

# **Atiyah-Patodi-Singer (APS) index theorem on a lattice**

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Collaboration with

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2019. 8. 21 Strings and Fields 2019@YITP

# Atiyah-Patodi-Singer (APS) index theorem and bulk-edge correspondence

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APS index theorem is important to understand bulk-edge correspondence in topological insulator.

[Witten 2015]

3+1-dim bulk — 2+1-dim edge

$$Z_{\text{edge}} \propto \exp\left(-i\pi\eta\left(iD^{3\text{D}}/2\right)\right) \quad \text{T-anomalous}$$

$$Z_{\text{bulk}} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0}d^4x\epsilon^{\mu\nu\rho\sigma}\text{tr}F_{\mu\nu}F_{\rho\sigma}\right) \quad \text{T-anomalous}$$

$$Z_{\text{edge}}Z_{\text{bulk}} \propto (-1)^{\mathcal{J}} \quad \text{T-symmetry is protected}$$

APS index

$$\mathcal{J} = \frac{1}{32\pi^2}\int_{x_4>0}d^4x\epsilon^{\mu\nu\rho\sigma}\text{tr}_c F_{\mu\nu}F_{\rho\sigma} - \frac{\eta\left(iD^{3\text{D}}\right)}{2}$$

# Eta-invariant

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$$\mathcal{J} = \frac{1}{32\pi^2} \int_{x_4 > 0} d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} - \boxed{\frac{\eta(iD^{3D})}{2}}$$

APS eta-invariant

Eta-invariant is defined by summing up all signs of eigenvalues.

$$\eta(H) = \sum_i \text{sgn} \lambda_i = \text{Tr} \frac{H}{\sqrt{H^2}} \quad H \phi_i = \lambda_i \phi_i$$

Eta-invariant in APS index

$$\frac{\eta(iD^{3D})}{2} = \frac{\text{CS}}{2} + \text{integer} \quad \text{CS} = \frac{1}{4\pi} \int d^3x \text{tr}_c \left[ \epsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) \right]$$

Eta-invariant = Chern-Simons term and integer(non-local effect)

# APS index theorem using domain-wall fermion

Topological insulator and domain-wall fermion [Kaplan 1992] share the same properties.

Massive fermion in the bulk + edge localized mode

The “new” APS index is formulated using domain-wall fermion in continuum theory.

[Fukaya-Onogi-Yamaguchi 2017]

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\text{DW}}) \quad H_{\text{DW}} = \gamma_5(D_{4\text{D}} - M\epsilon(x_4))$$



**Fujikawa method**

$$= \frac{1}{32\pi^2} \int_{x_4 > 0} d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} - \frac{\eta(iD^{3\text{D}})}{2}$$

# Index theorems on continuum theory

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Atiyah-Singer index theorem

$$\text{Index}(D) = -\frac{1}{2}\eta(\gamma_5(D - M)) \quad \leftarrow \text{Fukaya's talk}$$

Atiyah-Patodi-Singer index theorem

$$\text{Index}(D) = -\frac{1}{2}\eta(\gamma_5(D - M\epsilon(x_4)))$$

## How about on a lattice ?

AS index is related to chiral symmetry.

However we break the chiral symmetry on a lattice to avoid the fermion doubling problem. (Nielsen-Ninomiya theorem)

# Ginsparg-Wilson relation and Overlap Dirac operator

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Chiral symmetry on a lattice

$a$  : lattice spacing

$$\delta_\alpha \psi(x) = i\alpha \gamma_5 \left(1 - \frac{1}{2}aD\right) \psi(x) \quad \delta_\alpha \bar{\psi}(x) = i\alpha \bar{\psi}(x) \gamma_5 \left(1 - \frac{1}{2}aD\right)$$

Ginsparg-Wilson relation

[Ginsparg-Wilson 1982]

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D$$

Overlap Dirac operator

[Neuberger 1998]

$$D_{ov} = \frac{1}{a} \left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right) \quad H_W = \gamma_5 (D_W - M)$$

$D_W$  : Wilson Dirac operator

# AS index theorem on a lattice

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[Hasenfratz 1998]

AS index theorem on lattice

[Neuberger 1998]

Overlap Dirac operator

$$\begin{aligned} \text{Ind}(D_{ov}) &= \text{Tr} \gamma_5 \left( 1 - \frac{aD_{ov}}{2} \right) & D_{ov} &= \frac{1}{a} \left( 1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}} \right) \\ &= -\frac{1}{2} \text{Tr} \frac{H_W}{\sqrt{H_W^2}} & &= -\frac{1}{2} \eta(\gamma_5(D_W - M)) \end{aligned}$$

AS index can be written by eta-invariant of  
massive Wilson-Dirac operator.

Chiral symmetry is NOT important.

# Index theorems on continuum and on lattice

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D^{4D} - M))$	$-\frac{1}{2}\eta(\gamma_5(D_W - M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D^{4D} - M\epsilon(x_4)))$	$-\frac{1}{2}\eta(\gamma_5(D_W - M\epsilon(x_4)))$



# Eta-invariant is always an integer on 4-dim lattice

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Eta-invariant

$$\eta(H) = \sum_i \text{sgn} \lambda_i = \text{Tr} \frac{H}{\sqrt{H^2}} \quad H \phi_i = \lambda_i \phi_i$$

$-\frac{1}{2} \eta(H_{\text{DW}})$  is **always** an integer.

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Its variation is always zero.

$$H_{\text{DW}} = \gamma_5 \left( D_{\text{W}} - M \epsilon \left( x_4 - \frac{a}{2} \right) \right)$$

$$\begin{aligned} \delta (\eta(H_{\text{DW}})) &= \text{Tr} \left[ \delta H_{\text{DW}} (H_{\text{DW}}^2)^{-1/2} - \frac{1}{2} H_{\text{DW}} (H_{\text{DW}}^2)^{-3/2} (\delta H_{\text{DW}} H_{\text{DW}} + H_{\text{DW}} \delta H_{\text{DW}}) \right] \\ &= \mathbf{0} \end{aligned}$$

# Our goal

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Our goal is to show

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\text{DW}}) \quad H_{\text{DW}} = \gamma_5 \left( D_{\mathbf{W}} - M\epsilon \left( x_4 - \frac{a}{2} \right) \right)$$

$$= \frac{1}{32\pi^2} \int_{x_4 > 0} d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} - \frac{\eta(iD^{3\text{D}})}{2} + \mathcal{O}(a)$$

in continuum limit

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# How to evaluate eta-invariant

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Insert complete set

$$\eta(H_{\text{DW}}) = \text{Tr} \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} = a^4 \sum_{x,n} \boxed{\Phi_n^\dagger(x)} \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \boxed{\Phi_n(x)}$$

Step1: find a good complete set

Step2: perturbative calculation

# How to evaluate eta-invariant

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Insert complete set

$$\eta(H_{\text{DW}}) = \text{Tr} \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} = a^4 \sum_{x,n} \boxed{\Phi_n^\dagger(x)} \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \boxed{\Phi_n(x)}$$

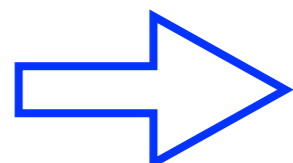
Step 1: find the complete set

$x_1, x_2, x_3$  -direction have no kink structure

Plane wave

$x_4$  -direction has kink structure

Plane wave and Edge localized mode

  $\Phi_n(x) = \phi_{\text{plane}}^{3\text{D}}(\vec{x}) \otimes \phi_{\pm}(x_4)$

# How to evaluate eta-invariant

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$$\eta(H_{\text{DW}}) = \text{Tr} \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} = a^4 \sum_{x,n} \Phi_n^\dagger(x) \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \Phi_n(x)$$

Step2: perturbative calculation

$$H_{\text{DW}}^2 = H_{\text{DW}0}^2(\text{free part}) + (\text{including } A_\mu)$$

Expand the denominator in gauge field

We choose  $\Phi_n(x)$  as eigenfunction of the free part.

$$a^2 H_{\text{DW}0}^2 \Phi_n(x) = \lambda^2 \Phi_n(x)$$

# Set up

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Domain-wall Dirac operator

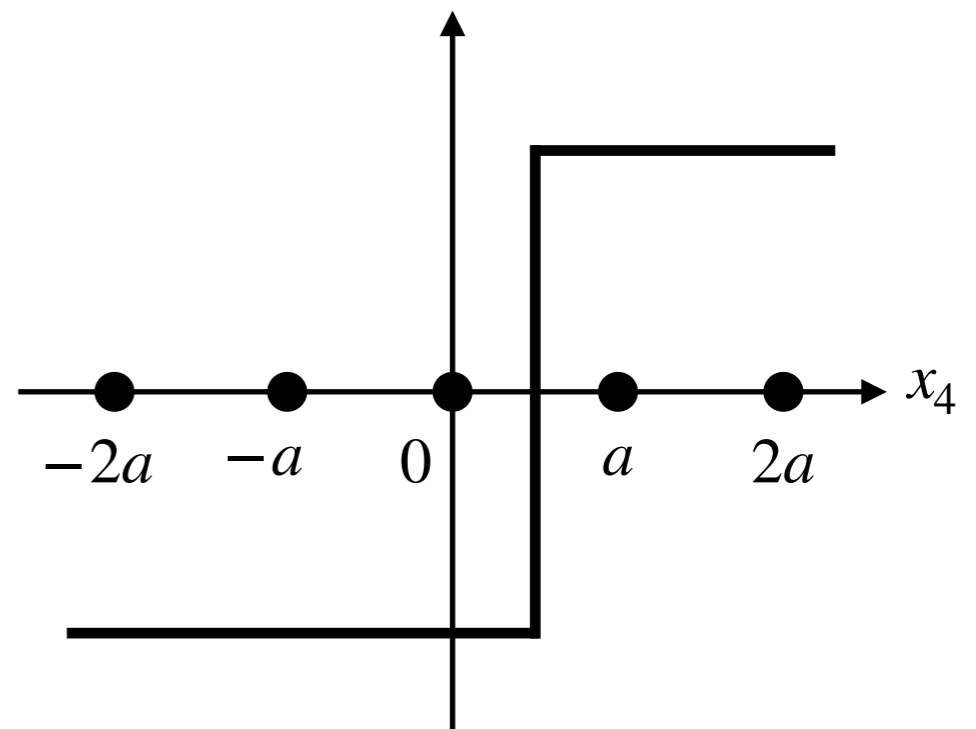
$$H_{\text{DW0}} = \gamma_5 \left( \gamma_\mu \partial_\mu + R - M \epsilon \left( x_4 - \frac{a}{2} \right) \right)$$

**Kink on Link**

Wilson term

$$R = -\frac{ar}{2} \nabla_\mu^* \nabla_\mu$$

We set Wilson parameter = 1



Domain-wall mass is asymmetric due to Wilson term.

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# Free domain-wall eigenfunction set

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We need to solve eigenvalue problem of the free domain-wall Dirac operator .

$$a^2 H_{\text{DW}0}^2 \Phi_n(x) = \lambda^2 \Phi_n(x)$$

Assuming the plane wave form in the  $x_1, x_2, x_3$ -directions.

$$\Phi_n(x) = \phi_{\text{plane}}^{3\text{D}}(\vec{x}) \otimes \phi_{\pm}(x_4)$$

Denoting  $\sin(p_i a) = s_i$        $\cos(p_i a) = c_i$

$$a^2 H_{\text{DW}0}^2 \left( a^2 H_{\text{DW}0}^2 \phi_{\pm}(x_4) = \lambda^2 \phi_{\pm}(x_4) \right)$$

$$= s_i^2 + M_{\pm}^2 - a^2 [1 + M_{\pm}] \nabla_4^* \nabla_4 + 2Ma \left[ P_+ \delta_{x_4,0} \hat{S}_+ - P_- \delta_{x_4,a} \hat{S}_- \right]$$

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$$M_{\pm} = \sum_i (1 - c_i) \mp Ma \quad P_{\pm} = \frac{1}{2}(1 \pm \gamma_4) \quad \hat{S}_{\pm} f(x) = f(x \pm a)$$

# Complete set

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Solve this difference equation

$$\begin{aligned} [s_i^2 + M_\epsilon] \phi_\pm(x_4) - [1 + M_\epsilon] [\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)] \\ + 2Ma [P_+ \delta_{x_4,0} \phi_\pm(x_4 + a) - P_- \delta_{x_4,a} \phi_\pm(x_4 - a)] = \lambda^2 \phi_\pm(x_4) \end{aligned}$$

# Complete set

Solve this difference equation

$$\begin{aligned} [s_i^2 + M_\epsilon] \phi_\pm(x_4) - [1 + M_\epsilon] [\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)] \\ + 2Ma [P_+ \delta_{x_4,0} \phi_\pm(x_4 + a) - P_- \delta_{x_4,a} \phi_\pm(x_4 - a)] = \lambda^2 \phi_\pm(x_4) \end{aligned}$$

(1) Edge localized mode

$$\gamma_4 u_\pm = \pm u_\pm$$

$$\phi_-^{\text{edge}}(x_4) = \begin{cases} Au_- e^{-\kappa_+(x_4-a)} & (x_4 \geq a), \\ Au_- e^{\kappa_-(x_4-a)} & (x_4 \leq 0), \end{cases}$$

$$A = \sqrt{\frac{M_+ M_- (2 + M_+) (2 + M_-)}{a(M_+ - M_-)(M_+ + M_- + 2)}}$$

Normalizable only if

$$|1 + M_+| < 1$$

$$e^{-\kappa_+ a} = 1 + M_+ \quad e^{\kappa_- a} = 1 + M_- \quad (0 < Ma < 2 \text{ in the continuum limit})$$

# Complete set

Solve this difference equation

$$\begin{aligned} [s_i^2 + M_\epsilon] \phi_\pm(x_4) - [1 + M_\epsilon] [\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)] \\ + 2Ma [P_+ \delta_{x_4,0} \phi_\pm(x_4 + a) - P_- \delta_{x_4,a} \phi_\pm(x_4 - a)] = \lambda^2 \phi_\pm(x_4) \end{aligned}$$

(2) Bulk mode 1 : Plane wave on both side of the kink

$$\phi_{1,\pm}^{\text{bulk}}(x_4) = \begin{cases} u_\pm T_1^\pm e^{i\omega_+ x_4} & (x_4 \geq a) \\ u_\pm [e^{-i\omega_- x_4} + R_1^\pm e^{-i\omega_- x_4}] & (x_4 \leq 0) \end{cases}$$

$$\phi_{2,\pm}^{\text{bulk}}(x_4) = \begin{cases} u_\pm [e^{-i\omega_+ x_4} + R_2^\pm e^{i\omega_+ x_4}] & (x_4 \geq a) \\ u_\pm T_2^\pm e^{-i\omega_- x_4} & (x_4 \leq 0) \end{cases}$$

orthogonal

$$R_1^\pm = -\frac{(1 + M_\pm) - (1 + M_\mp)e^{-i(\omega_+ - \omega_-)a}}{(1 + M_\pm) - (1 + M_\mp)e^{-i(\omega_+ + \omega_-)a}}$$

$$T_1^\pm = \frac{(1 + M_-)e^{-i\omega_+ a} 2i \sin(\omega_- a)}{(1 + M_\pm) - (1 + M_\mp)e^{-i(\omega_+ + \omega_-)a}}$$

$$R_2^\pm = -\frac{(1 + M_\mp) - (1 + M_\mp)e^{-i(\omega_+ - \omega_-)a}}{(1 + M_\mp) - (1 + M_\mp)e^{-i(\omega_+ + \omega_-)a}}$$

$$T_2^\pm = \frac{-(1 + M_+)e^{-i\omega_- a} 2i \sin(\omega_+ a)}{(1 + M_\mp) - (1 + M_\pm)e^{-i(\omega_+ + \omega_-)a}}$$

# Complete set

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Solve this difference equation

$$\begin{aligned} [s_i^2 + M_\epsilon] \phi_\pm(x_4) - [1 + M_\epsilon] [\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)] \\ + 2Ma [P_+ \delta_{x_4,0} \phi_\pm(x_4 + a) - P_- \delta_{x_4,a} \phi_\pm(x_4 - a)] = \lambda^2 \phi_\pm(x_4) \end{aligned}$$

(3) Bulk mode 2 : Plane wave on one side of the kink  
and localized mode on the other side

$$\phi_\pm^{\text{bulk}}(x_4) = \begin{cases} \frac{u_\pm}{\sqrt{2\pi}|A_\pm|} [A_\pm e^{i\omega x_4} - A_\pm^* e^{-i\omega x_4}] & (x_4 \geq a) \\ \frac{u_\pm}{\sqrt{2\pi}|A_\pm|} B_\pm e^{kx_4} & (x_4 \leq 0) \end{cases}$$

$$A_\pm = \frac{1 + M_\mp}{1 + M_\pm} e^{\kappa a} - e^{-i\omega a}$$

$$B_+ = e^{i\omega a} - e^{-i\omega a}$$

$$B_- = \frac{1 + M_+}{1 + M_-} [e^{i\omega a} - e^{-i\omega a}]$$

# Shamir-type domain-wall

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Introduce an additional mass  $M_2$

[Shamir 1993]

[Furman-Shamir 1995]

$$H_{\text{DW}0} = \gamma_5 \left[ \gamma_\mu \partial_\mu + R - M_1 \epsilon \left( x_4 - \frac{a}{2} \right) + M_2 \right]$$

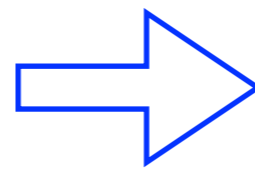
$$a^2 (H_{\text{DW}0})^2$$

$$= s_i^2 + M_\pm^2 - a^2 [1 + M_\pm] \nabla_4^* \nabla_4 + 2M_1 a \left[ P_+ \delta_{x_4,0} \hat{S}_+ - P_- \delta_{x_4,a} \hat{S}_- \right]$$

$M_1 + M_2 = \infty$  while  $M_1 - M_2 = M$  is fixed

$$M_+ = \sum (1 - c_i) - M_1 a + M_2 a$$

$$M_- = \sum (1 - c_i) + M_1 a + M_2 a$$



$$M_+ = \sum (1 - c_i) - M a$$

$$M_- = \infty$$

The index is independent of  $M_2$ .

[Fukaya-Onogi-Yamaguchi 2017]

# Shamir-type domain-wall

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Complete set is reduced to

(1) Edge localized mode

$$\gamma_4 u_{\pm} = \pm u_{\pm}$$

$$\phi_{-}^{\text{edge}}(x_4) = \sqrt{-\frac{1}{a}M_{+}(2 + M_{+})}u_{-}e^{-\kappa_{+}(x_4 - a)}$$

(2) Bulk mode

$$\phi_{+}^{\text{bulk}}(x_4) = \frac{1}{\sqrt{2\pi}}u_{+} [e^{i\omega x_4} - e^{-i\omega x_4}]$$

$$\phi_{-}^{\text{bulk}}(x_4) = \frac{1}{\sqrt{2\pi}}u_{-} [C_{\omega}e^{i\omega x_4} - C_{\omega}^{*}e^{-i\omega x_4}]$$

$$C_{\omega} = -\frac{(1 + M_{+}) - e^{-i\omega a}}{|(1 + M_{+}) - e^{-i\omega a}|}$$

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# Calculation of eta-invariant

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\text{DW}}) = -\frac{1}{2}\text{Tr}H_{\text{DW}}\frac{1}{\sqrt{H_{\text{DW}}^2}}$$

Bulk and edge decomposition

$$= -\frac{1}{2}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \int_0^{\frac{\pi}{a}} d\omega \sum_{\eta=\pm} \text{tr}_c \Phi_{\eta}^{\text{bulk}\dagger}(x) \left[ \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \right] \Phi_{\eta}^{\text{bulk}}(x)$$

bulk part

$$\Phi_{\eta}^{\text{bulk}}(x) = \phi_{\eta}^{\text{bulk}}(x_4) \otimes \phi_p^{3\text{D}}(\vec{x})$$

$$-\frac{1}{2}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \text{tr}_c \Phi_{-}^{\text{edge}\dagger}(x) \left[ \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \right] \Phi_{-}^{\text{edge}}(x)$$

edge part

$$\Phi_{-}^{\text{edge}}(x) = \phi_{-}^{\text{edge}}(x_4) \otimes \phi_p^{3\text{D}}(\vec{x})$$

# Calculation of eta-invariant (1) : bulk part

$$-\frac{1}{2}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3 p \int_0^{\frac{\pi}{a}} d\omega \sum_{\eta=\pm} \text{tr}_c \Phi_\eta^{\text{bulk}\dagger}(x) \left[ \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \right] \Phi_\eta^{\text{bulk}}(x)$$

$$H_{\text{DW}}^2 = H_{\text{DW}0}^2 + \Delta H_{\text{DW}}^2 \quad \Delta H_{\text{DW}}^2 \ni [\gamma^\mu, \gamma^\nu] [D_\mu, D_\nu] , [\gamma^\mu D_\mu, R]$$

$$\begin{aligned} [H_{\text{DW}}^2]^{-\frac{1}{2}} &= [H_{\text{DW}0}^2 + \Delta H_{\text{DW}}^2]^{-\frac{1}{2}} \\ &= \frac{1}{(H_{\text{DW}0}^2)^{\frac{1}{2}}} - \frac{1}{2} \Delta H_{\text{DW}}^2 \frac{1}{(H_{\text{DW}0}^2)^{\frac{3}{2}}} + \frac{3}{8} (\Delta H_{\text{DW}}^2)^2 \frac{1}{(H_{\text{DW}0}^2)^{\frac{5}{2}}} + \dots \end{aligned}$$

= 0 due to the spinor structure

# Calculation of eta-invariant (1) : bulk part

$$\begin{aligned}
 & -\frac{3}{16}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \int_0^{\frac{\pi}{a}} d\omega \sum_{\eta=\pm} \text{tr}_c \Phi_{\eta}^{\text{bulk}\dagger}(x) \left[ H_{\text{DW}} (\Delta H_{\text{DW}})^2 \frac{1}{(H_{\text{DW}0})^{\frac{5}{2}}} \right] \Phi_{\eta}^{\text{bulk}}(x) \\
 & = \frac{1}{32\pi^2} a^4 \sum_{\vec{x}, x_4 > a} I(M) \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} + \mathcal{O}(a)
 \end{aligned}$$

It is the same expression as [Suzuki 1999] which derives AS index on lattice by using perturbative expansion.

$$I(Ma) \rightarrow \theta(Ma) - 4\theta(Ma - 2) + 6\theta(Ma - 4) - 4\theta(Ma - 6) + \theta(Ma - 8)$$

We choose  $0 < Ma < 2$  in continuum limit.

[Kikukawa-Yamada 1998]

[Luscher 1998]

[Fujikawa 1998]

[Adams 1998]

$$\Rightarrow \frac{1}{32\pi^2} a^4 \sum_{\vec{x}, x_4 \geq a} \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} + \mathcal{O}(a)$$

# Calculation of eta-invariant (2) : edge part

Domain-wall Dirac operator in  $A_4 = 0$  gauge.

$$H_{\text{DW}} = \gamma_5 [-aP_- \nabla_4 + aP_+ \nabla_4^* + \gamma_i D_i(U_i(x)) + M_+(U_i(x))]$$

Expanding in  $\|U^\dagger \partial_{x_4} U\|/M$  (adiabatic expansion)

$\phi_-^{\text{edge}}(x_4)$  satisfies

$$-aP_- \nabla_4 \phi_-^{\text{edge}}(x_4) = -M_+ \phi_-^{\text{edge}}(x_4)$$

$\nabla_4$  : forward derivative

$\nabla_4^*$  : backward derivative

Then

$$\begin{aligned} H_{\text{DW}} \phi_-^{\text{edge}}(x_4) &= \gamma_5 \gamma_i D_i \phi_-^{\text{edge}}(x_4) \\ &= \begin{pmatrix} -i\sigma_i D_i & \\ & i\sigma_i D_i \end{pmatrix} \phi_-^{\text{edge}}(x_4) \\ &\longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_i D_i \end{pmatrix} \phi_-^{\text{edge}}(x_4) \end{aligned}$$

# Calculation of eta-invariant (2) : edge part

$$-\frac{1}{2}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3 p \text{tr}_c \Phi_-^{\text{edge}\dagger}(x) \left[ \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \right] \Phi_-^{\text{edge}}(x)$$

$$= -\frac{1}{2}a^4 \sum_{\vec{x}, x \geq a} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^3 p \text{tr}_c \phi_p^{3\text{D}\dagger}(\vec{x}) \left[ \frac{iD^{3\text{D}}}{\sqrt{(iD^{3\text{D}})^2}} \right] \underbrace{\phi_p^{3\text{D}}(\vec{x}) \phi_-^{\text{edge}}(x_4) \phi_-^{\text{edge}\dagger}(x_4)}_{-M_+(2 + M_+)e^{-2\kappa_+(x_4-a)}}$$

$$D^{3\text{D}}(x) = \sigma_i D_i(x)$$

In continuum and low-energy limit.

→  $\delta_{x_4, a}$

$$= -\frac{1}{2} \text{Tr}_{3\text{D}} \frac{iD^{3\text{D}}}{\sqrt{(iD^{3\text{D}})^2}} \Big|_{x_4=a} + \mathcal{O}(a) = -\frac{1}{2} \eta(iD^{3\text{D}}) \Big|_{x_4=a} + \mathcal{O}(a)$$

# Calculation of eta-invariant : Result

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\text{DW}}) = -\frac{1}{2}\text{Tr}H_{\text{DW}}\frac{1}{\sqrt{H_{\text{DW}}^2}}$$

$$= -\frac{1}{2}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \int_0^{\frac{\pi}{a}} d\omega \sum_{\eta=\pm} \text{tr}_c \Phi_{\eta}^{\text{bulk}\dagger}(x) \left[ \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \right] \Phi_{\eta}^{\text{bulk}}(x)$$

$$\Rightarrow \frac{1}{32\pi^2} a^4 \sum_{\vec{x}, x_4 \geq a} \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} + \mathcal{O}(a)$$

$$-\frac{1}{2}a^4 \sum_{\vec{x}, x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \text{tr}_c \Phi_{-}^{\text{edge}\dagger}(x) \left[ \frac{H_{\text{DW}}}{\sqrt{H_{\text{DW}}^2}} \right] \Phi_{-}^{\text{edge}}(x)$$

$$\Rightarrow -\frac{1}{2}\eta(iD^{3\text{D}}) + \mathcal{O}(a)$$

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# Summary

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- The eta invariant of massive Dirac operator gives a unified view of index theorems.

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D^{4D} - M))$	$-\frac{1}{2}\eta(\gamma_5(D_W - M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D^{4D} - M\epsilon(x_4)))$	$-\frac{1}{2}\eta(\gamma_5(D_W - M\epsilon(x_4)))$

- We confirmed that the eta-invariant of Wilson-Dirac operator with domain-wall mass converges to the APS index in the continuum limit.

$$\mathcal{J} = -\frac{1}{2}\eta(H_{DW}) = \frac{1}{32\pi^2} \int_{x_4 > 0} d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}_c F_{\mu\nu} F_{\rho\sigma} - \frac{\eta(iD^{3D})}{2} + \mathcal{O}(a)$$



# Outlook

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- Our computation was approximately done in  $x_4 \geq 0$  semi-infinite spacetime.
  - $x_4$  should be compactified with anti-domain-wall.
- Generalization to any even dimensional lattices.
- Application to odd dimensions.
- Discussion of locality and smoothness(admissibility condition).