The SL(K+3,C) Symmetry of the Bosonic String Scattering Amplitudes

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Linear Relations of SSA in the Hard Scattering Limit

- Symmetry:
  - Gauge theory: “Symmetry dictates interaction”
  - String theory:

    self-consistency condition → interactions $\overset{?}{\rightarrow}$ symmetry

- QFT: Spontaneously broken symmetries are:
  - Hidden at low-energy
  - Evident at high-energy

- String: Infinite symmetries are suggested by
  - Soft UV of hard string scatterings (exponential fall-off)
  - Field theory hard scatterings (power law divergent)
  - Infinite states without free parameters → Huge Symmetry Group or Relations among SSA !?

Linear Relations of SSA in the Hard Scattering Limit

- Conjectures: Saddle-point approximation (Gross, PRL, 1988)
- Existence of linear relations among SSA of different string states in the hard scattering limit (high-energy, fixed angle limit).
- We can use zero norm states (Lee, 1990, PRL) to derive the Ward-Takahashi identities which can be used to calculate the ratio of different SSA.

$$\langle V_1 V_{ZNS} V_3 V_4 \rangle = 0$$

Linear Relations of SSA in the Hard Scattering Limit

- Consider 4-point SSA in the hard scattering limit, we apply the zero norm state on one vertex and fix the other vertices to be certain states.
- By using decoupling ZNS, we can solve the ratio of the amplitudes of different string state.
- For example, there are 4 leading order SSA at mass level $M_2^2 = 4$ (Chan, Lee, 2004, NPB):

$$T_{TTT} : T_{LLT} : T_{(LT)} : T_{[LT]} = 8 : 1 : -1 : -1$$

Linear Relations of SSA in the Hard Scattering Limit

- The linear relations can be generalized to higher mass levels:

$$M_2^2 = 2(N-1)$$

- One first note that for each fixed mass level N, states of leading order in the Hard string scattering limit are of the following form (CHLTY, 2006, PRL).

$$V_2 \rightarrow |N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q}(\alpha_{-1}^L)^{2m}(\alpha_{-2}^L)^q|0;k\rangle$$

- The ratios among hard SSA of different string states at each fixed mass level N are (Decoupling of ZNS) (to all string loops!) (CHLTY, 2006, PRL)

$$\frac{A_{st}^{N,2m,q}}{A_{st}^{N,0,0}} = \left(\frac{-1}{M_2}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!$$

- At string-tree level, the above ratios can be explicitly calculated by saddle-point method.
- Gross conjectures were explicitly proved!!

Linear Relations of SSA in the Hard Scattering Limit

- The linear relations among SSA in hard scattering limit can also be extended to superstring case. In NS sector, for mass $M^2 = 2$, the ratios between the 4-point amplitudes associated with $(\alpha^T_{-1})\left(b^T_{-1/2}\right)|0;k\rangle,\ \ (\alpha^L_{-1})\left(b^L_{-1/2}\right)|0;k\rangle, \left(b^L_{-3/2}\right)|0;k\rangle$ are $1 : \frac{1}{M^2} : -\frac{1}{M}$ (Chan, Lee, Yang, NPB, 2005)

- Recently, we have calculated a class of polarized fermionic string scattering amplitudes (PFSSA) at arbitrary mass levels. We found that under hard scattering limit, the functional forms of the non-vanishing PFSSA at each fixed mass level are independent of the choices of spin polarizations. This result justifies and extends Gross's conjecture to the fermionic sector. (Lai, Lee, Yang, PLB, 2019)

Recurrence Relations of SSA in Regge Scattering Limit

- Let's consider another high energy limit, the Regge limit, which is also called fixed momentum transfer or high energy small angle regime:

$$s \to \infty, -t = \text{fixed}(\text{but} - t \neq \infty)$$

- The SSA of 1 arbitrary state and 3 tachyons can be written as the product of three parts, beta function, polarizations and the "first Appell function" (Lee, Yang, 2014, PLB):

$$|p_n, q_m, r_l\rangle = \prod_{n>0} (\alpha^T_{-n})^{p_n} \prod_{m>0} (\alpha^P_{-m})^{q_m} \prod_{l>0} (\alpha^L_{-l})^{r_l} |0, k\rangle,$$

$$A^{(p_n;q_m;r_l)} = \prod_{n=1} \left[(n-1)!\sqrt{-t}\right]^{p_n} \prod_{m=1} \left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m} \prod_{l=1} \left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l}$$

$$\cdot F_1\left(-\frac{t}{2} - 1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) \cdot B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right).$$

Recurrence Relations of SSA in Regge Scattering Limit

- In hard scattering limit, we have linear relations among different SSA. Analogously, there are four fundamental recurrence relations which link the contiguous functions in the Appell case:

$$
\begin{aligned}
&(a-b_1-b_2)\,F_1(a;b_1,b_2;c;x,y) - aF_1(a+1;b_1,b_2;c;x,y) \\
&\quad +b_1F_1(a;b_1+1,b_2;c;x,y) + b_2F_1(a;b_1,b_2+1;c;x,y) = 0, \qquad (1)\\
&cF_1(a;b_1,b_2;c;x,y) - (c-a)\,F_1(a;b_1,b_2;c+1;x,y) \\
&\quad -aF_1(a+1;b_1,b_2;c+1;x,y) = 0, \qquad (2)\\
&cF_1(a;b_1,b_2;c;x,y) + c\,(x-1)\,F_1(a;b_1+1,b_2;c;x,y) \\
&\quad -(c-a)\,xF_1(a;b_1+1,b_2;c+1;x,y) = 0, \qquad (3)\\
&cF_1(a;b_1,b_2;c;x,y) + c\,(y-1)\,F_1(a;b_1,b_2+1;c;x,y) \\
&\quad -(c-a)\,yF_1(a;b_1,b_2+1;c+1;x,y) = 0. \qquad (4)
\end{aligned}
$$

- These recurrence relations can be used to reduce the number of independent SSA from ∞ down to 1 under Regge limit (Lee, Yang, 2014).
- In the Regge limit, the string scattering amplitudes with the Appell function F_1 are associated with the symmetry group SL(5,C).

The D-type LSSA

- We calculated the 26-D open bosonic SSA of three tachyons and one arbitrary string state for **all** energies.

- We discovered that these SSA can be expressed in terms of the D-type Lauricella functions with associated SL(K+3,C) symmetry. (Lai, Lee, Yang, 2016, JHEP)

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \cdot \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}$$

$$\cdot B\left(-\frac{t}{2}-1, -\frac{s}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{u}{2}+2-N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right),$$

$$K = \sum_{\{\text{for all } r_j^T \neq 0\}} j + \sum_{\{\text{for all } r_j^P \neq 0\}} j + \sum_{\{\text{for all } r_j^L \neq 0\}} j .$$

- The results of the hard scattering limit, the Regge scattering limit can be rederived from the D-type Lauricella SSA.

The Recurrence Relations of D-type LSSA

- Both linear relations in hard scattering limit and the recurrence relations in Regge limit can be used to reduce the number of independent SSA from ∞ down to 1.
- To solve the LSSA, we will first generalize the 2+2 recurrence relations of the Appell functions to the K+2 recurrence relations of the D-type Lauricella functions.
- By using these extended recurrence relations, All LSSA can be solved and expressed in terms of one four tachyon amplitude (Lai, Lee, Lee, Yang, 2017, JHEP).

Hard scattering limit		Regge scattering limit
Linear relations **Gross**	**SL(K+3,C)** **recurrence relations**	**SL(5,C)** **recurrence relations**
←	**Lauricella SSA**	→ **Appell SSA**

The Recurrence Relations of D-type LSSA

- We have K+2 fundamental recurrence relations of the D-type Lauricella functions which are the extension of the 2+2 fundamental recurrence relations of Appell functions.

$$\left(\alpha - \sum_j \beta_j\right) F_D^{(K)} - \alpha F_D^{(K)}(\alpha+1) + \sum_j \beta_j F_D^{(K)}(\beta_j+1) = 0, \qquad (1)$$

$$\gamma F_D^{(K)} - (\gamma-\alpha) F_D^{(K)}(\gamma+1) - \alpha F_D^{(K)}(\alpha+1;\gamma+1) = 0, \qquad (2)$$

$$\gamma F_D^{(K)} + \gamma(x_m - 1) F_D^{(K)}(\beta_m+1) + (\alpha-\gamma)\, x_m F_D^{(K)}(\beta_m+1;\gamma+1) = 0. \quad (3)$$

- m can be chosen from 1 to K in third recurrence relation.

The Extended LSSA

- LSSA are not the basis functions of the representation space of SL(K+3,C) group. To further discuss and calculate the group representation of the LSSA for general K, we first define the extended LSSA (Miller, 1973).

$$
\begin{aligned}
& f_{ac}^{b_1 \cdots b_K}(\alpha; \beta_1, \cdots, \beta_K; x_1, \cdots, x_K) \\
=& B(\gamma-\alpha, \alpha) F_D^{(K)}(\alpha; \beta_1, \cdots, \beta_K; \gamma; x_1, \cdots, \cdots, x_K) a^\alpha b_1^{\beta_1} \cdots b_K^{\beta_K} c^\gamma .
\end{aligned}
$$

- By setting the a = 1 = c, the extended LSSA can be reduced to LSSA:

$$
A_{st}^{(r_n^T, r_m^P, r_l^L)} = f_{11}^{-(n-1)! k_3^P, -(l-1)! k_3^L}\left(-\frac{t}{2}-1; -R_n^T, -R_m^P, -R_l^L; \frac{u}{2}+2-N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right).
$$

The Operators of SL(K+3,C) Algebra

- We first introduce the $(K+3)^2-1$ generators of SL(K + 3, C) group.

$$E^{\alpha}=a\left(\sum_j x_j\partial_j+a\partial_a\right),\ E^{\beta_k}=b_k\left(x_k\partial_j+b_k\partial_{b_k}\right),E^{\gamma}=c\left(\sum_j(1-x_j)\partial_{x_j}+c\partial_c-a\partial_a-\sum_j b_j\partial_{b_j}\right):$$

$$E^{\alpha\gamma}=ac\left(\sum_j(1-x_j)\partial_{x_j}-a\partial_a\right),E^{\beta_k\gamma}=b_kc\left[(x_k-1)\partial_{x_k}+b_k\partial_{b_k}\right],E^{\alpha\beta_k\gamma}=ab_kc\partial_{x_k},$$

$$E_{\alpha}=\frac{1}{a}\left[\sum_j x_j(1-x_j)\partial_{x_j}+c\partial_c-a\partial_a-\sum_j x_jb_j\partial_{b_j}\right],$$

$$E_{\beta_k}=\frac{1}{b_k}\left[x_k(1-x_k)+x_k\sum_{j\neq k}x_j(1-x_j)\partial_{x_j}+c\partial_c-x_ka\partial_a-\sum_j b_j\partial_{b_j}\right],$$

$$E_{\gamma}=-\frac{1}{c}\left(\sum_j x_j\partial_{x_j}+c\partial_c-1\right),\ E_{\alpha\gamma}=\frac{1}{ac}\left[\sum_j x_j(1-x_j)\partial_{x_j}+c\partial_c-1-\sum_j x_jb_j\partial_{b_j}\right],$$

$$E_{\beta_k\gamma}=\frac{1}{b_kc}\left[x_k(x_k-1)\partial_{x_k}+\sum_{j\neq k}(x_j-1)x_j\partial_{x_j}+x_ka\partial_a-c\partial_c+1\right],$$

$$E_{\alpha\beta_k\gamma}=\frac{1}{ab_kc}\left[\sum_j x_j(x_j-1)\partial_{x_j}-c\partial_c+x_ka\partial_a+\sum_j x_jb_j\partial_{b_j}-x_k+1\right],$$

$$E^{\beta_k}_{\beta_p}=\frac{b_k}{b_p}\left[(x_k-x_p)\partial_{x_k}+b_k\partial_{b_k}\right],(k\neq p),\ J_{\alpha}=a\partial_a,\ J_{\beta_k}=b_k\partial_{b_k},\ J_{\gamma}=c\partial_c.$$

- It's straightforward to calculate the operations of these generators on the basis functions.
- We use the upper indices to denote the "raising operators" and the lower indices to denote the "lowering operators".
- We omit those arguments in $f_{ac}^{b_1 \dots b_K}$ which remain the same after the operation.
- $3K+3$ raising generators:
- 1 E^{α} , K E^{β_k} , 1 E^{γ} ,1 $E^{\alpha\gamma}$,K $E^{\beta_k\gamma}$ and K $E^{\alpha\beta_k\gamma}$,
- $3K+3$ lowering operators,
- $K(K-1)$ mixing operators: $E_{\beta_p}^{\beta_k}$,
- $K+2$ Cartan subalgebra: J,
- $(K+3)^2-1$ generators in total.

$$E^{\alpha} f_{ac}^{b_1 \cdots b_K}(\alpha) = (\gamma - \alpha - 1) f_{ac}^{b_1 \cdots b_K}(\alpha + 1),$$

$$E^{\beta_k} f_{ac}^{b_1 \cdots b_K}(\beta_k) = \beta_k f_{ac}^{b_1 \cdots b_K}(\beta_k + 1),$$

$$E^{\gamma} f_{ac}^{b_1 \cdots b_K}(\gamma) = \left(\gamma - \sum_j \beta_j\right) f_{ac}^{b_1 \cdots b_K}(\gamma + 1),$$

$$E^{\alpha\gamma} f_{ac}^{b_1 \cdots b_K}(\alpha;\gamma) = \left(\sum_j \beta_j - \gamma\right) f_{ac}^{b_1 \cdots b_K}(\alpha + 1;\gamma + 1),$$

$$E^{\beta_k\gamma} f_{ac}^{b_1 \cdots b_K}(\beta_k;\gamma) = \beta_k f_{ac}^{b_1 \cdots b_K}(\beta_k + 1;\gamma + 1),$$

$$E^{\alpha\beta_k\gamma} f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma) = \beta_k f_{ac}^{b_1 \cdots b_K}(\alpha + 1;\beta_k + 1;\gamma + 1),$$

$$E_{\alpha} f_{ac}^{b_1 \cdots b_K}(\alpha) = (\alpha - 1) f_{ac}^{b_1 \cdots b_K}(\alpha - 1),$$

$$E_{\beta_k} f_{ac}^{b_1 \cdots b_K}(\beta_k) = \left(\gamma - \sum_j \beta_j\right) f_{ac}^{b_1 \cdots b_K}(\beta_k - 1),$$

$$E_{\gamma} f_{ac}^{b_1 \cdots b_K}(\gamma) = (\alpha - \gamma + 1) f_{ac}^{b_1 \cdots b_K}(\gamma - 1),$$

$$E_{\alpha\gamma} f_{ac}^{b_1 \cdots b_K}(\alpha;\gamma) = (\alpha - 1) f_{ac}^{b_1 \cdots b_K}(\alpha - 1;\gamma - 1),$$

$$E_{\beta_k\gamma} f_{ac}^{b_1 \cdots b_K}(\beta_k;\gamma) = (\alpha - \gamma + 1) f_{ac}^{b_1 \cdots b_K}(\beta_k - 1;\gamma - 1),$$

$$E_{\alpha\beta_k\gamma} f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma) = (1 - \alpha) f_{ac}^{b_1 \cdots b_K}(\alpha - 1;\beta_k - 1;\gamma - 1),$$

$$E_{\beta_p}^{\beta_k} f_{ac}^{b_1 \cdots b_K}(\beta_k;\beta_p) = \beta_k f_{ac}^{b_1 \cdots b_K}(\beta_k + 1;\beta_p - 1),$$

$$J_{\alpha} f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma) = \alpha f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma),$$

$$J_{\beta_k} f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma) = \beta_k f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma),$$

$$J_{\gamma} f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma) = \gamma f_{ac}^{b_1 \cdots b_K}(\alpha;\beta_k;\gamma)$$

The Operators of SL(K+3,C) Algebra

States		LSSA		Extended LSSA
$(\alpha^T_{-1})^{r_1+1}\|0;k\rangle$	$\longleftrightarrow$	$(-k_3^T)^{r_1+1}BF_D(-r_1-1)$	$\overset{a=c=1}{\leftrightarrows}$	$f_{ab}^{(-k_3^T)}(-r_1-1)$
$\downarrow \; \uparrow$		$\downarrow \; \uparrow$		$E_{\beta_1} \uparrow \; \downarrow E^{\beta_1}$
$(\alpha^T_{-1})^{r_1}\|0;k\rangle$	$\longleftrightarrow$	$(-k_3^T)^{r_1}BF_D(-r_1)$	$\overset{a=c=1}{\leftrightarrows}$	$f_{ab}^{(-k_3^T)}(-r_1)$
$\downarrow \; \uparrow$		$\downarrow \; \uparrow$		$E_{\beta_1} \uparrow \; \downarrow E^{\beta_1}$
$(\alpha^T_{-1})^{r_1-1}\|0;k\rangle$	$\longleftrightarrow$	$(-k_3^T)^{r_1-1}BF_D(-r_1+1)$	$\overset{a=c=1}{\leftrightarrows}$	$f_{ab}^{(-k_3^T)}(-r_1+1)$

The Operators of SL(K+3,C) Algebra

- Rewrite the recurrence relations with the operators and extended LSSA defined in the previous slides, we get the corresponded Cartan subalgebra of SL(K+3,C) group.
- It's easy to see the number of the fundamental recurrence relations matches with the number of the Cartan subalgebras.

$$\left(\alpha-\sum_j\beta_j\right)F_D^{(K)}-\alpha F_D^{(K)}(\alpha+1)+\sum_j\beta_jF_D^{(K)}(\beta_j+1)=0,$$
$$\gamma F_D^{(K)}-(\gamma-\alpha)F_D^{(K)}(\gamma+1)-\alpha F_D^{(K)}(\alpha+1;\gamma+1)=0,$$
$$\gamma F_D^{(K)}+\gamma(x_m-1)F_D^{(K)}(\beta_m+1)+(\alpha-\gamma)x_mF_D^{(K)}(\beta_m+1;\gamma+1)=0.$$

fundamental recurrence relations

$$\left[(\alpha-J_\alpha)+\sum_j\left(\beta_j-J_{\beta_j}\right)\right]f_{ac}^{b_1\cdots b_K}=0.$$
$$\left[(\gamma-J_\gamma)-\sum_j\left(\beta_j-J_{\beta_j}\right)\right]f_{ac}^{b_1\cdots b_K}=0.$$
$$\left(\beta_m-J_{\beta_m}\right)f_{ac}^{b_1\cdots b_K}=0, m=1,2,...K.$$

Cartan subalgebras

The Operators of SL(K+3,C) Algebra

- There are K+2 fundamental recurrence relations among the Lauricella functions.
- The K+2 recurrence equations are equivalent to the Cartan subalgebra and the simple root system of SL(K+3, C) group.
- With the Cartan subalgebra and the simple roots, one can easily write down the whole Lie algebra of the SL(K+3, C) group.

The Operators of SL(K+3,C) Algebra

- With the following identifications:

$$E^{\alpha} = \mathcal{E}_{12}, E_{\alpha} = \mathcal{E}_{21}, E^{\beta_k} = \mathcal{E}_{k+3,3}, E_{\beta} = \mathcal{E}_{3,k+3},$$
$$E^{\gamma} = \mathcal{E}_{31}, E_{\gamma} = \mathcal{E}_{13}, E^{\alpha\gamma} = \mathcal{E}_{32}, E_{\alpha\gamma} = \mathcal{E}_{23},$$
$$E^{\beta_k\gamma} = -\mathcal{E}_{k+3,1}, E_{\beta_k\gamma} = -\mathcal{E}_{1,k+3}, E^{\alpha\beta_k\gamma} = -\mathcal{E}_{k+3,2},$$
$$E_{\alpha\beta_k\gamma} = -\mathcal{E}_{2,k+3}, J'_{\alpha} = \frac{1}{2}\left(\mathcal{E}_{11} - \mathcal{E}_{22}\right), J'_{\beta_k} = \frac{1}{2}\left(\mathcal{E}_{k+3,k+3} - \mathcal{E}_{33}\right), J'_{\gamma} = \frac{1}{2}\left(\mathcal{E}_{33} - \mathcal{E}_{11}\right).$$

- Lie algebra commutation relations of SL(K+3,C):

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{jk}\mathcal{E}_{il} - \delta_{li}\mathcal{E}_{kj}.$$

Summary

- The SSA of three tachyons and one arbitrary string state of 26-D open bosonic string theory can be expressed in terms of the Lauricella functions for all energy levels.
- The LSSA can reproduce the linear relations/recurrence relations (Appell functions) in the hard scattering/Regge limit respectively.
- Gross's conjecture on high energy SSA is not valid only in boson and NS sector but also holds in the R sector.
- We have to introduce the extended LSSA $f_{ac}^{b_1 \dots b_K}$ to discuss the SL(K+3, C) group algebra.
- The K+2 recurrence relations among the LSSA can be used to reproduce the Cartan subalgebra and simple root system of the SL(K+3, C) group.
- The SL(K+3, C) group can be used to solve all the LSSA and express them in terms of one amplitude

Thank you for your listening!

Appendices

- Gauss hypergeometric function: ${}_2F_1\left(a;b;c;z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$;

- Pochhammer symbol: $(q)_n = \begin{cases} 1, & n = 0 \\ q(q+1)\cdots(q+n-1), & n > 0 \end{cases}$;

- Appell function: $F_1\left(a;b_1,b_2;c;z_1,z_2\right) = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2} (b_1)_{n_1} (b_2)_{n_2}}{(c)_{n_1+n_2}} \frac{{z_1}^{n_1} {z_2}^{n_2}}{n_1! n_2!}$;

- D-type Lauricella function:

$$F_D^{(K)}\left(a;b_1,\cdots,b_k;c;z_1,\cdots,z_k\right) = \sum_{n_1,\cdots,n_k=0}^{\infty} \frac{(a)_{n_1+\cdots+n_k} (b_1)_{n_1} \cdots (b_k)_{n_k}}{(c)_{n_1+\cdots+n_k}} \frac{{z_1}^{n_1} \cdots {z_k}^{n_k}}{n_1! \cdots n_k!}$$

Appendices

- Appell function: $F_1\left(a; b_1, b_2; c; z_1, z_2\right) = \sum_{n_1, n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c)_{n_1+n_2}} \frac{{z_1}^{n_1}{z_2}^{n_2}}{n_1!n_2!}$;
- Kinematics:
- $k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0\right),$
- $k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0\right),$
- $k_3 = \left(-\sqrt{q^2 + M_3^2}, -q\cos\phi, -q\sin\phi\right),$
- $k_4 = \left(-\sqrt{q^2 + M_4^2}, +q\cos\phi, +q\sin\phi\right).$
- $e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t-M_2^2-M_3^2}{2M_2},$
- $e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t+M_2^2-M_3^2}{2M_2},$
- $e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t}.$

Appendices

- String state: $|p_n, q_m, r_l\rangle = \prod_{n>0}(\alpha_{-n}^{T})^{p_n} \prod_{m>0}(\alpha_{-m}^{P})^{q_m} \prod_{l>0}(\alpha_{-l}^{L})^{r_l} |0, k\rangle,$

- Appell SSA:

$$A^{(p_n;q_m;r_l)} = \prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{p_n} \prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m} \prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l}$$
$$\cdot F_1\left(-\frac{t}{2}-1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) \cdot B\left(-\frac{s}{2}-1, -\frac{t}{2}-1\right).$$

- $e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t-M_2^2-M_3^2}{2M_2},$

- $e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t+M_2^2-M_3^2}{2M_2},$

- $e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t}.$

Appendices

Given a string state:

$$\left|r_n^T, r_m^P, r_l^L\right\rangle = \prod_{n>0} \left(\alpha_{-n}^T\right)^{r_n^T} \prod_{m>0} \left(\alpha_{-m}^P\right)^{r_m^P} \prod_{l>0} \left(\alpha_{-l}^L\right)^{r_l^L} |0, k\rangle, \tag{1}$$

We can calculate the LSSA:

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \cdot \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}$$
$$\cdot B\left(-\frac{t}{2}-1, -\frac{s}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{u}{2}+2-N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right), \tag{2}$$

where we have defined $R_k^X \equiv \left\{-r_1^X\right\}^1, \cdots, \left\{-r_k^X\right\}^k$ with $\{a\}^n = \underbrace{a, a, \cdots, a}_{n}$,

$Z_k^X \equiv \left[z_1^X\right], \cdots, \left[z_k^X\right]$ with $\left[z_k^X\right] = z_{k0}^X, \cdots, z_{k(k-1)}^X$ and $z_k^X = \left|\left(-\frac{k_1^X}{k_3^X}\right)^{\frac{1}{k}}\right|$,

$z_{kk'}^X = z_k^X e^{\frac{2\pi i k'}{k}}$, $\tilde{z}_{kk'}^X \equiv 1 - z_{kk'}^X$ for $k' = 0, \cdots, k-1$.

We take the tensor state to be

$$|\text{state}\rangle = \left(\alpha_{-1}^{T}\right)^{r_1^T} \left(\alpha_{-1}^{P}\right)^{r_1^P} \left(\alpha_{-1}^{L}\right)^{r_1^L} |0, k\rangle. \tag{1}$$

The LSSA can be calculated to be

$$\begin{aligned} A_{st}^{(r_1^T, r_1^P, r_l^L)} &= \left(-k_3^T\right)^{r_1^T} \left(-k_3^P\right)^{r_1^P} \left(-k_3^L\right)^{r_1^L} B\left(-\frac{t}{2}-1, -\frac{s}{2}-1\right) \\ &\cdot F_D^{(3)}\left(-\frac{t}{2}-1; -r_1^T, -r_1^P, -r_1^L; \frac{u}{2}+2-N; \tilde{z}_{10}^T, \tilde{z}_{10}^P, \tilde{z}_{10}^L\right), \end{aligned} \tag{2}$$

where the arguments in $F_D^{(3)}$ are calculated to be

$$\begin{aligned} R_n^T &= \left\{-r_1^T\right\}^1, \cdots, \left\{-r_n^T\right\}^k = \left\{-r_1^T\right\}^1 = -r_1^T, \\ R_m^P &= \left\{-r_1^P\right\}^1, \cdots, \left\{-r_m^P\right\}^k = \left\{-r_1^P\right\}^1 = -r_1^P, \\ R_l^L &= \left\{-r_1^L\right\}^1, \cdots, \left\{-r_l^L\right\}^k = \left\{-r_1^L\right\}^1 = -r_1^L, \end{aligned} \tag{3}$$

$$\begin{aligned} \tilde{Z}_n^T &= \left[\tilde{z}_1^T\right], \cdots, \left[\tilde{z}_n^T\right] = \left[\tilde{z}_1^T\right] = \tilde{z}_{10}^T = 1 - z_{10}^T = 1 - z_k^T e^{\frac{2\pi i 0}{1}} = 1 - \left|-\frac{k_1^T}{k_3^T}\right|, \\ \tilde{Z}_n^P &= \left[\tilde{z}_1^P\right], \cdots, \left[\tilde{z}_n^P\right] = \left[\tilde{z}_1^P\right] = \tilde{z}_{10}^P = 1 - \left|-\frac{k_1^P}{k_3^P}\right|, \\ \tilde{Z}_n^L &= \left[\tilde{z}_1^L\right], \cdots, \left[\tilde{z}_n^L\right] = \left[\tilde{z}_1^L\right] = \tilde{z}_{10}^L = 1 - \left|-\frac{k_1^L}{k_3^L}\right| \end{aligned} \tag{4}$$

and

$$\begin{aligned} K &= \sum_{\{\text{for all } r_j^T \neq 0\}} j + \sum_{\{\text{for all } r_j^P \neq 0\}} j + \sum_{\{\text{for all } r_j^L \neq 0\}} j \\ &= 1 + 1 + 1 = 3. \end{aligned} \tag{5}$$