Sign Flip Triangulation of the Amplituhedron

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Based on

RK, JHEP 1904 (2019) 085 arxiv:1812.01822 [hep-th]

RK, Cameron Langer, Jaroslav Trnka and Minshan Zheng(UC Davis, QMAP), in progress

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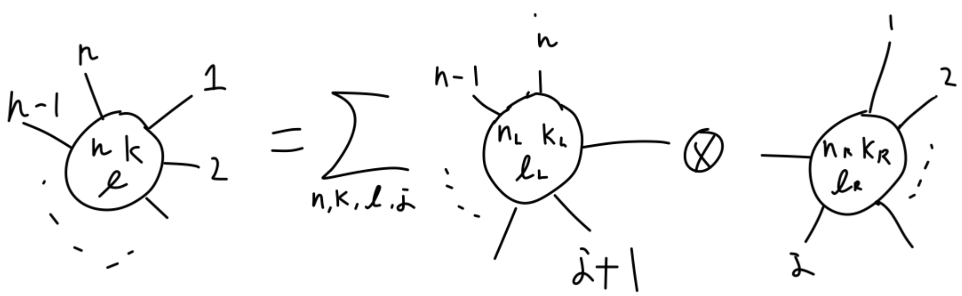
Introduction

- Scattering amplitude: Basic objects in QFT.
- Recently there are many progress about the understanding of the new structure of scattering amplitude.
- Studying and calculating scattering amplitudes became a new direction in theoretical physics.
- Major motivations:
    - Efficient calculations of the scattering amplitudes
    - Use amplitudes as a probe to explore quantum field theory

Introduction

- Efficient calculations of the scattering amplitudes
- Use amplitudes as a probe to explore quantum field theory

- Example: BCFW recursion relation (Britto, Cachazo, Feng, Witten 2005)



- Method for building higher-point amplitudes from lower-point

	$g+g\to 4g,$	$g+g\to 5g,$	$g+g\to 6g$
Feynman diagrams	220	2485	34300
BCFW	3	6	20

- It is possible to calculate all order amplitudes in Planar N=4 SYM from BCFW. (Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka 2010)

Introduction

- Efficient calculations of the scattering amplitudes
- Use amplitudes as a probe to explore quantum field theory

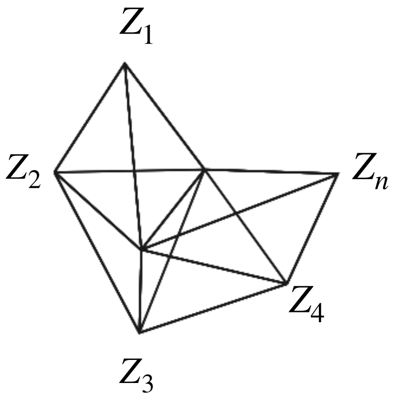
- Geometrization of scattering amplitudes

"Amplitude = Volume of a geometric object"

Example; Planar N=4 SYM

The amplituhedron $\mathcal{A}_{n,m,k}(Z)$ (N. Arkani-Hamed, J. Trnka 2013)

- k-plane in k+m dimensional space
- Defined as a generalization of convex polytope.
- m=4 ; physical amplituhedron $\mathcal{A}_{n,k}(Z)$



Introduction

- For each amplituhedron, we can define a volume form;

Canonical Form Ω

Ω has logarithmic singularities at all boundaries of $\mathcal{A}_{n,k}(Z)$

Canonical form Ω of $\mathcal{A}_{n,k}(Z)$ $\leftrightarrow$ n-point N^kMHV amplitude in Momentum twistor space Z

$$Z_i = (\lambda_i^a, y_i^{a\dot{a}}\lambda^a)$$

$$p_i^\mu \sigma_\mu^{a\dot{a}} = p_i^{a\dot{a}} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$$

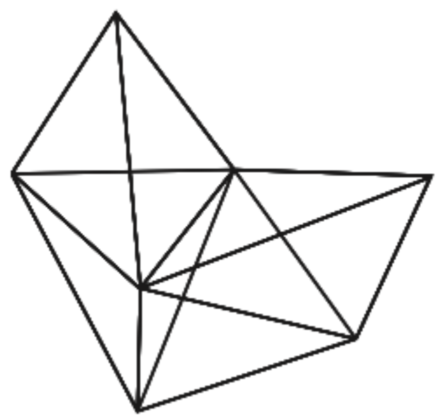
$$y_i^{\dot{a}a} - y_{i+1}^{\dot{a}a} = p_i^{\dot{a}a}$$

N^kMHV → amplitude with K+2 negative helicity particles

- Calculate the scattering amplitude = Construct the volume form of the amplituhedron: this is purely geometrical !

Introduction

Geometrization of scattering amplitudes



Amplituhedron

$\longrightarrow$ Ω

Canonical form

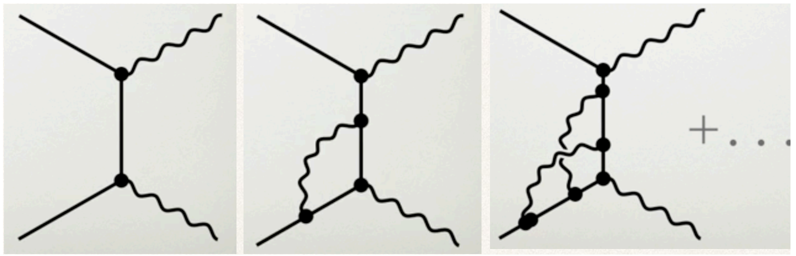
$\searrow$

$$A_n$$

Scattering Amplitude

$\mathscr{L}$ $\longrightarrow$

Lagrangian



Feynman diagram

$\nearrow$

Introduction

- Efficient calculations of the scattering amplitudes

  ⟶ BCFW recursion relation

- There are another recursion relations

  Local representation, Momentum twistor diagram···

Question

- What is the relation between these recursion relations?

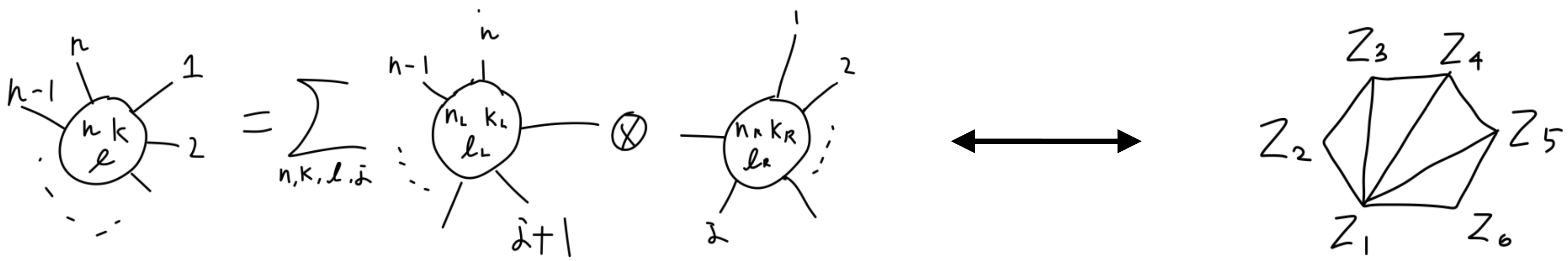
- Can we obtain another recursion relation?

↓

We use the geometrization of scattering amplitudes

Motivation

BCFW Recursion relation ⟷ One of the triangulation of the amplituhedron

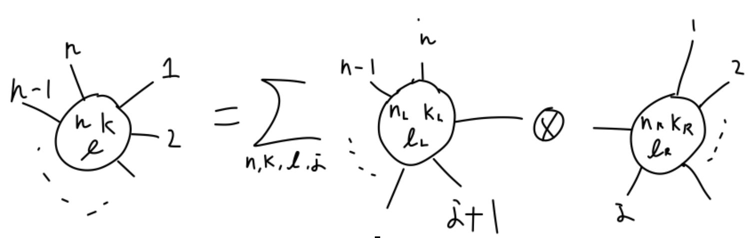


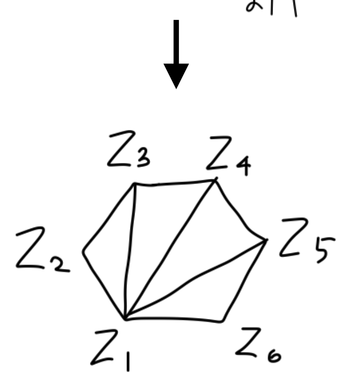
Of cause there are various way of the triangulation

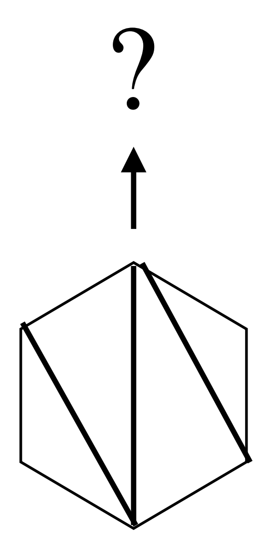
- Question:

Is it possible to obtain new recursion relations from the triangulation of the amplituhedron?

BCFW







- In this work, we found new recursion relations of the 2-loop MHV, 1-loop NMHV amplitude.

- [x] Introduction
- [ ] Review of the amplituhedron and sign flip
- [ ] New recursion relation of 2-loop MHV and 1-loop NMHV
- [ ] Summary

Review of the amplituhedron and sign flip

The definition of the amplituhedron $\mathcal{A}_{n,m,k}(Z)$

Y_a^I : k-plane in k+m dimensional space

$$Y_a^I = \sum_{i=1}^{n} c_{ai} Z_i^I \qquad a = 1, \cdots, k \qquad I = 1, \cdots, k+m$$

- $c_{ai} \in G_+(k,n)$; Positive Grassmannian: $k \times n$ matrix mod $GL(k)$

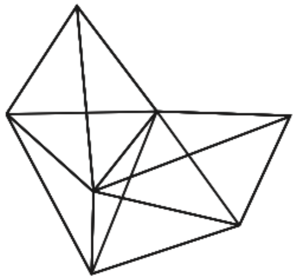
$$\langle c_{a_1} \cdots c_{a_k} \rangle > 0 \qquad a_1 < \cdots < a_k$$

- $Z_i^I \in M_+(k+m, n)$; Positive Matrix: $(k+m) \times n$ matrix

$$\langle Z_{a_1}, \cdots, Z_{a_n} \rangle > 0 \qquad a_1 < \cdots < a_n$$

- This space is the generalization of the interior of the convex polytope.

Convex polytope



$$Y^I = \sum_{i=1}^{n} c_i Z_i^I$$

$I = 1, \cdots, m+1$

$\langle Z_{i_1} Z_{i_2} \cdots Z_{i_{m+1}} \rangle > 0 \qquad c_i > 0$

generalize →

Amplituhedron $\mathcal{A}_{n,m,k}(Z)$

$$Y_a^I = \sum_{i=1}^{n} c_{ai} Z_i^I$$

$I = 1, \cdots, k+m \qquad a = 1, \cdots, k$

$\langle Z_{i_1} \cdots Z_{i_{k+m}} \rangle > 0 \qquad \langle c_{i_1} \cdots c_{i_k} \rangle > 0$

Review of the amplituhedron and sign flip

- Canonical form Ω : logarithmic singularities at all boundaries of $\mathcal{A}_{n,k}(Z)$
  - The simplest case: $\mathcal{A}_{m=4,k=1,n=5}$
  - Boundary $\langle Y1234\rangle$, $\langle Y2345\rangle$, $\langle Y3451\rangle$, $\langle Y4512\rangle$, $\langle Y5123\rangle$,
  - The canonical form is

$$\Omega = d\log\frac{\langle Y1234\rangle}{\langle Y5123\rangle} d\log\frac{\langle Y2345\rangle}{\langle Y5123\rangle} d\log\frac{\langle Y3451\rangle}{\langle Y5123\rangle} d\log\frac{\langle Y4512\rangle}{\langle Y5123\rangle}$$
$$= \frac{\langle Y dY dY dY dY\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

$$\langle Yijkl\rangle = \epsilon_{MIJKL} Y^M Z_i^I Z_j^J Z_k^K Z_l^L$$

- Let us rewrite Z_i as four-dimensional part and its complement

$$Z_i = \begin{pmatrix} z_i \\ \delta z_i \end{pmatrix} \quad \delta z_i = \eta_i \cdot \phi$$

$$\int d^4\phi \int \delta(Y - Y^*)\Omega = \frac{(\langle 1234\rangle\eta_5 + \langle 2345\rangle\eta_1 + \cdots + \langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

$$Y^* = (0,0,0,0,1)$$

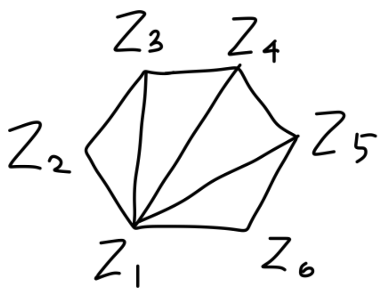
η, ϕ : auxiliary grassmann variables

⟶ 5pt tree amplitude

Review of the amplituhedron and sign flip

- To obtain the recursion relation, we need to triangulate the amplituhdron

- Example: Polygon



$$\Omega = \frac{\langle Y d^2 Y \rangle \langle 123 \rangle^2}{\langle Y 12 \rangle \langle Y 23 \rangle \langle Y 31 \rangle} + \frac{\langle Y d^2 Y \rangle \langle 134 \rangle^2}{\langle Y 13 \rangle \langle Y 34 \rangle \langle Y 41 \rangle} + \frac{\langle Y d^2 Y \rangle \langle 145 \rangle^2}{\langle Y 14 \rangle \langle Y 45 \rangle \langle Y 51 \rangle} + \frac{\langle Y d^2 Y \rangle \langle 156 \rangle^2}{\langle Y 15 \rangle \langle Y 56 \rangle \langle Y 61 \rangle}$$

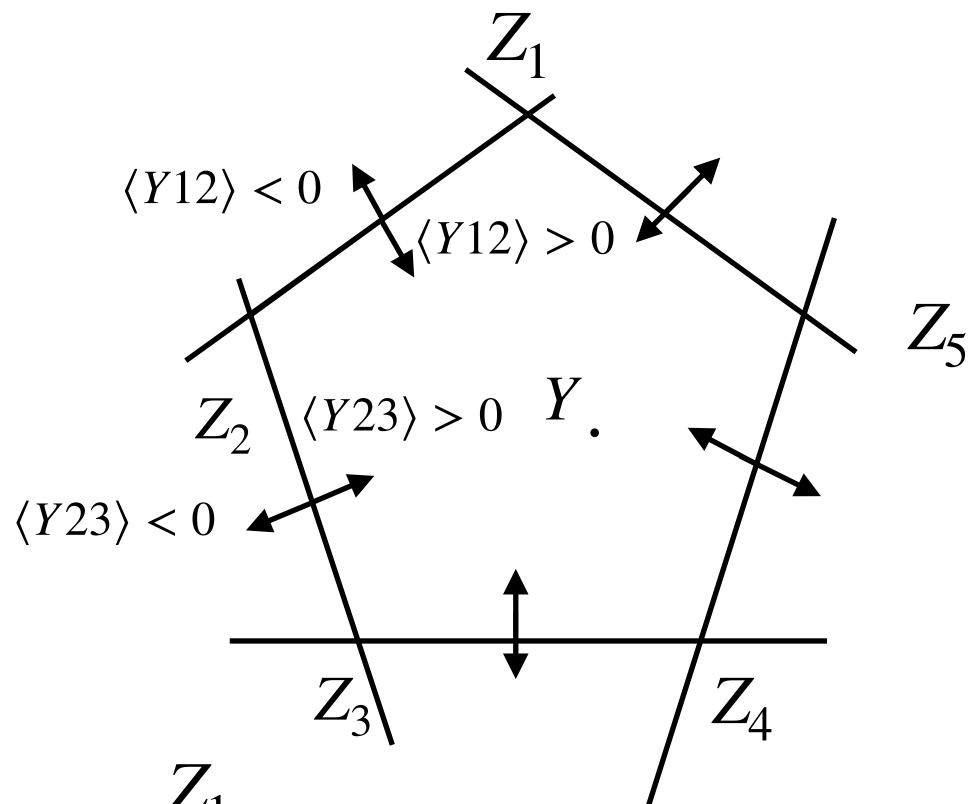
- More general case, it is difficult to triangulate.

- How to triangulate more general amplituhedron systematically?

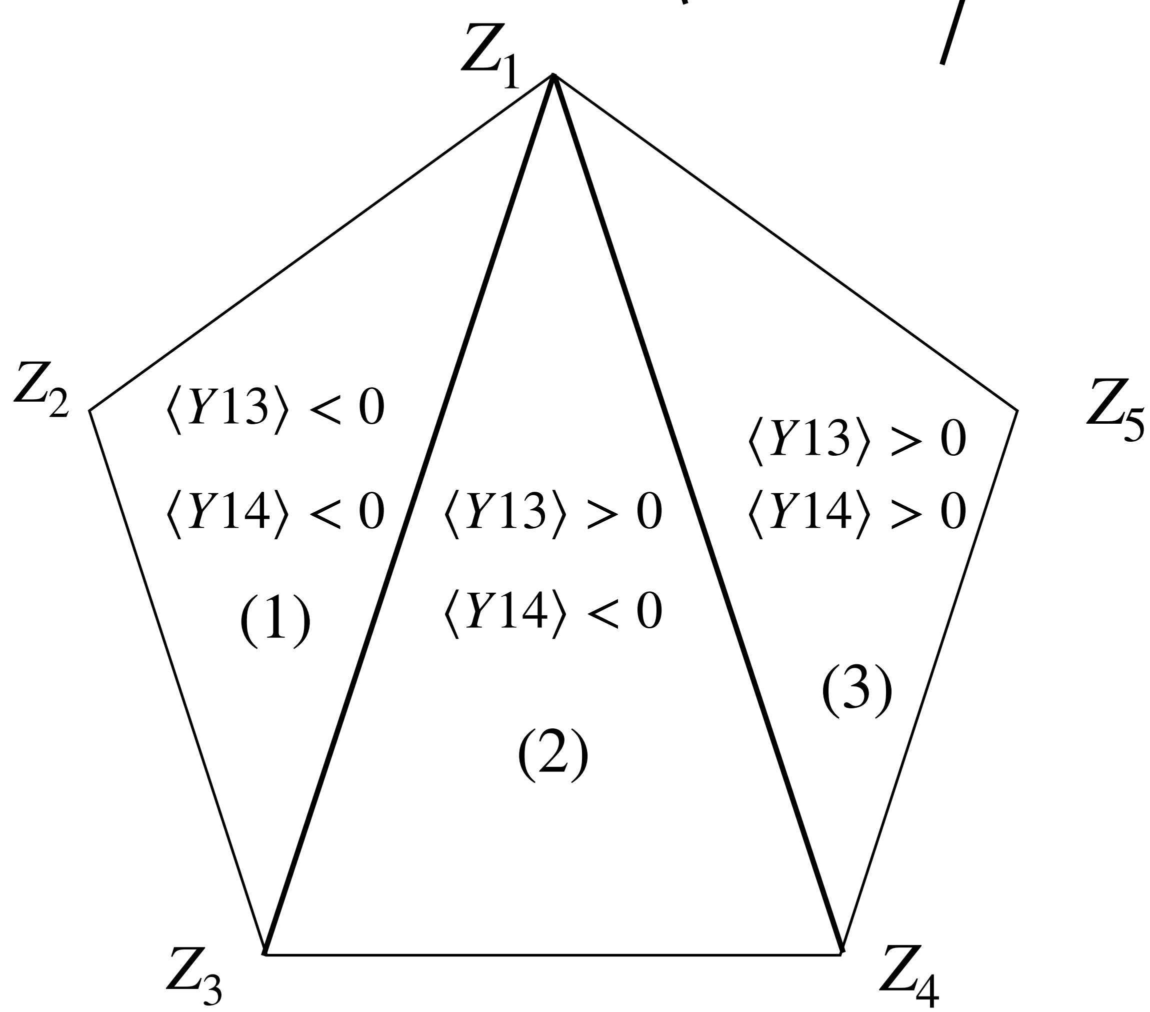
Sign flip triangulation

Review of the amplituhedron and sign flip

- $\langle Y12\rangle > 0, \langle Y23\rangle > 0, \langle Y34\rangle > 0, \langle Y45\rangle > 0, \langle Y51\rangle > 0$; Interior of this polygon



- Triangulate

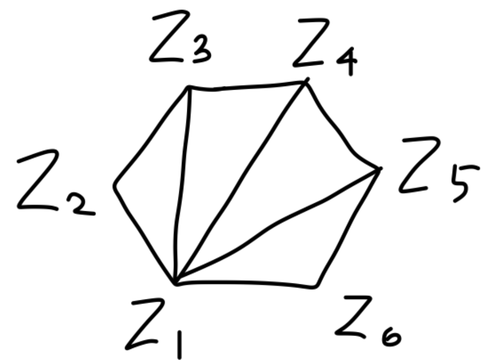


Each cell is specifies by the signs of $\langle Y1i\rangle$

$\langle Y12\rangle$	$\langle Y13\rangle$	$\langle Y14\rangle$	$\langle Y15\rangle$	
+	-	-	-	(1)
+	+	-	-	(2)
+	+	+	-	(3)

Review of the amplituhedron and sign flip

- m=2 k=1 n=6 tree amplituhedron: 4 patterns of signs



[Y12]	[Y13]	[Y14]	[Y15]	[Y16]
+	-	-	-	-
+	+	-	-	-
+	+	+	-	-
+	+	+	+	-

1 sign flip

- Generally,

> Each cell of the m=2 k,n amplituhedron is specified as
> $\langle Yii+1\rangle > 0$ and $\{\langle Y12\rangle, \cdots, \langle Y1n\rangle\}$has precisely k sign flips.

- Example; m=2 k=2 n=6

$\langle Y12\rangle$	$\langle Y13\rangle$	$\langle Y14\rangle$	$\langle Y15\rangle$	$\langle Y16\rangle$
+	-	+	+	+
+	-	-	+	+
+	-	-	-	+
+	+	-	+	+
+	+	-	-	+
+	+	+	-	+

→ 6 cells

- Sign flip triangulation of the amplituhedron

(N. Arkani-Hamed, H.Thomas, J. Trnka 2017)

Review of the amplituhedron and sign flip

- From each pattern, we can construct the canonical form.

- Sign flip takes place in i_1 , $\quad Y = Z_1 + xZ_{i_1} + yZ_{i_1+1}$

$$\Omega_{i_1} = \frac{dx}{x}\frac{dy}{y} = \frac{\langle Y d^2 Y\rangle\langle 1 i_1 i_1+1\rangle}{\langle Y 1 i_1\rangle\langle Y 1 i_1+1\rangle\langle Y i_1 i_1+1\rangle}$$

- The canonical form for this amplituhedron is

$$\Omega = \sum_{\text{all flips}} \Omega_{ij}$$

- Only m=1,2 case, we can triangulate from sign flip.

- m=4 physical amplituhedron or loop amplituhedron, there isn't simple relation between a particular cell and one of the flip pattern.

- [x] Introduction
- [x] Review of the amplituhedron and sign flip
- [ ] New recursion relation of 2-loop MHV and 1-loop NMHV
- [ ] Summary

New recursion relation of 2-loop MHV and 1-loop NMHV

- Only m=1,2 case, we can triangulate from sign flip.

- 1-loop MHV amplituhedron is isomorphic to the m=2 k=2 amplituhedron

$$\mathcal{A}(m=4,k=0,n;l=1) \simeq \mathcal{A}(m=2,k=2,n;l=0)$$

- From this, we can triangulate the 1-loop MHV amplituhedron from sign flips!

- 1-loop MHV from sign flips

$$\Omega_P = \sum_{i<j} \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(1ii+1)\cap(1jj+1)\rangle^2}{\langle AB1i\rangle\langle AB1i+1\rangle\langle ABii+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABjj+1\rangle}$$

- This is corresponding to the BCFW recursion relation.

New recursion relation of 2-loop MHV and 1-loop NMHV

- How about more general case? ⟶ 2-loop MHV amplituhedron

- 2-loop MHV from sign flips (RK 2018)

$$\Omega_{\text{MHV}}^{n\text{-pt 2-loop}} = \sum_{\substack{i,j,k,l=2,3,\cdots,n-1 \\ i<k<l<j}} \Omega_{ijkl}^{1} + \sum_{i<k<j<l} \Omega_{ijkl}^{2} + \sum_{i<j<k<l} \Omega_{ijkl}^{3} + \cdots + \sum_{k<l<i<j} \Omega_{ijkl}^{13}$$

$$\Omega_{ijkl}^{1} = \frac{\langle 1ii+1A_i\rangle\langle 1jj+1C_l\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB1i\rangle\langle AB1i+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABCD\rangle\langle CD1k\rangle\langle CD1k+1\rangle\langle CD1l\rangle\langle CD1l+1\rangle} \times \frac{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle + \langle A_iA_jC_kC_l\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}$$

$$\begin{aligned}
\langle ijkA_l\rangle &\equiv \langle AB1l+1\rangle\langle ijkl\rangle - \langle AB1l\rangle\langle ijkl+1\rangle = \langle ijk(AB)\cap(1ll+1)\rangle \\
\langle ijA_lC_k\rangle &\equiv \langle CD1k+1\rangle\langle ijA_lk\rangle - \langle CD1k\rangle\langle ijA_lk+1\rangle \\
\langle iA_lC_kE_j\rangle &\equiv \langle EF1j+1\rangle\langle iA_lC_kj\rangle - \langle EF1j\rangle\langle iA_lC_kj+1\rangle \\
\langle A_lC_kE_jG_i\rangle &\equiv \langle GH1i+1\rangle\langle A_lC_kE_ji\rangle - \langle GH1i\rangle\langle A_lC_kE_ji+1\rangle.
\end{aligned}$$

- This is not BCFW recursion relation → New recursion relation.

	4pt	5pt	6pt	7pt
BCFW	8	28	68	138
Sign flip	1	9	36	100

  - It is not obvious how to derive it from a quantum field theory argument.

New recursion relation of 2-loop MHV and 1-loop NMHV

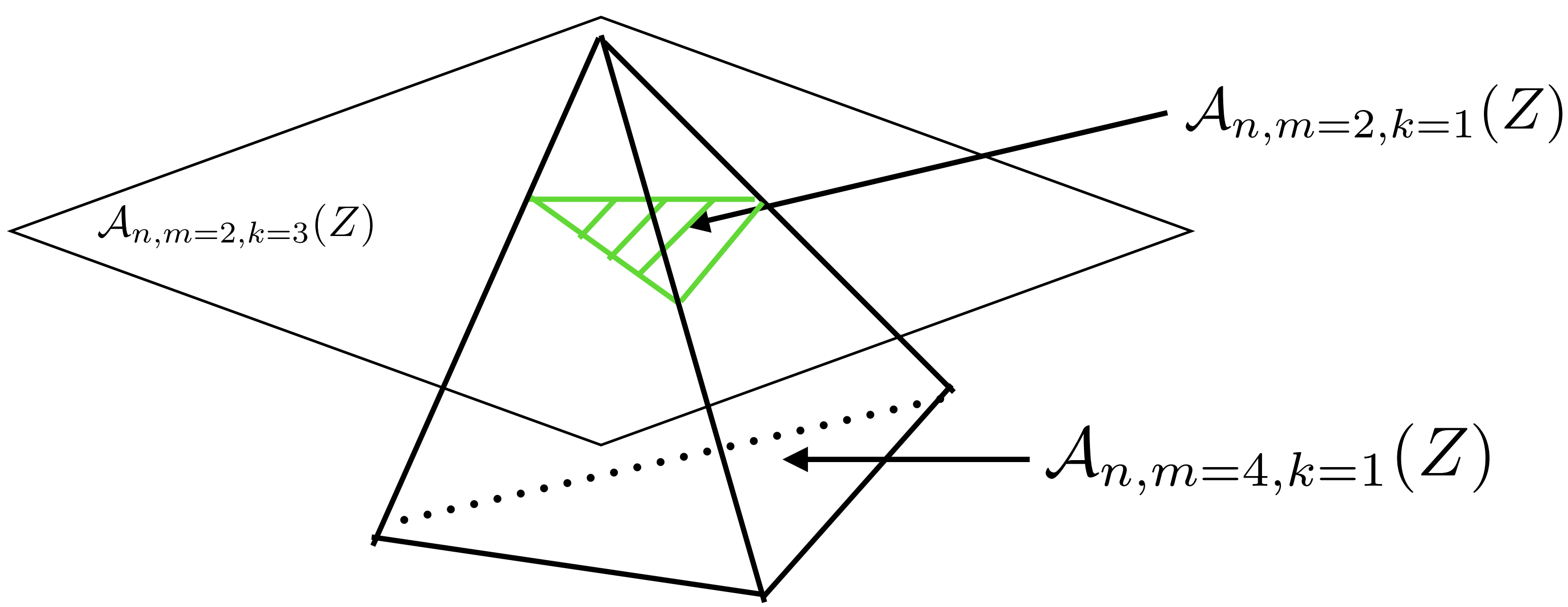
- The 1-loop NMHV case (RK, C. Langer, M. Zheng, J. Trnka, in progress)

$$\mathcal{A}_{n,l=1,k=1}(Z) = \mathcal{A}_{n,m=4,k=1}(Z) \times \mathcal{A}_{n,l=1,k=0}(Z)$$

- However, this amplituhedron is also constructed from m=2, k=3 tree amplituhedron and convex polygon (N. Arkani-Hamed, H.Thomas, J. Trnka 2017)

$$\mathcal{A}_{n,l=1,k=1}(Z) = \mathcal{A}_{n,m=2,k=3}(Z) \times \mathcal{A}_{n,m=2,k=1}(Z)$$

- Polygon is the intersection with the m=4 k=1 tree amplituhedron and $\mathcal{A}_{n,m=2,k=3}(Z)$

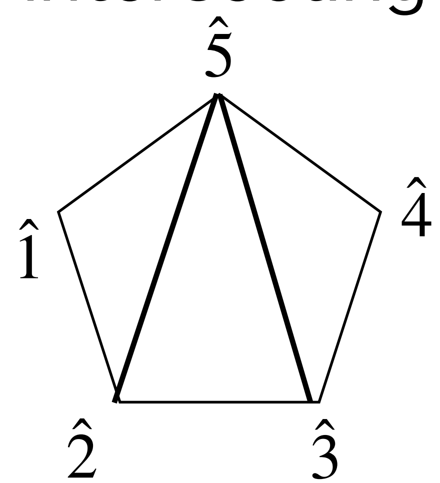


New recursion relation of 2-loop MHV and 1-loop NMHV

- 5-pt case
- m=2 k=3 n=5 amplituhedron

$$\Omega_6 = \frac{\langle 12345\rangle^2 \langle YABd^2Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB51\rangle}$$

- The intersecting polygon is



$$\begin{aligned}\hat{i} &= (YAB)\cap(i-1ii+1)\\ &= Z_A\langle YBi-1ii+1\rangle + Z_B\langle YAi-1ii+1\rangle\end{aligned}$$

$$\Omega_2 = \langle yd^2y\rangle \times \left(\frac{\langle\hat{5}\hat{1}\hat{2}\rangle^2}{\langle y\hat{5}\hat{1}\rangle\langle y\hat{1}\hat{2}\rangle\langle y\hat{5}\hat{2}\rangle} + \frac{\langle\hat{5}\hat{2}\hat{3}\rangle^2}{\langle y\hat{5}\hat{2}\rangle\langle y\hat{2}\hat{3}\rangle\langle y\hat{5}\hat{3}\rangle} + \frac{\langle\hat{5}\hat{3}\hat{4}\rangle^2}{\langle y\hat{5}\hat{3}\rangle\langle y\hat{3}\hat{4}\rangle\langle y\hat{5}\hat{4}\rangle}\right)$$

- The full form is

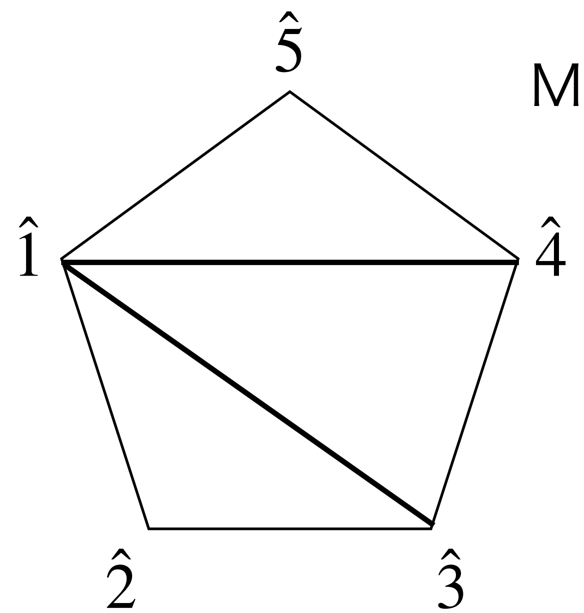
$$\begin{aligned}\Omega_6\times\Omega_2 &= \frac{\langle 12345\rangle^2 \langle YABd^2Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB51\rangle}\\ &\times\langle yd^2y\rangle\left(\frac{\langle\hat{5}\hat{1}\hat{2}\rangle^2}{\langle y\hat{5}\hat{1}\rangle\langle y\hat{1}\hat{2}\rangle\langle y\hat{5}\hat{2}\rangle} + \frac{\langle\hat{5}\hat{2}\hat{3}\rangle^2}{\langle y\hat{5}\hat{2}\rangle\langle y\hat{2}\hat{3}\rangle\langle y\hat{5}\hat{3}\rangle} + \frac{\langle\hat{5}\hat{3}\hat{4}\rangle^2}{\langle y\hat{5}\hat{3}\rangle\langle y\hat{3}\hat{4}\rangle\langle y\hat{5}\hat{4}\rangle}\right)\end{aligned}$$

- This is corresponding to the BCFW representation.

New recursion relation of 2-loop MHV and 1-loop NMHV

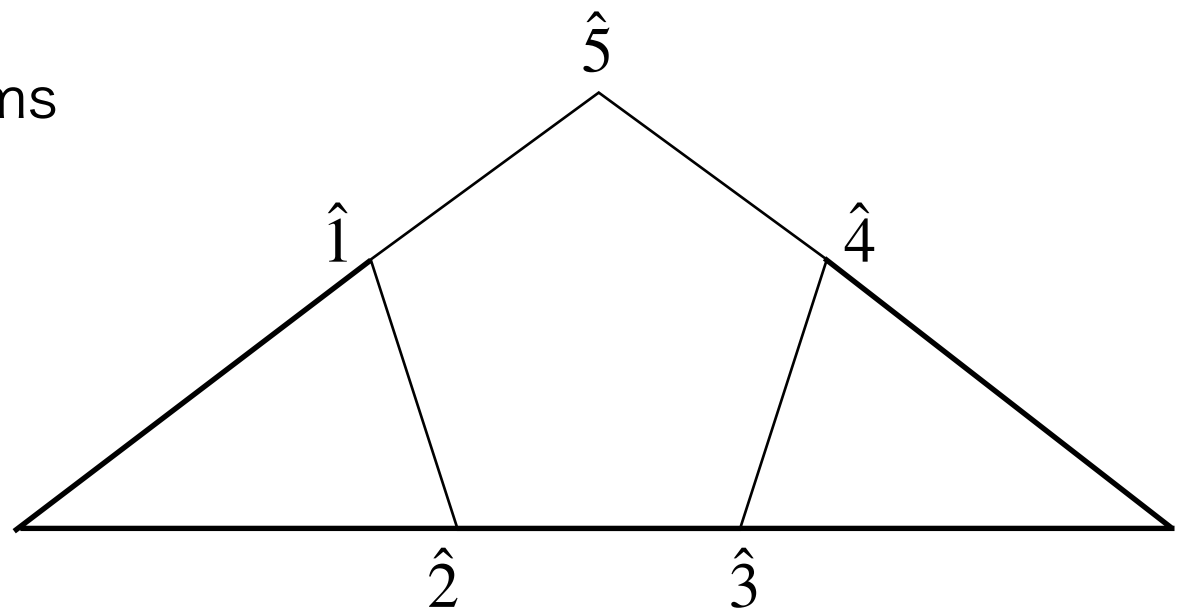
- In this 1-loop NMHV case, there are another recursion relations.

- We found that these recursion relations are corresponding to the another triangulations of the pentagon.



Recursion relation from Momentum twistor diagrams

(Y. Bai, S. He, T. Lam 2015)



Local representation

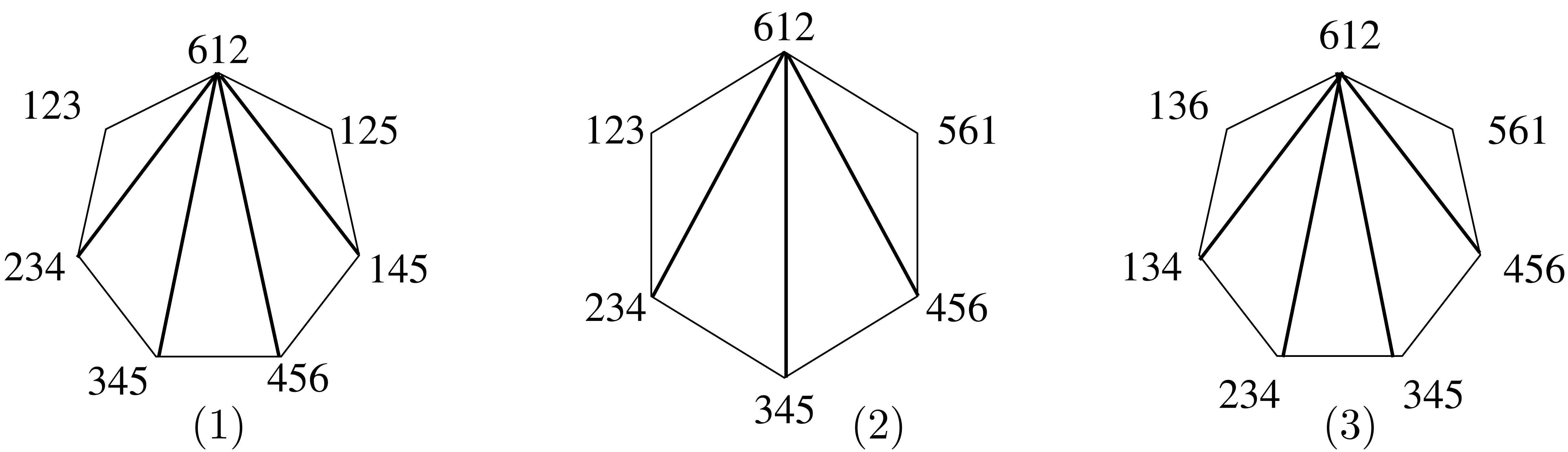
(N. Arkani-Hamed, J. Bourjaily, F. Cachazo, and J. Trnka 2010)

New recursion relation of 2-loop MHV and 1-loop NMHV

- 6-pt case
- m=2 k=3 n=6 amplituhedron ⋯ 4 cells

$$\Omega_6 = \Omega_6^{234} + \Omega_6^{235} + \Omega_6^{245} + \Omega_6^{345}$$

- For each cell, there is a pentagon



$$\Omega_6 = \Omega_6^{234} \times \Omega_2^{(1)} + (\Omega_6^{235} + \Omega_6^{245}) \times \Omega_2^{(2)} + \Omega_6^{345} \times \Omega_2^{(3)}$$

- This is not the BCFW recursion relation !

New recursion relation of 2-loop MHV and 1-loop NMHV

- In general

$$\Omega_n = \sum_{2\le i<j<k\le n} \Omega^6_{ijk} \times \Omega_2 + \sum_{1\le i<j<k<j<m\le n} \Omega^6_{ijklm} \times \Omega_2^{(ijk)(jkl)(klm)}$$

- where

$$\Omega^6_{ijk} = \langle YABd^2Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle$$
$$\times \frac{\begin{vmatrix} \langle YA1ii+1\rangle & \langle YA1jj+1\rangle & \langle YA1kk+1\rangle \\ \langle AB1ii+1\rangle & \langle AB1jj+1\rangle & \langle AB1kk+1\rangle \\ \langle BY1ii+1\rangle & \langle BY1jj+1\rangle & \langle BY1kk+1\rangle \end{vmatrix}^2}{\begin{matrix}\langle YAB1i\rangle\langle YAB1i{+}1\rangle\langle YABii{+}1\rangle\langle YAB1j\rangle\langle YAB1j{+}1\rangle\langle YABjj{+}1\rangle \\ \langle YAB1k\rangle\langle YAB1k{+}1\rangle\langle YABkk{+}1\rangle\end{matrix}}$$

$$\Omega^6_{abcde} = \frac{\langle YABd^2Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle\langle abcde\rangle^2}{\langle YABab\rangle\langle YABbc\rangle\langle YABcd\rangle\langle YABde\rangle\langle YABea\rangle}$$

$$\Omega_2^{(a)(b)(c)} = \frac{\langle abc\rangle^2\langle yd^2y\rangle}{\langle yab\rangle\langle ybc\rangle\langle yca\rangle}$$

- This is the new recursion relation of n-pt 1-loop NMHV amplitude.

Summary

- The amplituhedron, generalizing the interior of convex polygon, gives a complete definition of scattering amplitudes in planar N=4 SYM.

- From sign flip triangulation, we obtained new recursion relations for

  - 2-loop MHV amplitude

$$\Omega^{n\text{-pt 2-loop}}_{\text{MHV}} = \sum_{\substack{i,j,k,l=2,3,\cdots,n-1 \\ i<k<l<j}} \Omega^1_{ijkl} + \sum_{i<k<j<l} \Omega^2_{ijkl} + \sum_{i<j<k<l} \Omega^3_{ijkl} + \cdots + \sum_{k<l<i<j} \Omega^{13}_{ijkl}$$

  - 1-loop NMHV amplitude

$$\Omega_n = \sum_{2\leq i<j<k\leq n} \Omega^6_{ijk} \times \Omega_2 + \sum_{1\leq i<j<k<j<m\leq n} \Omega^6_{ijklm} \times \Omega_2^{(ijk)(jkl)(klm)}$$

  - Various recursion relations of the 1-loop NMHV are understood as various triangulations of the polygon.

- It is not obvious how to derive it from a quantum field theory argument.

Summary

- It is possible to generalize this result.

- 3-loop MHV: There are three 1-loop amplituhedron

$$\mathcal{A}(m=4,k=0,n;l=1) \simeq \mathcal{A}(m=2,k=2,n;l=0)$$

- 1-loop N^{k}MHV

$$\mathcal{A}_{n,l=1,k}(Z) = \mathcal{A}_{n,m=2,k+2}(Z) \times \mathcal{A}_{n,m=2,k}(Z)$$

Summary

- It is possible to generalize this result.

- 3-loop MHV: There are three 1-loop amplituhedron

$$\mathcal{A}(m=4,k=0,n;l=1) \simeq \mathcal{A}(m=2,k=2,n;l=0)$$

- 1-loop N^{k}MHV

$$\mathcal{A}_{n,l=1,k}(Z) = \mathcal{A}_{n,m=2,k+2}(Z) \times \mathcal{A}_{n,m=2,k}(Z)$$

Thank you