Accelerating cosmologies in an integrable model with noncommutative minisuperspace variables

arXiv:1903.07895 [gr-qc]

Nahomi Kan (NIT, Gifu College)
Masashi Kuniyasu, Kiyoshi Shiraishi, and Kohjiroh Takimoto
(Yamaguchi University)

§ 1. Introduction

*We study classical and quantum noncommutative cosmology with a Liouville-type scalar degree of freedom.

*The noncommutativity is imposed on the minisuperspace variables through a deformation of the Poisson algebra.

*We investigate the effects of noncommutativity of minisuperspace variables on the accelerating behavior of the cosmic scale factor.

*The probability distribution in noncommutative quantum cosmology is also studied and we propose a novel candidate for interpretation of the probability distribution in terms of noncommutative arguments.

§ 2. The model

The Liouville scalar model

$$S = \int d^D x \sqrt{-g} \left[R - \frac{1}{2}(\nabla\Phi)^2 - \frac{V}{2} e^{2\alpha\Phi} \right].$$

Assuming

$ds^2 = -e^{2n(t)} dt^2 + e^{2a(t)} d\mathrm{x}^2$ **,** $n(t) = (D-1)a(t)$ **,** **Φ is a fnc. of t**

↓

$$L = -\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{U}{2} e^{2\lambda x}$$

"Cosmological" effective Lagrangian

where

$$x(t) \equiv \sqrt{2(D-1)(D-2)}\left(a + \frac{\alpha}{D-1}\Phi\right), \quad y(t) \equiv \Phi + 2(D-2)\alpha a\,,$$

$$\lambda \equiv \sqrt{\frac{D-1}{2(D-2)}}, \quad U \equiv \Sigma \cdot V, \quad \Sigma \equiv 1 - 2\alpha^2 \frac{D-2}{D-1}.$$

This "Cosmological" effective Lagrangian can also be obtained from various theories, including f(R) theory, higher-dim. theory with compactification (with flux, or cosmological const., or Ricci-non-flat int. space,).

§ 3. Classical dynamics

Commutative Case

Lagrangian ➡ **Hamiltonian**

$$L = -\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{U}{2}e^{2\lambda x} \;\Rightarrow\; H = -\frac{1}{2}\pi_x^2 + \frac{1}{2}\pi_y^2 + \frac{U}{2}e^{2\lambda x}$$

$$\pi_x \equiv \frac{\partial L}{\partial \dot{x}} = -\dot{x}\,, \quad \pi_y \equiv \frac{\partial L}{\partial \dot{y}} = \dot{y}$$

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial \pi_x} = -\pi_x\,, \quad \dot{y} = \{y, H\} = \frac{\partial H}{\partial \pi_y} = \pi_y\,,$$

$$\dot{\pi}_x = \{\pi_x, H\} = -\frac{\partial H}{\partial x} = -\lambda U e^{2\lambda x}\,, \quad \dot{\pi}_y = \{\pi_y, H\} = -\frac{\partial H}{\partial y} = 0$$

$$\Rightarrow\; \ddot{x} - \lambda U e^{2\lambda x} = 0\,, \qquad \ddot{y} = 0\,.$$

We are considering a "cosmological" model, so
Remember the Hamiltonian constraint! $H = 0$

solution

$$y(t) = P(t - t_0) + y_0 \,,$$

- $U > 0$ $$x(t) = \frac{1}{2\lambda} \ln \frac{P^2}{U \sinh^2 \lambda P(t - t_0)} \,,$$
- $U < 0$ $$x(t) = \frac{1}{2\lambda} \ln \frac{P^2}{|U| \cosh^2 \lambda P(t - t_0)} \,.$$

(P, t_0, y_0 are constants)

Noncommutative Case

Hamiltonian: $H_\theta = -\frac{1}{2}\Pi_X^2 + \frac{1}{2}\Pi_Y^2 + \frac{U}{2}e^{2\lambda X}$

Poisson brackets: $\{X, \Pi_X\} = 1\,, \quad \{Y, \Pi_Y\} = 1\,, \quad \boxed{\{X, Y\} = \theta}$

Hamilton's equations:

$$\dot{X} = \{X, H_\theta\} = -\Pi_X\,, \quad \dot{Y} = \{Y, H_\theta\} = \Pi_Y - \theta\lambda U e^{2\lambda X}\,,$$

$$\dot{\Pi}_X = \{\Pi_X, H_\theta\} = -\lambda U e^{2\lambda X}\,, \quad \dot{\Pi}_Y = \{\Pi_Y, H_\theta\} = 0\,.$$

solution satisfying the constraint $H_\theta = 0$

- $U > 0$

$$X(t) = \frac{1}{2\lambda} \ln \frac{P^2}{U \sinh^2 \lambda P(t-t_0)}, \quad Y(t) = P(t-t_0) + y_0 + \theta P \coth \lambda P(t-t_0),$$

- $U < 0$

$$X(t) = \frac{1}{2\lambda} \ln \frac{P^2}{|U| \cosh^2 \lambda P(t-t_0)}, \quad Y(t) = P(t-t_0) + y_0 + \theta P \tanh \lambda P(t-t_0)$$

which is originally found by

G. D. Barbosa and N. Pinto-Neto, Phys. Rev. **D70** (2004) 103512

Noncommtativity from Commutative variables

Let us identify:

$$X = x - \frac{\theta - \rho}{2}\pi_y\,, \quad Y = y + \frac{\theta + \rho}{2}\pi_x\,, \quad \Pi_X = \pi_x\,, \quad \Pi_Y = \pi_y$$

ρ : an arbitrary constant.

Then, $$H_\theta = -\frac{1}{2}\pi_x^2 + \frac{1}{2}\pi_y^2 + \frac{U}{2}e^{2\lambda[x-(\theta-\rho)\pi_y/2]}$$

Hamilton's equations

$$\dot{x} = \frac{\partial H_\theta}{\partial \pi_x} = -\pi_x\,, \quad \dot{y} = \frac{\partial H_\theta}{\partial \pi_y} = \pi_y - \frac{\theta - \rho}{2}\lambda U e^{2\lambda[x-(\theta-\rho)\pi_y/2]}\,,$$

$$\dot{\pi}_x = -\frac{\partial H_\theta}{\partial x} = -\lambda U e^{2\lambda[x-(\theta-\rho)\pi_y/2]}\,, \quad \dot{\pi}_y = -\frac{\partial H_\theta}{\partial y} = 0\,,$$

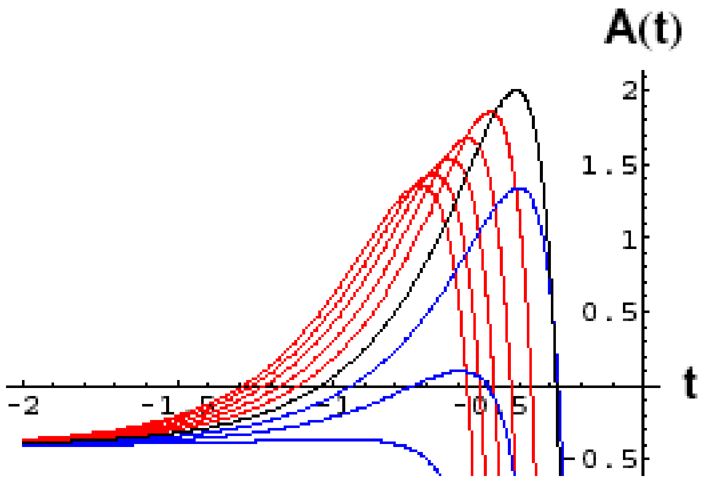
recovers the same equations for X, Y, Π_X, Π_Y, and the same solutions, *for any* ρ.

§ 4. Accelerating universe

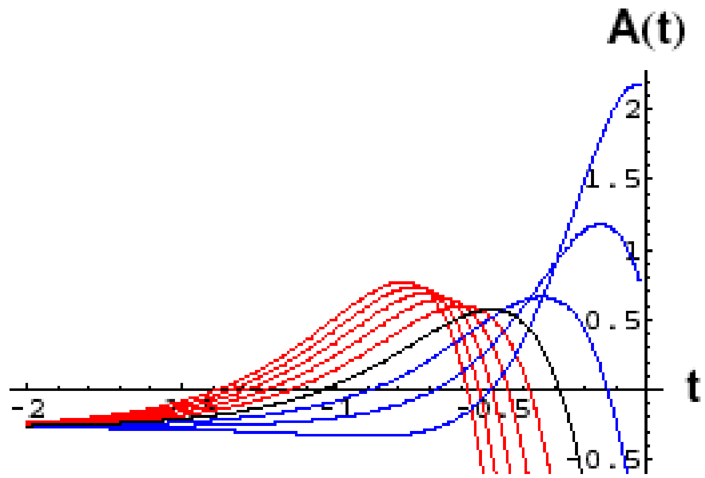
$$ds^2 = -d\eta^2 + S^2(\eta)d\mathbf{x}^2, \quad S(\eta) = e^{a(t)}, \quad d\eta = \pm e^{(D-1)a(t)}dt = \pm S^{D-1}dt$$

$$\frac{dS}{d\eta} = S^{1-D}\frac{dS}{dt} = -\frac{1}{D-2}\frac{dS^{2-D}}{dt}, \quad \frac{d^2S}{d\eta^2} = -\frac{1}{D-2}S^{1-D}\frac{d^2S^{2-D}}{dt^2}.$$

If $A(t) \equiv -S^{D-2}(t)\frac{d^2S^{2-D}(t)}{dt^2}$ >0, expansion is accelerating.



U>0



U<0

red curves: $\theta = 0.1, 0.2, 0.3, 0.4, 0.5$, **blue curves:** $\theta = -0.1, -0.2, -0.3$

§ 5. Wave function of the Universe

To obtain Wheeler-DeWitt equation (for the minisuperspace),

replace momenta as $\pi_x \to -i\frac{\partial}{\partial x}$ and $\pi_y \to -i\frac{\partial}{\partial y}$.

Express WDW eq. in Noncommutative case by **commutative variables**:

$$X \to \hat{X} = x + i\frac{\theta-\rho}{2}\frac{\partial}{\partial y}, \quad Y \to \hat{Y} = y - i\frac{\theta+\rho}{2}\frac{\partial}{\partial x}$$

$$\Pi_X \to \hat{\Pi}_X = -i\frac{\partial}{\partial x}, \quad \Pi_Y \to \hat{\Pi}_Y = -i\frac{\partial}{\partial y}.$$

EX: Confirm $[\hat{X}, \hat{\Pi}_X] = i\,, \quad [\hat{Y}, \hat{\Pi}_Y] = i\,, \quad [\hat{X}, \hat{Y}] = i\theta$ **!**

Note that if $\rho = -\theta$, $Y = y$!

Now, WDW eq. of Noncommutative Quantum Cosmology becomes:

$$\left\{\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+U\exp\left[2\lambda\left(x+i\frac{\theta-\rho}{2}\frac{\partial}{\partial y}\right)\right]\right\}\Psi(x,y)=0$$

The solution of the WDW eq.

$$\Psi(x,y)=\int_{-\infty}^{\infty}d\nu\,\mathcal{A}_\nu\,\Psi_\nu(x,y)=\int_{-\infty}^{\infty}d\nu\,\mathcal{A}_\nu\,\psi_\nu(x)\,e^{i\nu(y-y_0)}$$

where $\psi_\nu(x)=c_1F_{i\nu/\lambda}(\sqrt{U}e^{\lambda[x-\nu(\theta-\rho)/2]}/\lambda)+c_2G_{i\nu/\lambda}(\sqrt{U}e^{\lambda[x-\nu(\theta-\rho)/2]}/\lambda)\,,\quad (U>0)$

with $F_\nu(z)\equiv\frac{1}{2\cos(\nu\pi/2)}[J_\nu(z)+J_{-\nu}(z)]\,,\quad G_\nu(z)\equiv\frac{1}{2\sin(\nu\pi/2)}[J_\nu(z)-J_{-\nu}(z)]$

and $\psi_\nu(x)=c_3K_{i\nu/\lambda}(\sqrt{|U|}e^{\lambda[x-\nu(\theta-\rho)/2]}/\lambda)\,.\quad (U<0)$

We are interested in $\Psi(X,Y)$, instead of $\Psi(x,y)$!

(We want to see some correspondence with class. sol.)

if $\rho = -\theta$, $Y = y$ ←common variable both for C & NC

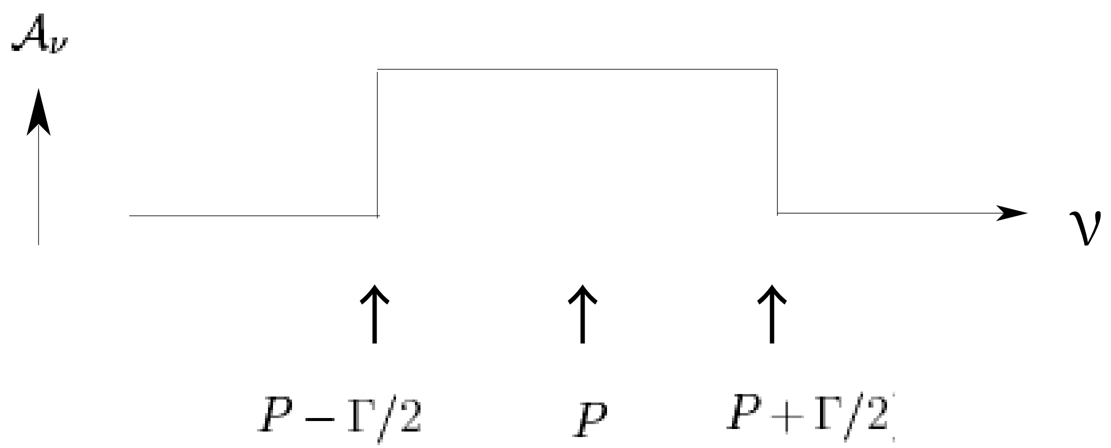
then $\hat{\Pi}_Y \Psi_\nu = \nu \Psi_\nu$.

Thus, for a wave packet peaking around $\nu \sim P$,

we can regard $X \sim x - \theta P$ approximately.

Hereafter, we consider

$$\mathcal{A}_\nu = \Gamma^{-1/2} \quad (P - \Gamma/2 < \nu < P + \Gamma/2), \text{ and } \mathcal{A}_\nu = 0 \quad (\nu < P - \Gamma/2,\ P + \Gamma/2 < \nu)$$

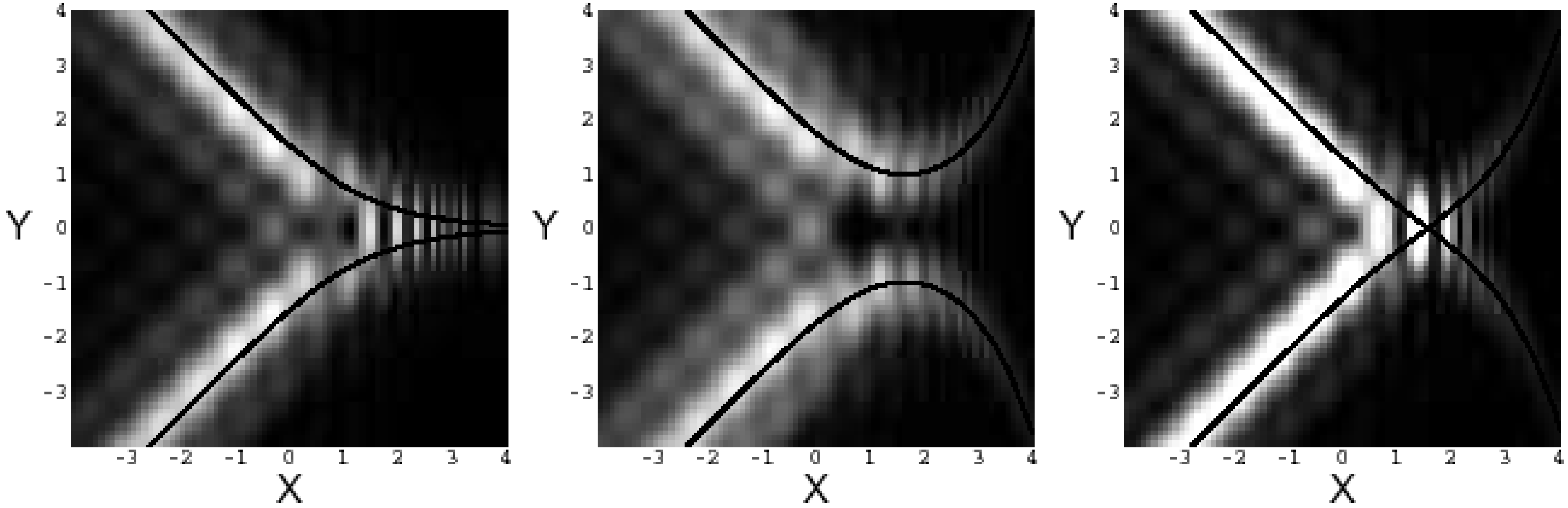


(rectangular amplitude)

U>0

$$c_1 = \sqrt{\frac{\nu/\lambda}{\tanh(\nu\pi/(2\lambda))}}, \quad c_2 = 0$$

$$\varrho(X,Y) \equiv |\Psi(X,Y)|^2$$



(a) (b) (c)

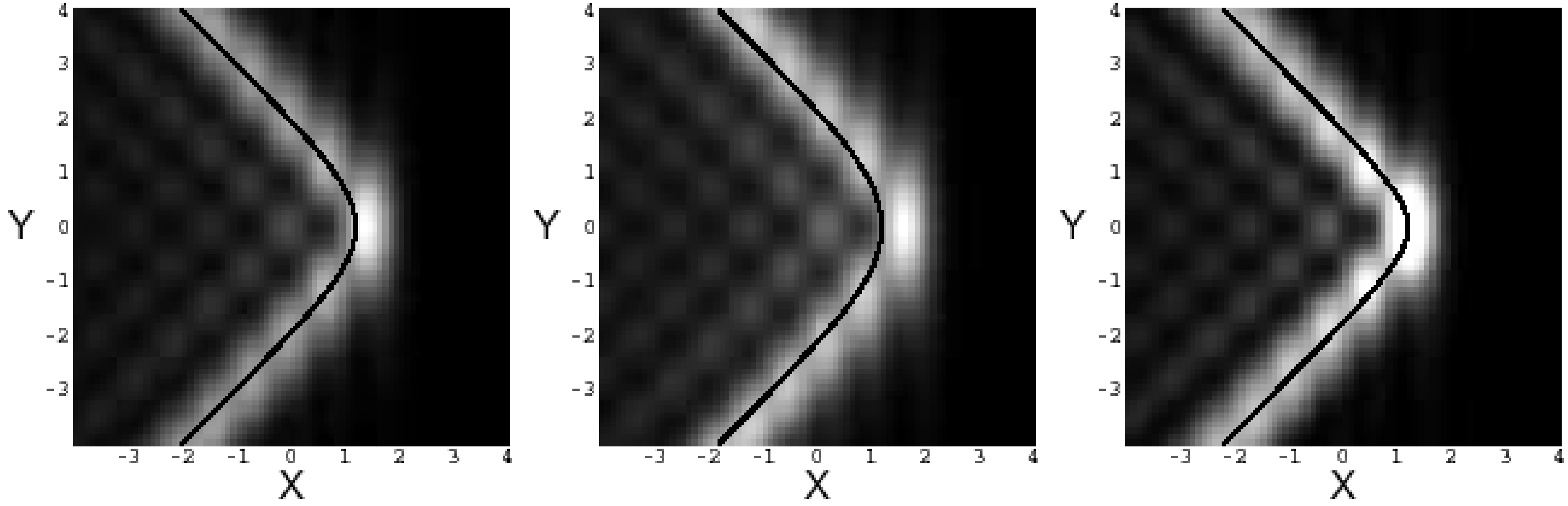
θ=0 θ=0.1 θ=-0.1

bold curves indicate classical solutions!

U<0

$$c_3 = \sqrt{\frac{2(\nu/\lambda)\sinh(\pi\nu/\lambda)}{\pi}}$$

$$\varrho(X,Y) \equiv |\Psi(X,Y)|^2$$



(a) θ =0

(b) θ =0.1

(c) θ =-0.1

bold curves indicate classical solutions!

§ 6. Wigner function of the Universe

For a wave function $\phi(q)$, the **Wigner function** is defined as:

$$W(q,p) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, \phi^*\left(q-\frac{u}{2}\right)\phi\left(q+\frac{u}{2}\right)e^{-ipu}$$

properties: $\int_{-\infty}^{\infty} dp\, W(q,p) = |\phi(q)|^2\,, \quad \int_{-\infty}^{\infty} dq\, W(q,p) = |\tilde{\phi}(p)|^2$

where $\tilde{\phi}(p)$ is the Fourier transform of $\phi(q)$.

For our wavefunction:

$$\begin{aligned} W(x,y,p_y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dv\, \Psi^*\left(x,y-\frac{v}{2}\right)\Psi\left(x,y+\frac{v}{2}\right)e^{-ip_y v} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu' \int_{-\infty}^{\infty} d\nu\, \mathcal{A}^*_{\nu'}\psi^*_{\nu'}(x)\mathcal{A}_{\nu}\psi_{\nu}(x)e^{i(\nu-\nu')(y-y_0)}\int_{-\infty}^{\infty} dv\, e^{-i[p_y-(\nu'+\nu)/2]v} \\ &= \int_{-\infty}^{\infty} d\nu\, \mathcal{A}^*_{p_y-\nu/2}\psi^*_{p_y-\nu/2}(x)\mathcal{A}_{p_y+\nu/2}\psi_{p_y+\nu/2}(x)e^{i\nu(y-y_0)}\,. \end{aligned}$$

Its Fourier transform:

$$\tilde{W}(x,\nu,p_y) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, W(x,y,p_y) e^{-i\nu(y-y_0)}$$
$$= \mathcal{A}^*_{p_y-\nu/2}\psi^*_{p_y-\nu/2}(x)\mathcal{A}_{p_y+\nu/2}\psi_{p_y+\nu/2}(x)$$

Our idea:

define $X \equiv x - \theta p_y\,, \quad \tilde{X} \equiv \theta x + p_y$ and

integrate out $\tilde{X}$.

$$\tilde{\varrho}_W(X,\nu) \equiv \frac{1}{1+\theta^2}\int_{-\infty}^{\infty} d\tilde{X}\, \tilde{W}(x(X,\tilde{X}),\nu,p_y(X,\tilde{X}))$$

where $x(X,\tilde{X}) = \frac{1}{1+\theta^2}(X+\theta\tilde{X})\,, \quad p_y(X,\tilde{X}) = \frac{1}{1+\theta^2}(\tilde{X}-\theta X)$

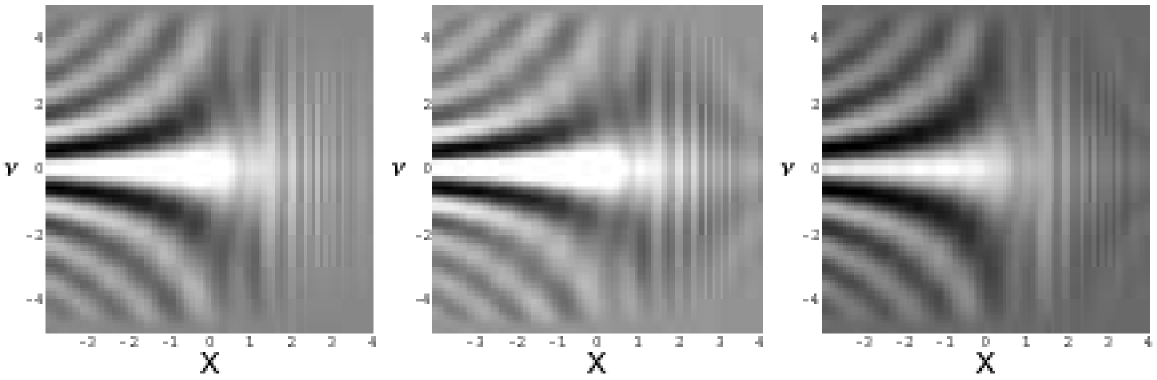
Check and Confirm our idea

Compare $\tilde{\varrho}(X,\nu)$ with the Fourier transform of the density obtained in the previous section:

$$\tilde{\varrho}(X,\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\nu'' \int_{-\infty}^{\infty} d\nu' \, \mathcal{A}^*_{\nu''}\psi^*_{\nu''}(X+\theta P)\mathcal{A}_{\nu'}\psi_{\nu'}(X+\theta P)e^{i(\nu'-\nu'')(y-y_0)}\, e^{-i\nu(y-y_0)}$$
$$= \int_{-\infty}^{\infty} d\nu' \, \mathcal{A}^*_{\nu'-\nu/2}\psi^*_{\nu'-\nu/2}(X+\theta P)\mathcal{A}_{\nu'+\nu/2}\psi_{\nu'+\nu/2}(X+\theta P)\,.$$

U>0

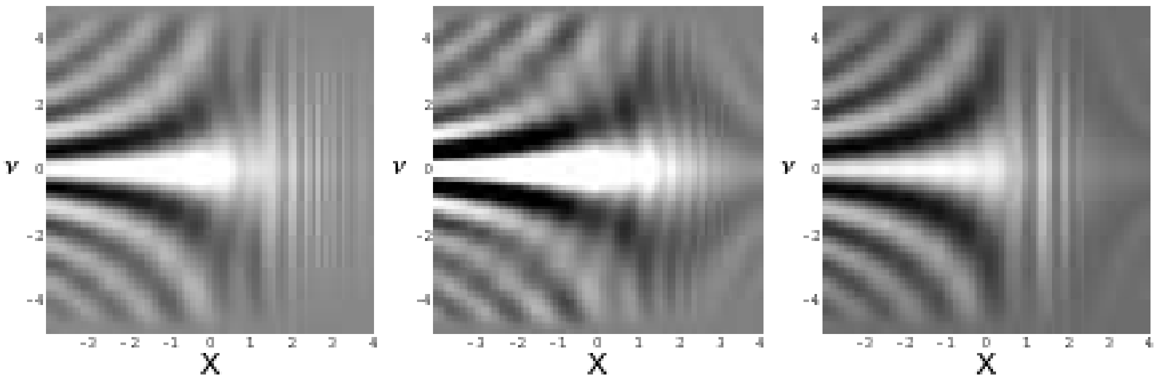
$\tilde{\varrho}_W(X,\nu)$



(a) (b) (c)

θ =0 **θ =0.1** **θ =-0.1**

$\tilde{\varrho}(X,\nu)$

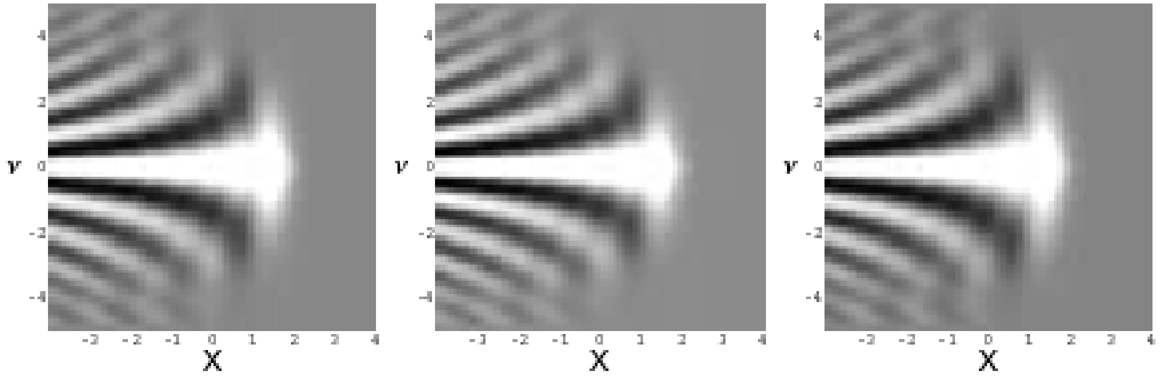


(a) (b) (c)

almost indistinguishable

U<0

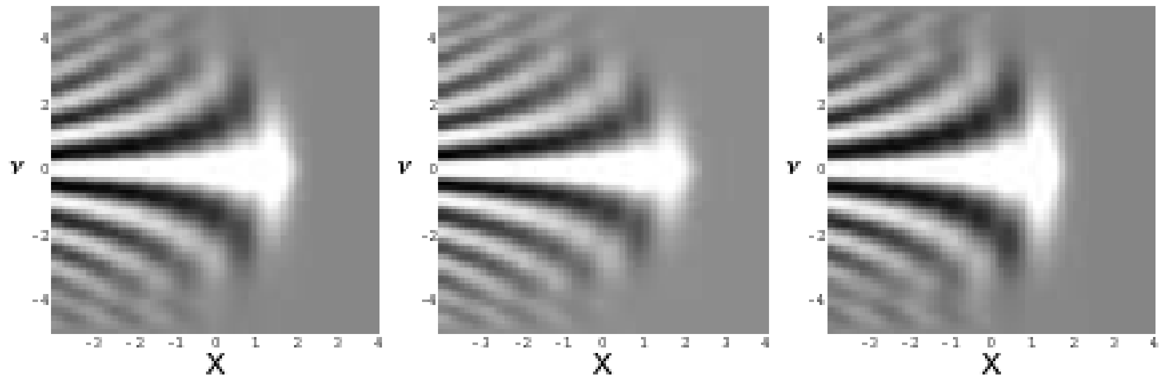
$\tilde{\varrho}_W(X,\nu)$



(a) (b) (c)

θ=0 **θ=0.1** **θ=-0.1**

$\tilde{\varrho}(X,\nu)$



(a) (b) (c)

almost indistinguishable

§ 7. Discussion and Outlook

- A NonCommutative (**NC**) deformation of the minisuperspace variables is studied by means of an integrable model. Its analytical solutions are obtained in classical and quantum cosmology.

- We showed that the peak of the wave packet reproduces the classical trajectory by using exact solutions with an interpretation of the **NC** variables in the present model.
- We proposed a new probability distribution in **NC** quantum cosmology constructed from the Wigner function. Its validity in the present solvable model is confirmed numerically.

- In future study, we will investigate general **NC** cosmology by using the probability distribution function. General deformations of minisuperspace variables should be studied further.
- The model with a phantom scalar field and/or a phantom gauge field may also be worth studying in the context of **NC** cosmology.