} & S^{M=1}_{(-3|2)}\\ S^{M=0}_{(-5|-2)} & S^{M=1}_{(-5|4)} & S^{M=1}_{(-5|2)}\\ S^{M=-1}_{(2|-2)} & S^{M=0}_{(2|4)} & S^{M=0}_{(2|2)}\end{pmatrix}$$

[Matsumoto, Moriyama 2014] [Kubo, Moriyama 2018]

We can describe any 2-point function by **four hook** diagram types.

We construct the **fermion formula** for Shifted Giambelli identity.



2D Toda Lattice Hierarchy (2DTL)

2DTL equation : $\frac{\partial^2 u_n}{\partial t_1 \partial t_{-1}} = e^{u_n - u_{n-1}} - e^{u_{n+1}-u_n}$

$u_n = u_n(\cdots, t_{-2}, t_{-1}, t_1, t_2, \cdots) = u_n(t_+; t_-) \qquad n \in \mathbb{Z}$

hierarchy : a set of infinite symmetries of soliton equation

Hirota Method

D_i : Hirota derivative of t_i

soliton equations
↓
bilinear form

2DTL equation : $D_1 D_{-1} \tau_n \cdot \tau_n = -2\tau_{n+1}\tau_{n-1}$

$u_n = \ln\left({}^{\tau_{n+1}}/_{\tau_n}\right)$ (τ_n : τ-function)

A solution to a soliton hierarchy is written only by one function. (**τ-function**)

If τ-function is expanded by **Schur functions** s_λ, then the coefficients $C_{\lambda\bar{\mu}}$ satisfy the **Plücher relations.**

$$\tau_n(t_+; t_-) = \sum_{\lambda\mu} C_{\lambda\bar{\mu}}(n) s_\lambda(t_+) s_\mu(-t_-)$$

[Kyoto School], and see also [Alexandorov, Kazakov, Leurent, Tsuboi, Zabrodin 2012] for recent reviews.

(ϕ : trivial Young diagram)

2DTL $\xrightarrow{\mu \to \phi}$ mKP

2DTL becomes another hierarchy (mKP) by taking the trivial diagram.

Fermion Formula in 2DTL

$\{\psi_i, \psi_j^*\} = \delta_{ij}$, $\{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0$, for $i, j \in \mathbb{Z}$

$gl(\infty)$ act to fermions :

$$\begin{cases} G\psi_i G^{-1} = \sum_{j\in\mathbb{Z}} (R^{-1})_{ij}\psi_j \\ G\psi_i^* G^{-1} = \sum_{j\in\mathbb{Z}} \psi_j^* R_{ji} \end{cases} \qquad \left(\begin{array}{l} G = e^{b_{ij}\psi_i^*\psi_j} \\ R_{ij} = (e^b)_{ij} \\ i, j \in \mathbb{Z} \end{array} \right)$$

$$\sum_{k\in\mathbb{Z}} \psi_k G \otimes \psi_k^* G = \sum_{k\in\mathbb{Z}} G\psi_k \otimes G\psi_k^*$$: (**bilinear**) identity

two dimensional Toda Lattice Hierarchy

$$\tau_n(t_+; t_-) = \langle n | e^{J_+(t_+)} G e^{-J_-(t_-)} | n \rangle$$

($J_k = \sum_{j\in\mathbb{Z}} : \psi_j \psi_{j+k}^* :$, $J_\pm(t_\pm) = \sum_{k=1}^\infty J_{\pm k} t_{\pm k}$)

$C_{\lambda\bar{\mu}}(n) = \langle \lambda, n | G | \mu, n \rangle$ ($|\lambda, n\rangle$: excited state)

Summary

- In ABJM/2DTL correspondence, the generationg function of 2-point functions is τ-function.
- In ABJM matrix model, the generalization to 2-point function matches the integrable structure well.
- In ABJM matrix model, we found the fermionic structure for vacuum expectation values.

Future Work

- We don't know the correspondence for partition function.

Does our study relate to Painvele [Bonelli, Grassi, Tanzini 2018] ?

- We don't understaund the condition to parameters b_{ij}.

ABJM/2DTL correspondence

Main Result

We construct the correspondence between ABJM and 2DTL not to break shifted Giambelli identity.

ABJM Matrix Model		Integrable system
2-point function	⟷	2DTL Hierarchy
$\mu \to \phi$ ↓		$\mu \to \phi$ ↓
1-point function	⟷	mKP Hierarchy

ABJM/2DTL

We construct the 2-point functions in terms of vacuum expectation values of fermion.

ABJM ⟵ **2DTL**

$$S^{M=0}_{(\bar{A}|\bar{B})} = \frac{\langle 0|G\psi^*_{\bar{A}}\psi_{-\bar{B}-1}|0\rangle}{\langle 0|G|0\rangle} \qquad S^{M=1}_{(\bar{A}|B)} = \frac{\langle 0|\psi_{-B-1}G\psi^*_{\bar{A}}|0\rangle}{\langle 0|G|0\rangle}$$

$$S^{M=-1}_{(A|\bar{B})} = \frac{\langle 0|\psi^*_A G\psi_{-\bar{B}-1}|0\rangle}{\langle 0|G|0\rangle} \qquad S^{M=0}_{(A|B)} = \frac{\langle 0|\psi^*_A\psi_{-B-1}G|0\rangle}{\langle 0|G|0\rangle}$$

ABJM ⟶ **2DTL**

$$R_{--} = \mathcal{H}_{--}\left(1 + \sqrt{(\mathcal{H}_{--})^{-1}\mathcal{H}_{-+}(\mathcal{H}_{++})^{-1}\mathcal{H}_{+-}}\right)$$

$$R_{-+} = -\mathcal{H}_{-+}(\mathcal{H}_{++})^{-1}$$

$$R_{+-} = (\mathcal{H}_{-+})^{-1}\mathcal{H}_{--}\sqrt{(\mathcal{H}_{--})^{-1}\mathcal{H}_{-+}(\mathcal{H}_{++})^{-1}\mathcal{H}_{+-}}$$

$$R_{++} = -(\mathcal{H}_{++})^{-1}$$

$$\left((R_{ij}) = \begin{pmatrix} R_{--} & R_{-+} \\ R_{+-} & R_{++} \end{pmatrix} \begin{array}{c} -\infty \\ \updownarrow \\ +\infty \end{array} \quad \begin{pmatrix} S^{M=0}_{(\bar{A}|\bar{B})} & S^{M=1}_{(\bar{A}|B)} \\ S^{M=-1}_{(A|\bar{B})} & S^{M=0}_{(A|B)} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{--} & \mathcal{H}_{-+} \\ \mathcal{H}_{+-} & \mathcal{H}_{++} \end{pmatrix} \right)$$

$-\infty \longleftrightarrow +\infty$

Result **Wick theorem**

Generalize from [Alexandorov, Kazakov, Leurent, Tsuboi, Zabrodin 2012]

We can obtain many identities from Wick theorem!

- **Shifted Giambelli identity** for $C_{\lambda\bar{\mu}}(-M)$

$$\left\{ \begin{array}{ll} \langle W_\lambda \bar{W}_\mu \rangle^{GC}_{k,M} = C_{\lambda\bar{\mu}}(-M) & \textbf{: ABJM/2DTL} \text{ correspondence} \\ \langle W_\lambda \rangle^{GC}_{k,M} = C_\lambda(-M) & \textbf{: ABJM/mKP} \text{ correspondence} \end{array} \right.$$

- **Giambelli identity for 2-point functions (New identity)**
- **Giambelli identity for 1-point functions**
- **Jacobi-Trudi identity for 1-point functions**

Result

- The Wick theorem allows us to prove above identities simply.
- The ABJM/2DTL correspondence is consistent with shifted Giambelli identity.