Black holes with baryonic charge and $\mathcal{I}$-extremization

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Based on 1904.05344 with Nakwoo Kim
See also 1904.04269 (HZ) and 1904.04282 (GMS)

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Recently, there has been progress on the microscopic understanding of the AdS black hole entropy such as

- the magnetically charged static AdS black holes and the topologically twisted index
- the electrically charged rotating AdS black holes and the superconformal index with complex chemical potentials

(See Morteza's overview talk)

In this talk, I will discuss the magnetically charged static AdS black holes with baryonic flux from the viewpoint of the extremization principle and toric geometry.

Four Extremizations

It is known that there are four extremization principles in supersymmetric gauge theories.

a-maximization	F-maximization
[Intrilligator, Wecht 03]	[Jafferis 10]
4 d, $\mathcal{N}$=1	3 d, $\mathcal{N}$=2
central charge $a_{\rm trial}(\Delta_a)$	S^3 free energy $F(\Delta_a)$

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✓gravity dual

AdS$_5\times$ Y$_5$, $a_{\text{trial}} \sim \frac{1}{\text{vol}(\text{Y}_5)}$ AdS$_4\times$ Y$_7$, $F \sim \frac{1}{\sqrt{\text{vol}(\text{Y}_7)}}$

Geometric dual of a- & F-maximization

: volume minimization [Martelli, Sparks, Yau 05]

Compactify theories on Σ_g with a topological twist

c-extremization	$\mathcal{I}$-extremization
[Benini, Bobev 12]	[Benini, Hristov, Zaffaroni 15]
2 d, $\mathcal{N}$=(0,2)	1 d, $\mathcal{N}$=2
central charge $c_r(\Delta_a, \mathbf{n}_a)$	topologically twisted index $\mathcal{I}(\Delta_a, \mathbf{n}_a)$

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✓gravity dual

$\mathrm{AdS}_3 \times \Sigma_g$,	$\mathrm{AdS}_2 \times \Sigma_g$ the entropy of the magnetically charged static AdS black hole

Aim : Finding a geometric dual of c- and $\mathcal{I}$-extremization.

The geometric dual of c-extremization was studied.

[Couzens, Gauntlett, Martelli, Sparks 1810; GMS 1812 ; Hosseini, Zaffaroni 1901]

In this talk, I will focus on the $\mathcal{I}$-extremization.

[HZ 1901; HZ; GMS; KK 1904]

AdS solutions from wrapped D3- and M2-branes

AdS_3 solutions in type IIB [Nakwoo Kim 05]

$$ds^2_{10} = L^2 e^{-B/2} \left(ds^2(AdS_3) + ds^2(Y_7) \right),$$
$$F_5 = -L^4 \left(\text{vol}_{AdS_3} \wedge F + *_7 F \right).$$

AdS_2 solutions in d=11 supergravity [N. Kim, Jong-Dae Park 06]

$$ds^2_{11} = L^2 e^{-2B/3} \left(ds^2(AdS_2) + ds^2(Y_9) \right),$$
$$F_5 = L^3 \text{vol}_{AdS_2} \wedge F.$$

$*$ SUSY requires a Killing vector ξ in Y_{2n+1} and the foliation Y_{2n} to be a Kähler manifold.
$**$ The 2n-dimensional Kähler metrics satisfy the gauge field equation of motion

$$\Box_{2n} R - \frac{1}{2} R^2 + R_{ij} R^{ij} = 0,$$

where $n = 3$ for IIB and $n = 4$ for d=11.

CGMS-Extremization

Imposing the supersymmetry condition (∗) and relaxing the equation of motion (∗∗), the supersymmetric solution can be obtained by extremizing (2n+1)-dimensional action [Couzens, Gauntlett, Martelli, Sparks 1810]

$$S_{\text{SUSY}} = \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!}.$$

For n=3, the central charge

$$c_{\text{sugra}} = \frac{3L}{2G_3} = \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S_{\text{SUSY}}|_{\text{on-shell}}.$$

For n=4, the Bekenstein-Hawking entropy

$$S_{\text{BH}} = \frac{1}{4G_2} = \frac{4\pi L^9}{(2\pi)^8 \ell_p^9} S_{\text{SUSY}}|_{\text{on-shell}}.$$

OLD & NEW extremizations

✓ $AdS_5 \times SE_5$ and $AdS_4 \times SE_7$ solutions

- $C(X_{2n-1})$ is Kähler : X_{2n-1} is Sasakian.
- volume minimization : relax Einstein conditions and extremize the Sasakian volume $V(b_i)$. [Martelli, Sparks, Yau 05]
- $\vec{b}$ is a Killing vector, called Reeb vector, which is dual to a U(1) R-symmetry in the field theory.
- It corresponds to a geometric dual of a- and F-maximization

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✓ $AdS_3 \times Y_7$ and $AdS_2 \times Y_9$ solutions

- $C(Y_{2n+1})$ is not Kähler : Y_{2n+1} is no longer Sasakian.
- Focus on a special case where $Y_{2n-1} \to Y_{2n+1} \to \Sigma_g$ and $C(Y_{2n-1})$ is toric. [Gauntlett, Martelli, Sparks 1812]
- For a given toric data of $C(Y_{2n-1})$, we can calculate a master volume $\mathcal{V}(b_i; \{\lambda_a\})$ where λ_a is the transverse Kähler class.
- Extremizing $2n+1$-dimensional action $S_{\rm SUSY}$ corresponds to a geometric dual of c- and $\mathcal{I}$-extremization.

GMS-Extremization with the master volume

Step 1. Construct the master volume $\mathcal{V}(b_i;\{\lambda_a\})$ for a given toric data.

Step 2. Solve the constraint equation and the flux quantization conditions for λ_a, A

$$A\sum_{a,b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b}=2\pi n^1\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}-2\pi b_1\sum_{i=1}^{4}n^i\sum_{a=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i},$$

$$N=-\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a},\quad \mathbf{n}_aN=-\frac{A}{2\pi}\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b}-b_1\sum_{i=1}^{4}n^i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}.$$

Step 3. Obtain the entropy functional and the R-charges of baryonic operators

$$S(b_i,\mathbf{n}_a)=-8\pi^2\left(A\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}+2\pi b_1\sum_{i=1}^{4}n^i\frac{\partial\mathcal{V}}{\partial b_i}\right)\Bigg|_{\lambda_a,A},$$

$$\tilde{R}_a(b_i,\mathbf{n}_a)=-\frac{2}{N}\ \frac{\partial\mathcal{V}}{\partial\lambda_a}\Bigg|_{\lambda_a,A}.$$

Step 4. Extremize the entropy functional $S(b_i,\mathbf{n}_a)$ with respect to b_2, b_3 and b_4 after setting $b_1=1$.

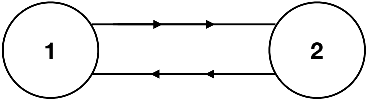
Topologically twisted indices, black hole entropy and entropy functional : ABJM case

[Benini, Hristov, Zaffaroni 15]
[Hosseini, Zaffaroni 1901]

ABJM theory

ABJM theory is a 3-dimensional $U(N)_k \times U(N)_{-k}$ Chern-Simons theory

- 4 bi-fundamental chiral multiplets
- the quartic superpotential
  $W \propto \mathrm{tr}(\epsilon_{ab}\epsilon^{cd} Z^a W_c Z^b W_d)$



The dual gravity theory is

- the $\mathrm{AdS}_4 \times \mathrm{S}^7$ solution of D=11 supergravity
- the SO(8)-invariant vacuum of D=4, $\mathcal{N}=8$ SO(8) gauged supergravity

Topological twisted index and black hole entropy

The topologically twisted index is the partition function on $\Sigma_g \times S^1$ with magnetic fluxes $\mathbf{n}_a$ on Σ_g. In the large N-limit, it reduce to

$$\mathcal{I}\left(\Delta_a, \mathbf{n}_a\right) = -\frac{\pi}{3} N^{3/2} \sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4} \left(\sum_{a=1}^{4} \frac{\mathbf{n}_a}{\Delta_a}\right)$$

where

$$\sum_{a=1}^{4} \Delta_a = 2, \quad \sum_{a=1}^{4} \mathbf{n}_a = 2 - 2\mathfrak{g}.$$

The entropy of magnetically charged static D=4 AdS black holes solution is

$$S_{\mathrm{BH}} = -\frac{\pi L^2}{G_4} \sqrt{X_1 X_2 X_3 X_4} \left(\sum_{a=1}^{4} \frac{\mathbf{n}_a}{X_a}\right).$$

[Benini, Hristov, Zaffaroni 15]

$\mathcal{I}$-extremization

Extremizing the twisted index and the black hole entropy w.r.t Δ_a and X_a, respectively, leads to

$$\mathcal{I}|_{\Delta_a=\bar{\Delta}_a}(\mathbf{n}_a) = S_{\rm BH}|_{X=X(r_h)}(\mathbf{n}_a)\,.$$

Comments

- The entropy is a function of magnetic flux.
- The topologically twisted index successfully reproduces the entropy of the black hole.
- The extremization procedure

  on the field theory side is called $\mathcal{I}$-extremization.

  on the gravity side corresponds to the attractor mechanism.

They agree even before extremization! (off-shell)

Entropy functional

Using the toric data of $\mathbb{C}^4$, ▸ MV-M111

$$v_1 = (1,0,0,0), v_2 = (1,1,0,0), v_3 = (1,0,1,0), v_4 = (1,0,0,1),$$

the master volume for S^7 is easily obtained as [Hosseini, Zaffaroni 1901]

$$\mathcal{V}(b_i, \lambda_a) = \frac{8\pi^4 \left(\lambda_1(b_2+b_3+b_4-b_1) - \lambda_2 b_2 - \lambda_3 b_3 - \lambda_4 b_4\right)^3}{3 b_2 b_3 b_4 \left(b_1 - b_2 - b_3 - b_4\right)}.$$

The entropy functional and R-charges are

$$S(b_i, \mathfrak{n}_a) = -\frac{2\pi\sqrt{2}N^{3/2}}{3}\sqrt{\frac{b_2 b_3 b_4 (b_1 - b_2 - b_3 - b4)}{b_1}} \times \left(\frac{\mathfrak{n}_1}{b_1 - b_2 - b_3 - b_4} + \frac{\mathfrak{n}_2}{b_2} + \frac{\mathfrak{n}_3}{b_3} + \frac{\mathfrak{n}_4}{b_4}\right),$$

$$\Delta_1(b_i) = \frac{2\left(b_1 - b_2 - b_3 - b_4\right)}{b_1}, \quad \Delta_2 = \frac{2b_2}{b_1}, \quad \Delta_3 = \frac{2b_3}{b_1}, \quad \Delta_4 = \frac{2b_4}{b_1}.$$

The entropy functional exactly agrees with the topologically twisted index.

$$S(b_i, \mathfrak{n}_a) = \mathcal{I}(\Delta_a, \mathfrak{n}_a)|_{\Delta_a(b_i)}.$$

Comments

- It is the first example of the geometric dual of $\mathcal{I}$-extremization.
- In computing the entropy functional all we need is only the toric data.
- We do not need to know the explicit metric.
- Nonetheless, the existence of the explicit solutions of gravity theory and the IR fixed point of field theory are important.

Black holes with baryonic charge : $M^{1,1,1}$ case

[HK, N. Kim 1904]

$\mathrm{AdS}_4 \times M^{111}$

A homogeneous Sasaki-Einstein seven-manifold M^{111} is an U(1) fibration over $\mathrm{S}^2 \times \mathbb{CP}^2$.

- It preserves $\mathcal{N} = 2$ supersymmetry.
- There is a non-trivial 2-cycle: $b_2(M^{111}) = 1$.

The bulk massless vector fields come from

- the isometries of Y_7
- the reduction of A_3 potential on non-trivial two-cycles in Y_7.

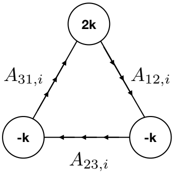
They are related to the mesonic and baryonic global symmetries in dual field theories, respectively.

- The 4-form flux through this cycle gives one Betti vector multiplet which is related to a baryonic symmetry in the dual field theory.
- $b_2(S^7) = 0$: In ABJM theory, there is no baryonic symmetry.
  $b_2(Q^{111}) = 2$: Q^{111} is an U(1) fibration over $S^2 \times S^2 \times S^2$.

Dual field theory

3-dimensional $U(N)^3$ Chern-Simons theory [Martelli, Sparks 08]

- CS levels (2k,-k,-k)
- 9 bifundamental fields
- superpotential $W = \epsilon_{ijk}\mathrm{tr}A_{12,i}A_{23,j}A_{31,k}$
- $SU(3) \times SU(2) \times U(1)_R$ symmetry
- One can assign baryonic charges $(1, -2, 1)$.



Matrix model

A simple method to calculate the S^3 free energy (by computing the matrix integral) at large N was devised.

[Herzog, Klebanov, Pufu, Tesileanu 10]

It successfully applies to various $\mathcal{N} = 2$ theories, especially for non-chiral models which consist of the bi-fundamental fields in a real representation.

[Martelli, Sparks; Cheon, HK, N. Kim; Jafferis, Klebanov, Pufu, Safdi 11]

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Field theory dual to $M^{1,1,1}$ has two problems. [Jafferis, Klebanov, Pufu, Safdi 11]

☹ The trial R-charge is a linear combination of all $U(1)$ charges. But the free energy functional is independent of the baryonic mixing parameter δ_B due to the existence of the flat directions, i.e. $F = F(\Delta_i)$.

$$\Big(\text{e.g. } \tilde{R}[A_{12,1}] = \underbrace{\frac{2}{3} + \delta_1 + \delta_2}_{\Delta_1} + \delta_B,\ \tilde{R}[A_{12,2}] = \underbrace{\frac{2}{3} - \delta_1 + \delta_2}_{\Delta_2} + \delta_B,\ \tilde{R}[A_{12,3}] = \underbrace{\frac{2}{3} - 2\delta_2}_{\Delta_3} + \delta_B.\Big)$$

☹ Chiral model : The matrix model is not working.

The long-range forces between the eigenvalues do not cancel. The free energy is proportional to N^2.

Operator counting

Operator counting method provides us a prescription to obtain the S^3 free energy at large N . [Gulotta, Herzog, Pufu 11]

By counting the gauge invariant operators $T_m A_{12}^{mk_1+s} A_{23}^{m(k_1+k_2)+s} A_{31}^{s}$, we can obtain $\rho(x)$ and $y(x)$ needed in computing a free energy functional $F\left[\rho(x), y(x)\right]$.

The S^3 free energy at large N is written as

$$F = 4\pi \, \frac{\Delta_1\Delta_2\Delta_3}{\sqrt{\Delta_1\Delta_2 + \Delta_2\Delta_3 + \Delta_3\Delta_1}} N^{3/2} k^{1/2}.$$

- Maximizing F gives the correct free energy $F = \frac{16\pi}{9\sqrt{3}} k^{1/2} N^{3/2}$ and R-charges $R_a = \frac{2}{3}$.

Operator counting method also calculate the volume of the non-trivial five-cycles. Then, the R-charges of the baryonic operators becomes

$$\tilde{R} = \frac{\pi}{6} \frac{\mathrm{Vol}(\Sigma_5)}{\mathrm{Vol}(Y)}.$$

1. Topologically twisted index with mesonic flux

In the large-N limit, the topologically twisted index can be expressed in terms of S^3 free energy as [Hosseini, Zaffaroni 16]

$$\mathcal{I}\left(\Delta_i, \mathbf{m}_i\right) = \frac{1}{2}\sum_{i=1}^{3} \mathbf{m}_i \frac{\partial F_{S^3}\left(\Delta_i\right)}{\partial \Delta_i}.$$

Extremizing the index w.r.t Δ_i, we obtain the index and the fluxes. (We consider $\Delta_1 = \Delta_3$ case for simplicity.) [HK, N. Kim 1904]

$$\mathcal{I} = \frac{8\pi}{3}\left(\mathfrak{g}-1\right)\frac{N^{3/2}\,\Delta_1^2\left(\Delta_1^2 + 6\Delta_1\Delta_2 + 3\Delta_2^2\right)}{\sqrt{\Delta_1^2 + 2\Delta_1\Delta_2}\left(4\Delta_1^3 + 8\Delta_1^2\Delta_2 + 4\Delta_1\Delta_2^2 - \Delta_2^3\right)}. \quad (1)$$

$$\mathbf{m}_1 = \mathbf{m}_3 = \left(\mathfrak{g}-1\right)\frac{2\Delta_1\left(5\Delta_1^2 + 7\Delta_1\Delta_2 + 3\Delta_2^2\right)}{3\left(4\Delta_1^3 + 8\Delta_1^2\Delta_2 + 4\Delta_1\Delta_2^2 - \Delta_2^3\right)},$$

$$\mathbf{m}_2 = \left(\mathfrak{g}-1\right)\frac{2\left(2\Delta_1^3 + 10\Delta_1^2\Delta_2 + 6\Delta_1\Delta_2^2 - 3\Delta_2^3\right)}{3\left(4\Delta_1^3 + 8\Delta_1^2\Delta_2 + 4\Delta_1\Delta_2^2 - \Delta_2^3\right)}.$$

The index is independent of the baryonic flux.

2. Black holes in $\mathrm{AdS}_4 \times M^{111}$ with baryonic flux

A consistent truncation of M-theory on M^{111} leads to D=4, $\mathcal{N}=2$ gauged supergravity coupled to a Betti vector multiplet, a massive vector multiplet and a hypermultiplet. [Cassani, Koerber, Varela 12]

- AdS black holes charged under the Betti vector field are known. [Halmagyi, Petrini, Zaffaroni 13]
- The entropy of a magnetically charged AdS black hole in M^{111} is

$$S_{\mathrm{BH}} = \frac{4\pi}{9\sqrt{3}} \frac{v_1(9 - 2v_1^2 + v_1^4)}{(1+v_1^2)} N^{3/2} |\mathfrak{g} - 1|. \tag{2}$$

- v_1 is the imaginary part of the vector multiplet scalar.
- The magnetic charges are $P_1 = -\frac{1}{2\sqrt{2}}$, $P_2 = -\frac{\sqrt{3}(-1+v_1^2)^2}{8(1+v_1^2)}$.
- Setting $v_1 = 1$, we can turn off the Betti vector multiplet.

A consistent truncation, which keeps the vectors associated with the isometry, is not known. In other words, the explicit solution with the mesonic flux is not known.

A status report : Puzzle

	topologically twisted index $\mathcal{I}$	black hole entropy $S_{\rm BH}$
with mesonic flux	eq. (1)	no known sol.
with baryonic flux	$\mathcal{I}$ is indep. of $\mathbf{m}_B$	eq. (2)

"A particularly puzzling feature is that in supergravity the background flux for baryonic U(1) symmetries affects the details of the AdS2 vacuum and thus the black hole entropy. On the other hand, it seems that such baryonic magnetic fluxes do not change the large N limit of the topologically twisted index."

[Azzurli, Bobev, Crichigno, Min, Zaffaroni 17]

3. Extremization principle

	topologically twisted index $\mathcal{I}$	black hole entropy $S_{\rm BH}$	GMS extremization
with mesonic flux	eq. (1)	no known sol.	✓
with baryonic flux	$\mathcal{I}$ is indep. of $\mathbf{m}_B$	eq. (2)	✓

We study the topologically twisted index with the mesonic flux and the entropy of the black hole with the baryonic flux from the viewpoint of GMS extremization principle.
We successfully reproduce these quantities by using toric data of $M^{1,1,1}$,

$$w_1 = (1,0,0,0), \quad w_2 = (1,1,0,0), \quad w_3 = (1,0,1,0),$$
$$w_4 = (1,-1,-1,3k), \quad w_5 = (1,0,0,2k).$$

▸ MV-S7 ▸ MV-M111

Final results

✓ topologically twisted index with the mesonic flux

- Reeb vector : $\vec{b} = (1, 0, b_3, 1 - b_3)$
- flux identifications :

$$\begin{array}{ll} \mathfrak{n}_1 = \mathfrak{n}_5 & \mathbf{n}_2 + \frac{2}{3}\mathbf{n}_5 \equiv -\mathbf{m}_1 \\ \mathfrak{n}_2 = \mathfrak{n}_4 & \mathbf{n}_3 + \frac{2}{3}\mathbf{n}_5 \equiv -\mathbf{m}_2 \\ \text{one constraint on } \mathbf{n}_a & \mathbf{n}_4 + \frac{2}{3}\mathbf{n}_5 \equiv -\mathbf{m}_3 \end{array}$$

$$S\left(b_3, \mathfrak{n}_a(b_3)\right) = \mathcal{I}\left(\Delta_a, \mathfrak{m}_a(\Delta_a)\right)|_{\Delta_a(b_3)}$$

✓ black holes with the baryonic flux

- Reeb vector : $\vec{b} = (1, 0, 0, 1)$
- flux identifications :

$$\mathfrak{n}_1 = \mathfrak{n}_5 \equiv \frac{4\sqrt{2}}{3} P_2 \left(1 - \mathfrak{g}\right) = \frac{1}{3}\left(1 - \mathfrak{g}\right) + 3B$$

$$\mathfrak{n}_2 = \mathfrak{n}_3 = \mathfrak{n}_4 \equiv \frac{16\sqrt{2}}{9} P_1 \left(1 - \mathfrak{g}\right) = \frac{4}{9}\left(1 - \mathfrak{g}\right) - 2B$$

$$S\left(\mathfrak{n}_a\right)|_{\mathfrak{n}_a(v_1)} = S_{\rm BH}(P_\alpha(v_1))$$

Concluding remarks

We have studied the $\mathcal{I}$-extremization and its geometric dual for $M^{1,1,1}$.

- Since there is a non-trivial two-cycle in $M^{1,1,1}$, baryonic symmetry is important.
- On the field theory side, we do not know how to include the effect of the baryonic flux to the index. However, on the gravity side, we only know the black holes with baryonic charges. Using the extremization principle, we can reproduce the index with mesonic flux and the entropy of the black hole with baryonic charge.
- We hope that the extremization principles give us some hints to resolve this puzzle.

There are many questions to be answered.

- Can we apply this method to inhomogeneous Sasaki-Einstein manifolds, for example, $Y^{p,k}\left(\mathbb{CP}^2\right)$?
- Dyonic black holes and the twisted indices are known. Do we incorporate the electric charges in the variational problem?
- chiral quiver, non-convex toric cones, $\cdots$

Thank you!!

Appendix

Master volume for $M^{1,1,1}$

In[300]:= **vol**

Out[300]=

$$(8 \pi^4 ((2 b1 - 2 b2 - 2 b3 - b4) (2 b1 + b2 - 2 b3 - b4) (2 b1 - 2 b2 + b3 - b4) \lambda_1 (3 (b1 - b2 - b3 - b4) \lambda_1 + (3 b2 + b4) \lambda_2 + 3 b3 \lambda_3 + b4 \lambda_3 + b4 \lambda_4)^2 + b4 (3 b2 + b4)$$
$$(3 b3 + b4) \lambda_5 ((-2 b1 - b2 + 2 b3 + b4) \lambda_2 + (-2 b1 + 2 b2 - b3 + b4) \lambda_3 - 2 b1 \lambda_4 + 2 b2 \lambda_4 + 2 b3 \lambda_4 + b4 \lambda_4 + 3 b1 \lambda_5 - 3 b2 \lambda_5 - 3 b3 \lambda_5 - 3 b4 \lambda_5)^2 +$$
$$(2 b1 - 2 b2 + b3 - b4) (3 b3 + b4) \lambda_3 \left(3 \left(4 b1^3 + 2 b2^3 + b2^2 b4 - (b3 + b4) (2 b3 + b4)^2 - 2 b1^2 (3 b2 + 6 b3 + 4 b4) - b2 \left(6 b3^2 + 7 b3 b4 + 2 b4^2\right) +\right.\right.$$
$$\left.\left. b1 \left(12 b2 b3 + 12 b3^2 + 7 b2 b4 + 16 b3 b4 + 5 b4^2\right)\right) \lambda_1^2 + 2 b2 (2 b1 + b2 - 2 b3 - b4) (3 b2 + b4) \lambda_2^2 + 12 b1 b3^2 \lambda_3^2 + 6 b2 b3^2 \lambda_3^2 -$$
$$12 b3^3 \lambda_3^2 - 4 b1 b2 b4 \lambda_3^2 + 4 b2^2 b4 \lambda_3^2 + 8 b1 b3 b4 \lambda_3^2 - 4 b2 b3 b4 \lambda_3^2 - 8 b3^2 b4 \lambda_3^2 + 2 b2 b4^2 \lambda_3^2 - 4 b3 b4^2 \lambda_3^2 - 8 b1 b2 b4 \lambda_3 \lambda_4 +$$
$$8 b2^2 b4 \lambda_3 \lambda_4 + 8 b1 b3 b4 \lambda_3 \lambda_4 - 8 b3^2 b4 \lambda_3 \lambda_4 + 4 b2 b4^2 \lambda_3 \lambda_4 - 4 b3 b4^2 \lambda_3 \lambda_4 - 4 b1 b2 b4 \lambda_4^2 + 4 b2^2 b4 \lambda_4^2 + 4 b2 b3 b4 \lambda_4^2 +$$
$$2 b2 b4^2 \lambda_4^2 + 2 \left(4 b1^2 - 2 b2^2 + b2 (2 b3 + b4) + (2 b3 + b4)^2 - 2 b1 (b2 + 4 b3 + 2 b4)\right) \lambda_1 ((3 b2 + b4) \lambda_2 + (3 b3 + b4) \lambda_3 + b4 \lambda_4) +$$
$$12 b1 b2 b4 \lambda_3 \lambda_5 - 12 b2^2 b4 \lambda_3 \lambda_5 + 6 b2 b3 b4 \lambda_3 \lambda_5 + 4 b1 b4^2 \lambda_3 \lambda_5 - 10 b2 b4^2 \lambda_3 \lambda_5 + 2 b3 b4^2 \lambda_3 \lambda_5 - 2 b4^3 \lambda_3 \lambda_5 + 12 b1 b2 b4 \lambda_4 \lambda_5 -$$
$$12 b2^2 b4 \lambda_4 \lambda_5 - 12 b2 b3 b4 \lambda_4 \lambda_5 + 4 b1 b4^2 \lambda_4 \lambda_5 - 10 b2 b4^2 \lambda_4 \lambda_5 - 4 b3 b4^2 \lambda_4 \lambda_5 - 2 b4^3 \lambda_4 \lambda_5 - 9 b1 b2 b4 \lambda_5^2 + 9 b2^2 b4 \lambda_5^2 +$$
$$9 b2 b3 b4 \lambda_5^2 - 3 b1 b4^2 \lambda_5^2 + 12 b2 b4^2 \lambda_5^2 + 3 b3 b4^2 \lambda_5^2 + 3 b4^3 \lambda_5^2 + 2 (2 b1 + b2 - 2 b3 - b4) (3 b2 + b4) \lambda_2 (2 b3 \lambda_3 + b4 \lambda_5)) +$$
$$(2 b1 - 2 b2 - 2 b3 - b4) b4 \lambda_4 \left(3 \left(4 b1^3 + 2 b2^3 + 2 b3^3 + b3^2 b4 - 2 b3 b4^2 - b4^3 + b2^2 (-3 b3 + b4) - 2 b1^2 (3 b2 + 3 b3 + 4 b4) -\right.\right.$$
$$\left.\left. b2 \left(3 b3^2 + 7 b3 b4 + 2 b4^2\right) + b1 \left(9 b2 b3 + 7 b2 b4 + 7 b3 b4 + 5 b4^2\right)\right) \lambda_1^2 + 2 (b2 - b3) (2 b1 + b2 - 2 b3 - b4) (3 b2 + b4) \lambda_2^2 -$$
$$12 b1 b2 b3 \lambda_3^2 + 12 b2^2 b3 \lambda_3^2 + 12 b1 b3^2 \lambda_3^2 - 18 b2 b3^2 \lambda_3^2 + 6 b3^3 \lambda_3^2 - 4 b1 b2 b4 \lambda_3^2 + 4 b2^2 b4 \lambda_3^2 + 4 b1 b3 b4 \lambda_3^2 - 4 b3^2 b4 \lambda_3^2 + 2 b2 b4^2 \lambda_3^2 -$$
$$2 b3 b4^2 \lambda_3^2 - 24 b1 b2 b3 \lambda_3 \lambda_4 + 24 b2^2 b3 \lambda_3 \lambda_4 - 12 b2 b3^2 \lambda_3 \lambda_4 - 8 b1 b2 b4 \lambda_3 \lambda_4 + 8 b2^2 b4 \lambda_3 \lambda_4 + 8 b2 b3 b4 \lambda_3 \lambda_4 + 4 b2 b4^2 \lambda_3 \lambda_4 -$$
$$12 b1 b2 b3 \lambda_4^2 + 12 b2^2 b3 \lambda_4^2 + 12 b2 b3^2 \lambda_4^2 - 4 b1 b2 b4 \lambda_4^2 + 4 b2^2 b4 \lambda_4^2 - 4 b1 b3 b4 \lambda_4^2 + 8 b2 b3 b4 \lambda_4^2 + 4 b3^2 b4 \lambda_4^2 + 2 b2 b4^2 \lambda_4^2 +$$
$$2 b3 b4^2 \lambda_4^2 + 2 \left(4 b1^2 - 2 b2^2 - 2 b3^2 + b3 b4 + b4^2 + b2 (5 b3 + b4) - 2 b1 (b2 + b3 + 2 b4)\right) \lambda_1 ((3 b2 + b4) \lambda_2 + (3 b3 + b4) \lambda_3 + b4 \lambda_4) +$$
$$36 b1 b2 b3 \lambda_3 \lambda_5 - 36 b2^2 b3 \lambda_3 \lambda_5 + 18 b2 b3^2 \lambda_3 \lambda_5 + 12 b1 b2 b4 \lambda_3 \lambda_5 - 12 b2^2 b4 \lambda_3 \lambda_5 + 12 b1 b3 b4 \lambda_3 \lambda_5 - 24 b2 b3 b4 \lambda_3 \lambda_5 +$$
$$6 b3^2 b4 \lambda_3 \lambda_5 + 4 b1 b4^2 \lambda_3 \lambda_5 - 10 b2 b4^2 \lambda_3 \lambda_5 - 4 b3 b4^2 \lambda_3 \lambda_5 - 2 b4^3 \lambda_3 \lambda_5 + 36 b1 b2 b3 \lambda_4 \lambda_5 - 36 b2^2 b3 \lambda_4 \lambda_5 - 36 b2 b3^2 \lambda_4 \lambda_5 +$$
$$12 b1 b2 b4 \lambda_4 \lambda_5 - 12 b2^2 b4 \lambda_4 \lambda_5 + 12 b1 b3 b4 \lambda_4 \lambda_5 - 42 b2 b3 b4 \lambda_4 \lambda_5 - 12 b3^2 b4 \lambda_4 \lambda_5 + 4 b1 b4^2 \lambda_4 \lambda_5 - 10 b2 b4^2 \lambda_4 \lambda_5 -$$
$$10 b3 b4^2 \lambda_4 \lambda_5 - 2 b4^3 \lambda_4 \lambda_5 - 27 b1 b2 b3 \lambda_5^2 + 27 b2^2 b3 \lambda_5^2 + 27 b2 b3^2 \lambda_5^2 - 9 b1 b2 b4 \lambda_5^2 + 9 b2^2 b4 \lambda_5^2 - 9 b1 b3 b4 \lambda_5^2 + 45 b2 b3 b4 \lambda_5^2 +$$
$$9 b3^2 b4 \lambda_5^2 - 3 b1 b4^2 \lambda_5^2 + 12 b2 b4^2 \lambda_5^2 + 12 b3 b4^2 \lambda_5^2 + 3 b4^3 \lambda_5^2 - 2 (2 b1 + b2 - 2 b3 - b4) (3 b2 + b4) \lambda_2 (2 b3 \lambda_4 - (3 b3 + b4) \lambda_5)) +$$
$$(2 b1 + b2 - 2 b3 - b4) (3 b2 + b4) \lambda_2 \left(3 \left(4 b1^3 - 4 b2^3 + 2 b3^3 + b3^2 b4 - 2 b3 b4^2 - b4^3 - 2 b2^2 (3 b3 + 4 b4) -\right.\right.$$
$$\left.\left. 2 b1^2 (6 b2 + 3 b3 + 4 b4) - b2 b4 (7 b3 + 5 b4) + b1 \left(12 b2^2 + 4 b2 (3 b3 + 4 b4) + b4 (7 b3 + 5 b4)\right)\right) \lambda_1^2 +$$
$$2 \left(-6 b2^3 + b2^2 (3 b3 - 4 b4) - 2 b2 b4 (b3 + b4) + b3 b4 (2 b3 + b4) + b1 \left(6 b2^2 + 4 b2 b4 - 2 b3 b4\right)\right) \lambda_2^2 + 12 b1 b3^2 \lambda_3^2 - 12 b2 b3^2 \lambda_3^2 +$$
$$6 b3^3 \lambda_3^2 + 4 b1 b3 b4 \lambda_3^2 - 4 b2 b3 b4 \lambda_3^2 - 4 b3^2 b4 \lambda_3^2 - 2 b3 b4^2 \lambda_3^2 - 4 b1 b3 b4 \lambda_4^2 + 4 b2 b3 b4 \lambda_4^2 + 4 b3^2 b4 \lambda_4^2 + 2 b3 b4^2 \lambda_4^2 +$$
$$2 \left(4 b1^2 + 4 b2^2 - 2 b3^2 + b3 b4 + b4^2 + 2 b2 (b3 + 2 b4) - 2 b1 (4 b2 + b3 + 2 b4)\right) \lambda_1 ((3 b2 + b4) \lambda_2 + (3 b3 + b4) \lambda_3 + b4 \lambda_4) + 12 b1 b3 b4 \lambda_3 \lambda_5 -$$
$$12 b2 b3 b4 \lambda_3 \lambda_5 + 6 b3^2 b4 \lambda_3 \lambda_5 + 4 b1 b4^2 \lambda_3 \lambda_5 - 4 b2 b4^2 \lambda_3 \lambda_5 - 4 b3 b4^2 \lambda_3 \lambda_5 - 2 b4^3 \lambda_3 \lambda_5 + 12 b1 b3 b4 \lambda_4 \lambda_5 - 12 b2 b3 b4 \lambda_4 \lambda_5 - 12 b3^2 b4 \lambda_4 \lambda_5 +$$
$$4 b1 b4^2 \lambda_4 \lambda_5 - 4 b2 b4^2 \lambda_4 \lambda_5 - 10 b3 b4^2 \lambda_4 \lambda_5 - 2 b4^3 \lambda_4 \lambda_5 - 9 b1 b3 b4 \lambda_5^2 + 9 b2 b3 b4 \lambda_5^2 + 9 b3^2 b4 \lambda_5^2 - 3 b1 b4^2 \lambda_5^2 + 3 b2 b4^2 \lambda_5^2 + 12 b3 b4^2 \lambda_5^2 +$$
$$3 b4^3 \lambda_5^2 + 2 \lambda_2 (2 b2 (2 b1 - 2 b2 + b3 - b4) (3 b3 + b4) \lambda_3 - 2 (b2 - b3) b4 (-2 b1 + 2 b2 + 2 b3 + b4) \lambda_4 + (2 b1 + b2 - 2 b3 - b4) b4 (3 b3 + b4) \lambda_5)))) /$$
$$(3 (2 b1 + b2 - 2 b3 - b4) (2 b1 - 2 b2 + b3 - b4) b4 (3 b2 + b4) (-2 b1 + 2 b2 + 2 b3 + b4)$$
$$(3 b3 + b4))$$

◂ return