# 5d Dirac fermion on quantum graph

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## Summary
### —Key features of our model—

| Mysteries of the Standard Model | | Our model |
|---|---|---|
| | | • Extra dimension = Quantum graph |
| **Generation problem** Why does nature provide three copies of quarks and leptons ? | ← | **Degeneracy** of zero mode solutions |
| **Fermion mass hierarchy** Why are fermion masses different exponentially ? **The origin of flavor mixing** Where does the flavor mixing come from ? | ← | **Localized** zero mode solutions |
| **The origin of CP phase** Where does the CP phase come from ? | ← | Zero mode solutions are genuine **complex functions**. |

## Why quantum graph ?

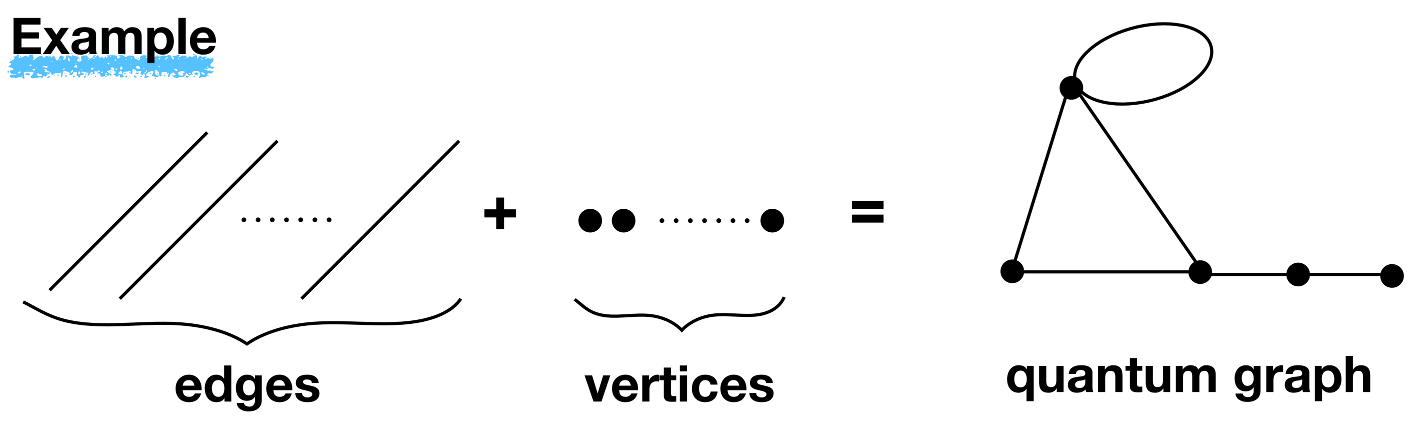
| $S^1/Z_2$ orbifold model | Quantum graph model |
|---|---|
| • No degeneracy ✗ | • Multi degeneracy ○ |
| • Not localized ✗ | • Localization ○ |
| • Real function ✗ | • Complex function ○ |

**Quantum graph can realize all of our requirements !**

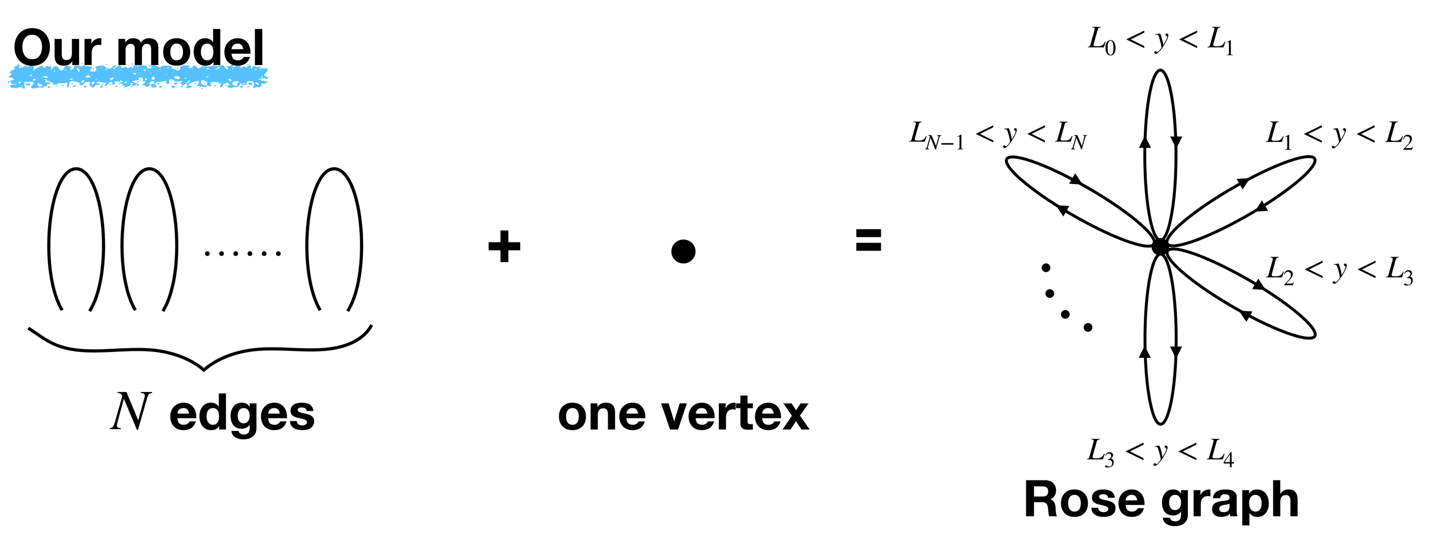
## What quantum graph ?

**A graph constructed by any edges and any vertices, and equipped with a Hermitian differential operator (Hamiltonian).**

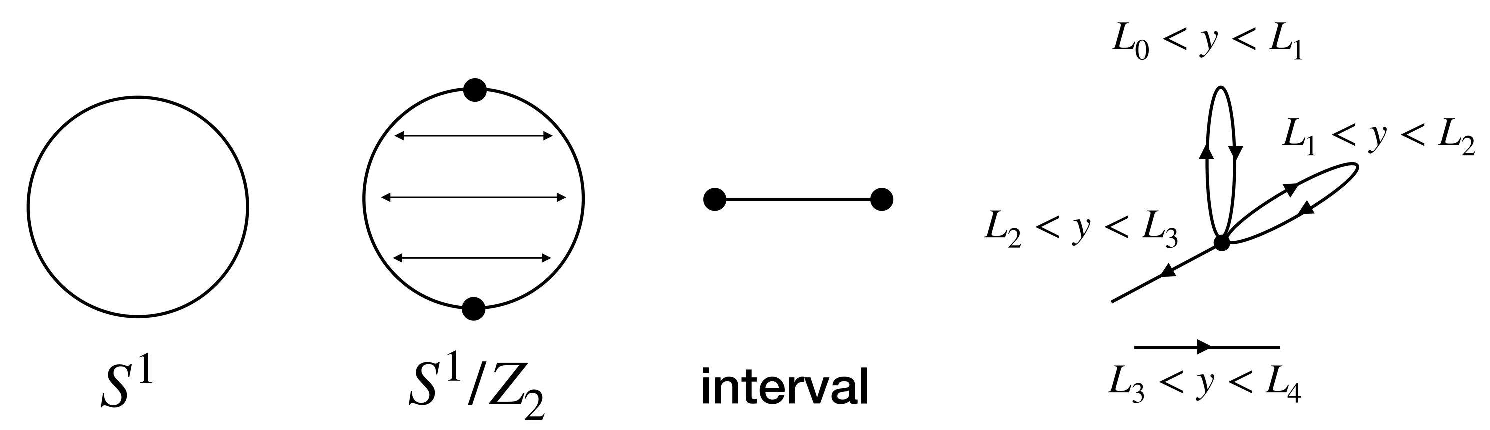
**Example**




**Our model**



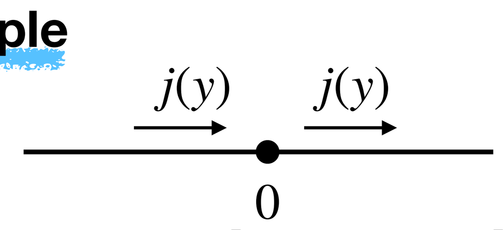

- **The rose graph contains various graphs :**




## Boundary conditions on quantum graph vertex

**equipped with Hamiltonian** ⟷ **current conservation** ⟶ **boundary condition**

**Example**




$$j(-0) = j(+0)$$

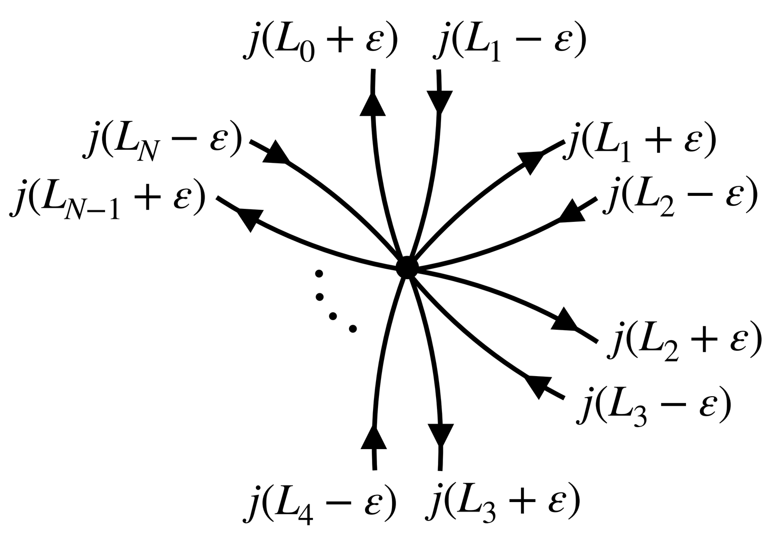
$$j(y) = -i\left[\varphi^*\varphi' - \varphi'^*\varphi\right]$$

$$\longrightarrow (I_2 - U)\Phi + L_0(I_2 + U)\Phi' = 0 \quad \longrightarrow$$

$$\Phi = \begin{pmatrix}\varphi(+0)\\ \varphi(-0)\end{pmatrix}, \quad \Phi' = \begin{pmatrix}\varphi'(+0)\\ -\varphi'(-0)\end{pmatrix}$$

**This BC contains**
- **Dirichlet BC**
- **Neumann BC**
- **Periodic BC**
- **...etc.**

**Our model**




enlarged center point of the rose graph

outgoing current, incoming current

$$\sum_{a=1}^{N} j(L_a - \varepsilon) = \sum_{a=1}^{N} j(L_{a-1} + \varepsilon)$$

( $\varepsilon$ is an infinitesimal positive constant )

$$\longrightarrow \begin{aligned} &(I_{2N} - U)\,\vec{F}_n^{(i)} = \vec{0} \\ &(I_{2N} + U)\,\vec{G}_m^{(j)} = \vec{0} \\ &(U \in U(2N), \quad U^2 = I_{2N}) \end{aligned}$$

$$\vec{F}_n^{(i)} \equiv \begin{pmatrix} f_n^{(i)}(L_0+\varepsilon) \\ f_n^{(i)}(L_1-\varepsilon) \\ f_n^{(i)}(L_1+\varepsilon) \\ f_n^{(i)}(L_2-\varepsilon) \\ \vdots \\ f_n^{(i)}(L_{a-1}+\varepsilon) \\ f_n^{(i)}(L_a-\varepsilon) \\ \vdots \\ f_n^{(i)}(L_{N-1}+\varepsilon) \\ f_n^{(i)}(L_N-\varepsilon) \end{pmatrix} \qquad \vec{G}_m^{(j)} \equiv \begin{pmatrix} g_m^{(j)}(L_0+\varepsilon) \\ -g_m^{(j)}(L_1-\varepsilon) \\ g_m^{(j)}(L_1+\varepsilon) \\ -g_m^{(j)}(L_2-\varepsilon) \\ \vdots \\ g_m^{(j)}(L_{a-1}+\varepsilon) \\ -g_m^{(j)}(L_a-\varepsilon) \\ \vdots \\ g_m^{(j)}(L_{N-1}+\varepsilon) \\ -g_m^{(j)}(L_N-\varepsilon) \end{pmatrix}$$

## Setup

**We consider 5d Dirac fermions on the rose graph as an extra dimension.**

**Action :**

$$S = \int d^4x \sum_{a=1}^{N} \int_{L_{a-1}}^{L_a} dy\, \overline{\Psi}(x,y)[i\gamma^\mu\partial_\mu + \gamma^5\partial_y + M]\Psi(x,y)$$

(5d Dirac fermion: $\Psi$; bulk mass: $M$; $\sum_{a=1}^{N}\int_{L_{a-1}}^{L_a}$: integral over the Rose graph)

**Kaluza-Klein expansion :**

$$\Psi(x,y) = \sum_i \sum_n \psi_{\mathrm{R},n}^{(i)}(x)\, f_n^{(i)}(y) + \sum_i \sum_m \psi_{\mathrm{L},m}^{(i)}(x)\, g_m^{(i)}(y)$$

($\sum_i$: Sum of the degeneracy of n-th KK mode; $\sum_n$: Sum of KK modes; $f_n^{(i)}, g_m^{(i)}$: mode function; $\psi_{\mathrm{L},m}^{(i)}$: 4d left-handed chiral fermion; $\psi_{\mathrm{R},n}^{(i)}$: 4d right-handed chiral fermion)

**Eigenvalue equations :**

$$(-\partial_y^2 + M^2) f_n^{(i)}(y) = m_n^2 f_n^{(i)}(y)$$

$$(-\partial_y^2 + M^2) g_n^{(i)}(y) = m_n^2 g_n^{(i)}(y)$$

($m_n$: 4d mass)

# Solving the generation problem

**Generation problem**
Why does nature provide three copies of quarks and leptons ?

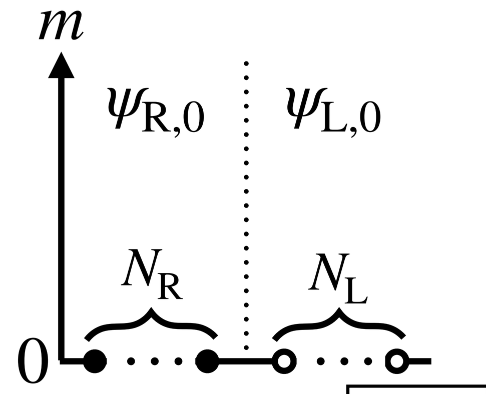
← **Degeneracy of zero mode solutions**

**Boundary condition** / **The number of zero modes**

(A) Type $(N-3, N+3)$

$$U = V\begin{pmatrix} I_{N-3} & 0 \\ 0 & -I_{N+3} \end{pmatrix} V^{\dagger}$$

$$(V \in U(2N))$$

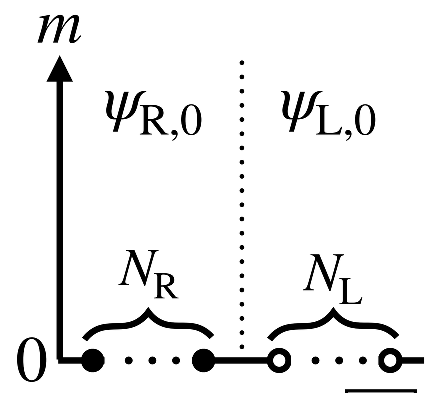



$N_R - N_L = \boxed{-3}$
Witten index
↓
$\psi_{L,0}$ is three-fold degeneracy.

(B) Type $(N+3, N-3)$

$$U = V\begin{pmatrix} I_{N+3} & 0 \\ 0 & -I_{N-3} \end{pmatrix} V^{\dagger}$$

$$(V \in U(2N))$$




$N_R - N_L = \boxed{3}$
Witten index
↓
$\psi_{R,0}$ is three-fold degeneracy.

**One generation of 5d Dirac fermion** ⟶ **Three generations of 4d chiral fermions**

$\mathcal{Q}_{\text{doublet}}$ —BC of (A)→ $\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}$

$\mathcal{U}_{\text{singlet}}$ —BC of (B)→ $u_R, c_R, t_R$

$\mathcal{D}_{\text{singlet}}$ —BC of (B)→ $d_R, s_R, b_R$

**Three generations are realized !**

# How to solve the rest problems

**Generation problem was solved above, then we tackle the rest problems.**

**Fermion mass hierarchy**
Why are fermion masses different exponentially ?

**The origin of flavor mixing**
Where does the flavor mixing come from?

← **Localized zero mode solutions**

**The origin of CP phase**
Where does the CP phase come from?

← **Zero mode solutions are genuine complex functions.**

# Solving the rest problems

**We consider Yukawa interaction term in 5d spacetime.**

$$S = \int d^4x \int dy \left[ g_Y \overline{\mathcal{Q}}_d \begin{pmatrix} v \\ 0 \end{pmatrix} \mathcal{U}_s + g'_Y \overline{\mathcal{Q}}_d \begin{pmatrix} 0 \\ v \end{pmatrix} \mathcal{D}_s \right]$$

↓
$\mathcal{Q}_d$ : BC of (A)
$\mathcal{U}_s$ : BC of (B)
$\mathcal{D}_s$ : BC of (B)
} **Only one generation of 5d Dirac fermion**

**We get effective Yukawa interaction term in 4d spacetime.**

$$S = \int d^4x \sum_{i,j=1}^{\boxed{3}} \left[ m_{ij}\overline{u_{iL}}u_{jR} + (m_{ij})^{\dagger}\overline{u_{jR}}u_{iL} + \widetilde{m}_{ij}\overline{d_{iL}}d_{jR} + (\widetilde{m}_{ij})^{\dagger}\overline{d_{jR}}d_{iL} \right]$$

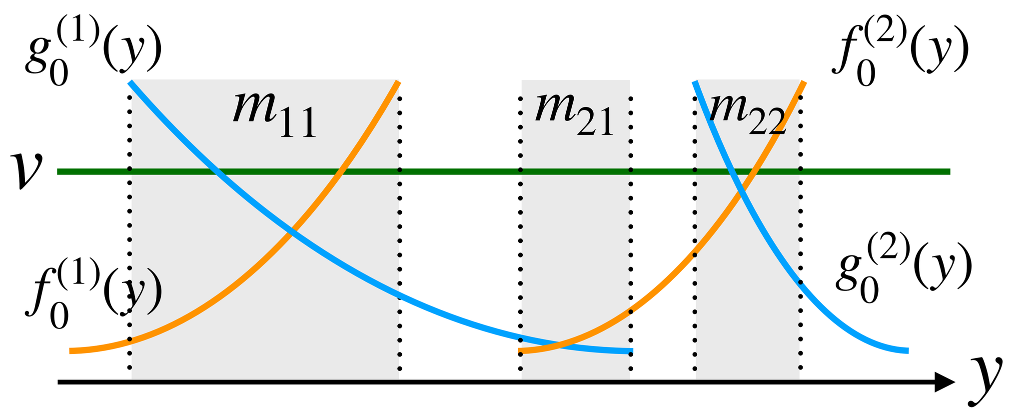
**Three generations of 4d chiral fermion**

**Mass matrix :**

$$m_{ij} = g_Y v \int_0^{L_N} dy\, g_0^{(i)*}(y) f_0^{(j)}(y)$$

$v$ : Higgs VEV
$g_Y$ : Yukawa coupling
$i, j = 1,2,3$

**ex)**




## Fermion mass hierarchy

**Zero mode solutions are localized.**

**The overlap integrals can realize hierarchical values !**

**The fermion masses are different exponentially !**

## Flavor mixing

**Overlap integrals between different generations are non-vanishing.**

**The mass matrices can get off-diagonal elements.**

**The flavor mixing is realized !**

## CP phase

**Boundary conditions depend on unitary matrix $U$.**

**Zero mode solutions can be genuine complex functions.**

**The mass matrices contain a CP phase !**

# Future work

- **We have to analyze our model numerically whether the model can produce experimental values or not.**
- **Since the rose graph has rich and non-trivial geometry, it would be of great interest to study non-abelian Berry phase, extended quantum-mechanical SUSY, … etc.**