Hanany-Witten Transition in Quantum Curves

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Based on: [NK-Moriyama, 1907.04971]

YUKAWA INSTITUTE FOR
THEORETICAL PHYSICS

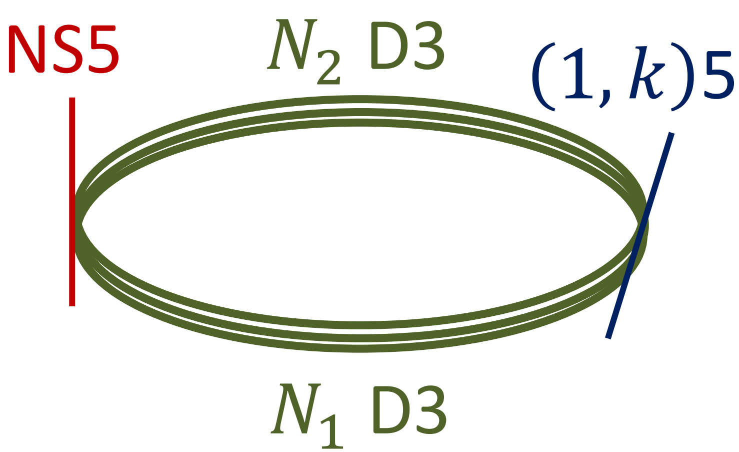
(Strings and Fields 2019)

M2-breanes

- M2-brane is one of the most important object in the M-theory.

M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$

Dimensional reduction



Type IIB string theory

Worldvolume theory

ABJ(M) theory

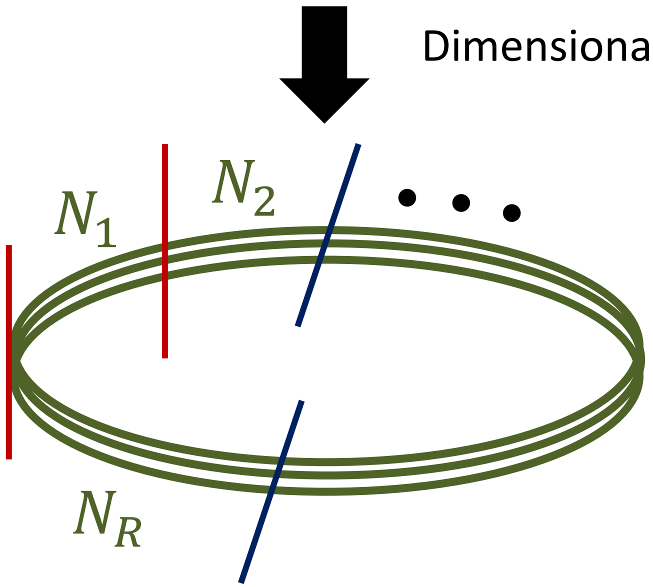
3d $\mathcal{N}=6$ $U(N_1)_k \times U(N_2)_{-k}$ super Chern-Simons theory

[Aharony-Bergman-Jafferis-Maldacena 2008]
[Aharony-Bergman-Jafferis 2008]

Quiver gauge theories

M2-branes on $\mathbb{C}^4/\Gamma$

Dimensional reduction



Worldvolume theory

Quiver $U(N)^R$ $\mathcal{N}=4$ superconformal Chern-Simons theory

[Imamura-Kimura 2008]

- A partition function on S^3 reduces to a partition function of matrix model by using the localization technique.

$$Z\left(N_1, N_2, \ldots, N_R\right) \propto \int \prod_a d^{N_a}\lambda_a e^{ik_a \sum_i \lambda_{a,i}^2} Z_{N_a,N_{a+1}}(\lambda_a, \lambda_{a+1})$$

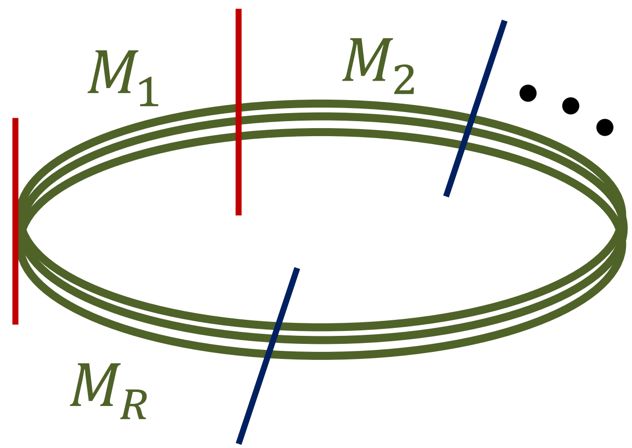
$$Z_{N_a,N_{a+1}}(\mu,\nu) = \frac{\prod_{i<j}^{N_a} 2\sinh\frac{\mu_i-\mu_j}{2} \prod_{i<j}^{N_{a+1}} 2\sinh\frac{\nu_i-\nu_j}{2}}{\prod_{i<j}^{N_a,N_{a+1}} 2\cosh\frac{\mu_i-\nu_j}{2}}$$

Grand partition function

- A grand partition function plays an important role.

$$\Xi_n(\vec{M};\kappa)$$
$$=\sum_N \kappa^{N+M_n}\left|Z\left(N+M_1,N+M_2,\ldots,N+M_R\right)\right|$$

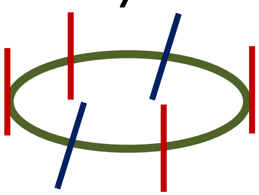
Only relative ranks are important in this expression.

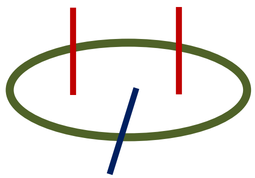


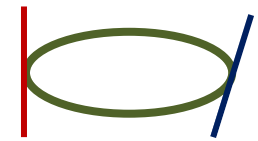
Hidden group structure

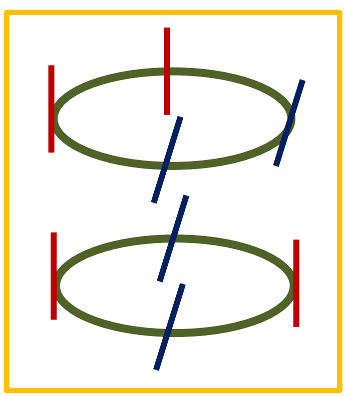
- Large N behavior of a grand partition function can be expressed by a character of corresponding exceptional type group. [Moriyama-Nosaka-Yano 2017]

$$E_7 - E_6 - D_5 - A_4 - (A_2 + A_1) - (A_1 + A_1) - A_1$$









D_5 theory

Can we realize this symmetry? → We could carry out.

[NK-Moriyama-Nosaka 2018]

Realization of symmetry

Conjecture:

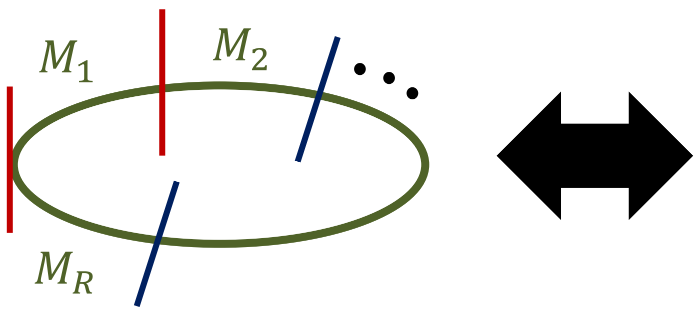
$$\Xi(\vec{M};\kappa) = \mathrm{Det}\left(1+\kappa \widehat{H}(\vec{e};\widehat{Q},\widehat{P})^{-1}\right)$$

$$\widehat{Q}=e^{\widehat{q}},\ \ \widehat{P}=e^{\widehat{p}},\ \ [\widehat{q},\widehat{p}]=2\pi i k$$

[Grassi-Hatsuda-Marino 2014]

$$\widehat{H}(\vec{e};\widehat{Q},\widehat{P}) = \sum_{i,j} e_{ij}\widehat{Q}^i\widehat{P}^j$$ (Quantum curve)

(No parameter identification)



$$\widehat{H}(\vec{e};\widehat{Q},\widehat{P})$$

Important for the realization

For D_5 theory,

$$\widehat{H}=\widehat{Q}\widehat{P}+\widehat{Q}+\widehat{Q}\widehat{P}^{-1}+e_1\widehat{P}+e_2\widehat{P}^{-1}+e_3\widehat{Q}^{-1}\widehat{P}+e_4\widehat{Q}^{-1}+e_5\widehat{Q}^{-1}\widehat{P}^{-1}$$

Realization of symmetry

$$\Xi(\vec{M};\kappa) = \mathrm{Det}\left(1+\kappa\widehat{H}^{-1}\right)$$

We introduced maps generated by similarity transformations.

$$w : \widehat{H} \to \widehat{G}\widehat{H}\widehat{G}^{-1}$$

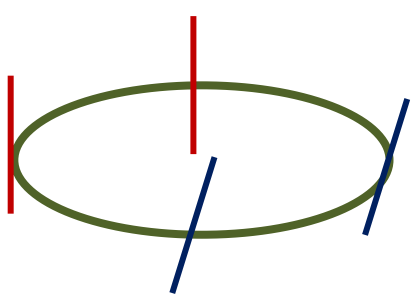
↑

A set of these maps have Weyl group structure!

[NK-Moriyama-Nosaka 2018]

- The results are consistent with the large N results.

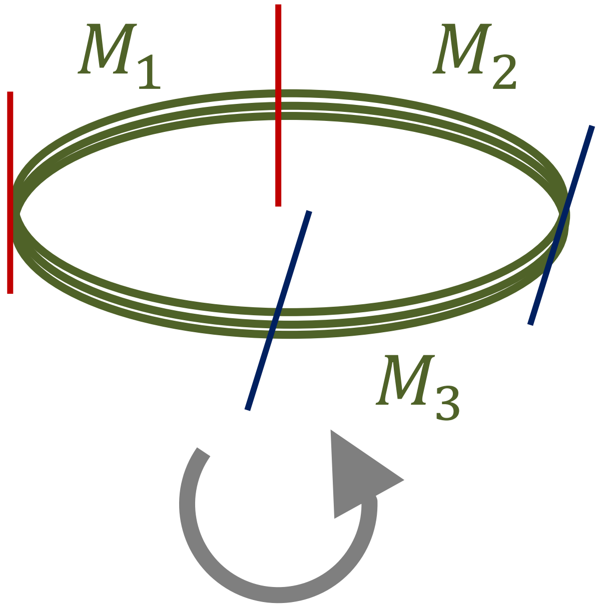
D_5 theory



⟷ $W(D_5)$

(Weyl group of D_5)

Correspondence?



$\Leftrightarrow \widehat{H}(\vec{e}; \widehat{Q}, \widehat{P})$

Symmetries? $\Leftrightarrow$ $W(D_5)$

Parameter identification of $\vec{M}$ and $\vec{e}$

→ We can interpret the Weyl group as symmetries of branes.

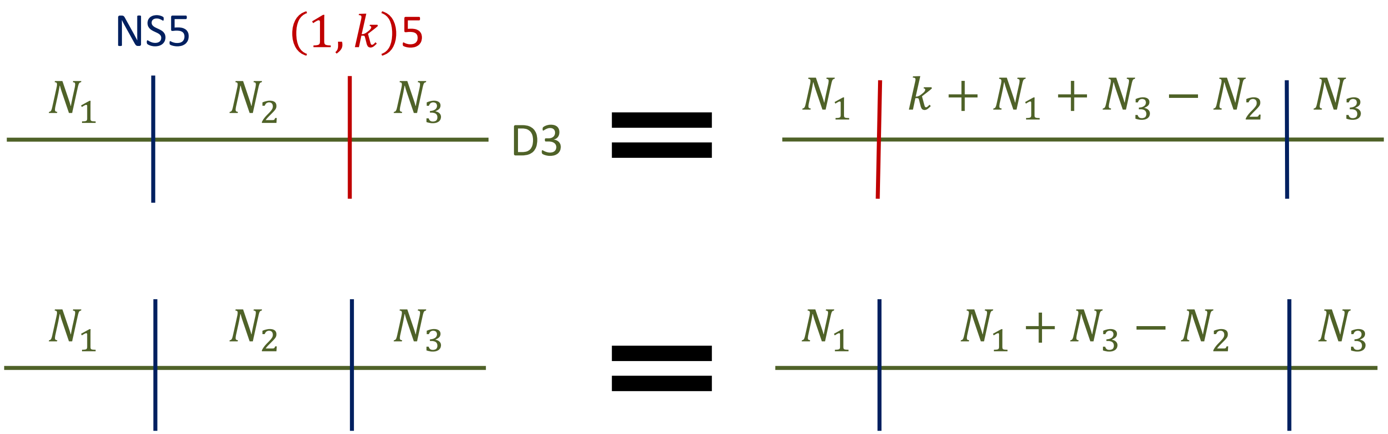
Contents

- Introduction
- Symmetries of branes
- Parameter Identification
- Summary

Symmetries of branes

Branes of type IIB string theory possess many symmetries.

- Hanany-Witten transitions



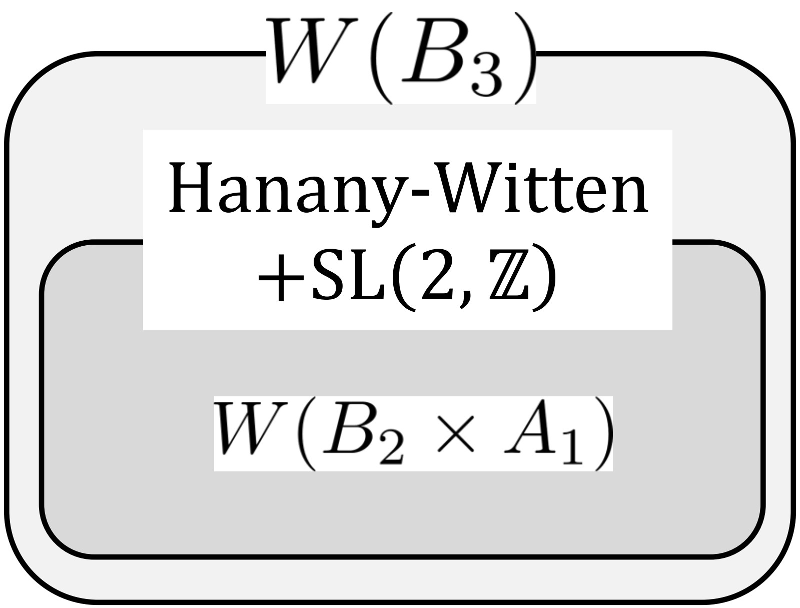
- $\mathrm{SL}(2,\mathbb{Z})$ transformations

$$\mathrm{NS5} \leftrightarrow (1,k)\,5$$

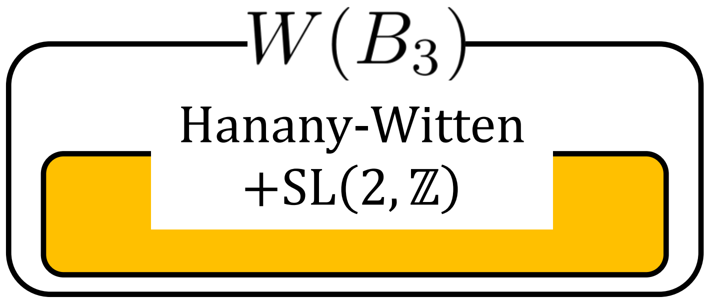
with $\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and flipping the sign of RR charge of D5-branes.

Results

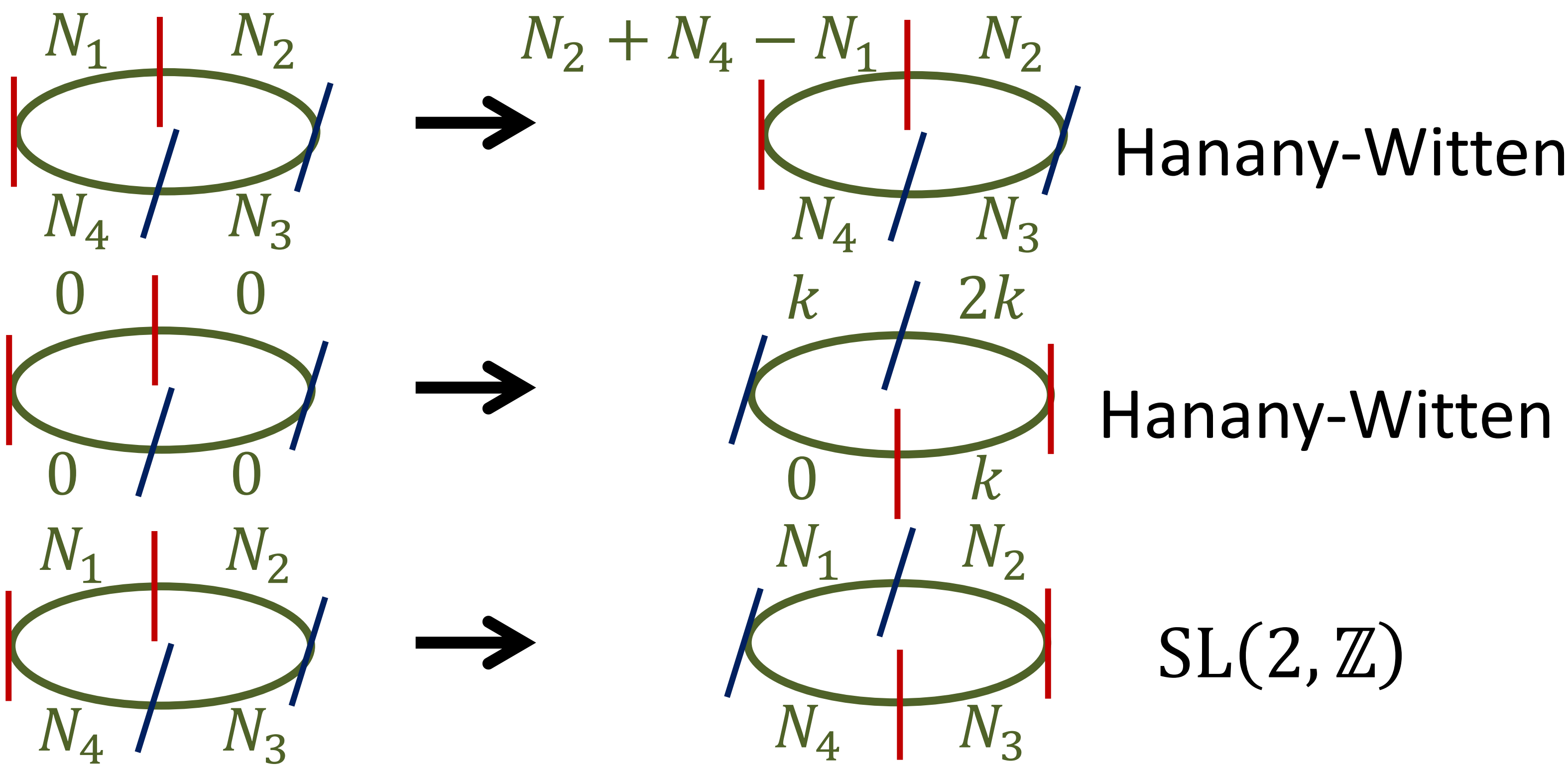
- Only $W(B_3) \subset W(D_5)$ works as symmetries of brane configurations.
- $W(B_3)$ is bigger than Hanany-Witten+$\mathrm{SL}(2,\mathbb{Z})$.



Results

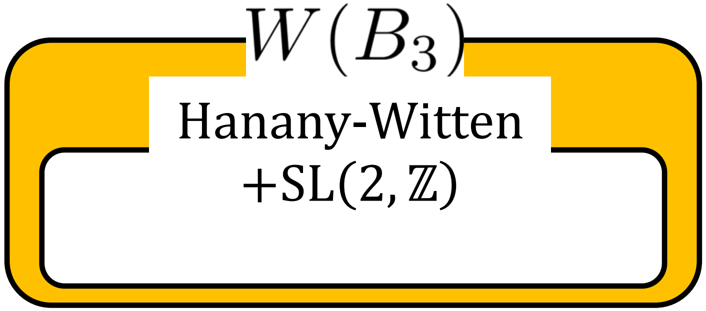


ex)

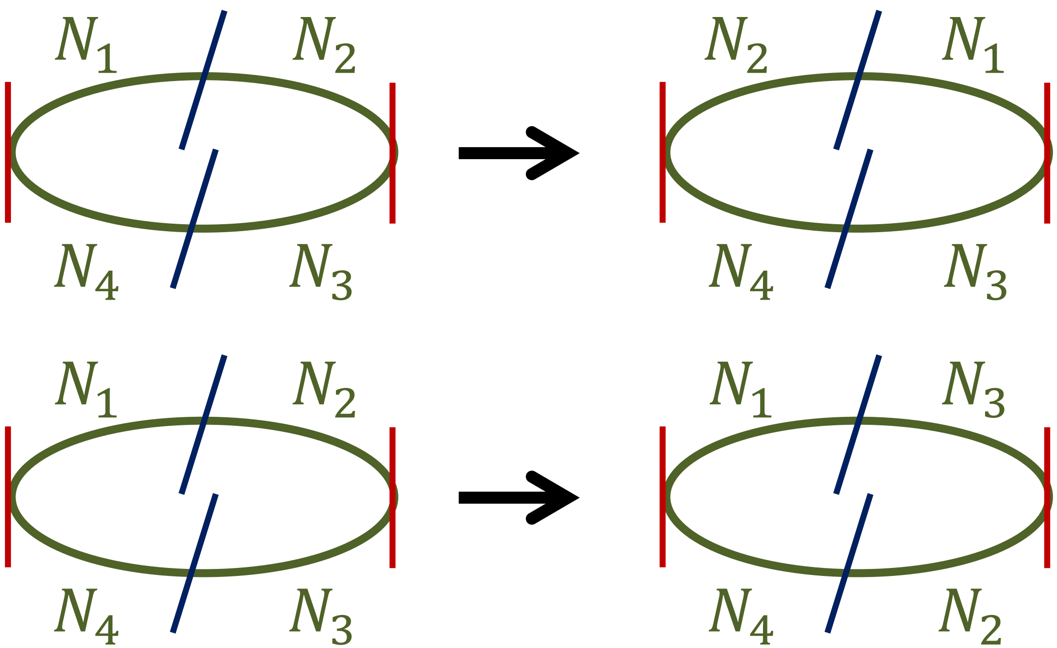


We could unify Hanany-Witten+$\mathrm{SL}(2,\mathbb{Z})$ into a Weyl group.

Results



ex)



New brane symmetries?

Contents

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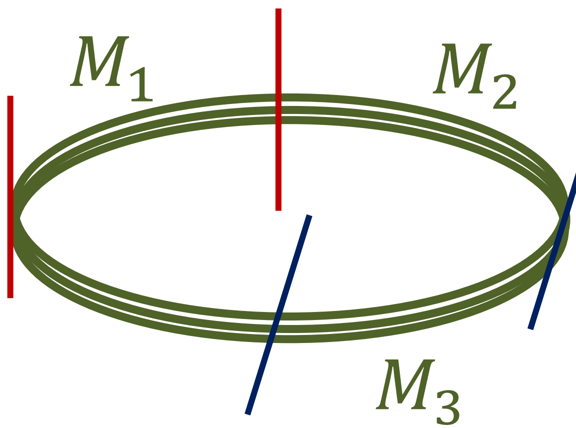
Parameters

$$\Xi(\vec{M};\kappa) = \mathrm{Det}\left(1+\kappa\widehat{H}(\vec{e};\widehat{Q},\widehat{P})^{-1}\right)$$

- Brane configurations



(M_1, M_2, M_3)

3 parameters

Relation?

- Quantum curves

$$\widehat{Q}\widehat{P}+\widehat{Q}+\widehat{Q}\widehat{P}^{-1}+e_1\widehat{P}+e_2\widehat{P}^{-1}$$
$$+e_3\widehat{Q}^{-1}\widehat{P}+e_4\widehat{Q}^{-1}+e_5\widehat{Q}^{-1}\widehat{P}^{-1}$$

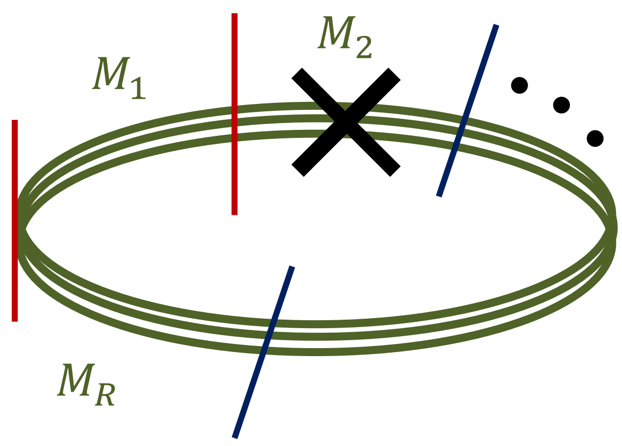
$(e_1, e_2, e_3, e_4, e_5)$

5 parameters

Reference frame

- A concept "reference frame" is important.
- Technically, "reference frame" appear in the definition of grand partition function.

$$\Xi(\vec{M};\kappa) = \sum_{N} \kappa^{N+M_2} Z\left(N+M_1, N+M_2, \ldots, N+M_R\right)$$



Identification

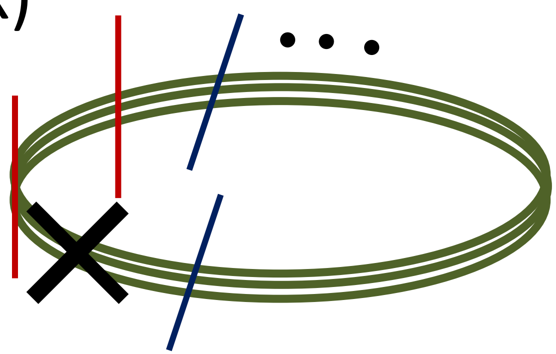
- The relation was proved for no rank deformations cases. (Fermi gas formalism)

[Marino-Putrov 2012]
[Moriyama-Nosaka 2014]

 $\Rightarrow (\widehat{P}^{1/2} + \widehat{P}^{-1/2})$

 $\Rightarrow (\widehat{Q}^{1/2} + \widehat{Q}^{-1/2})$

ex)

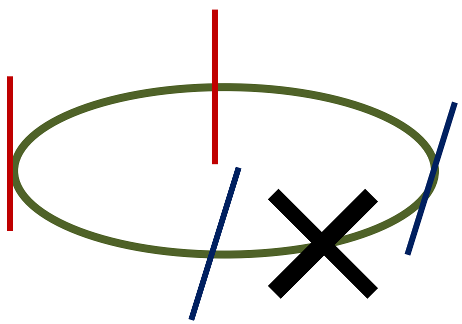


$$\Rightarrow (\widehat{P}^{1/2} + \widehat{P}^{-1/2})^2(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}) \cdots$$

Identification

Non-trivial rank deformations from no rank deformations.

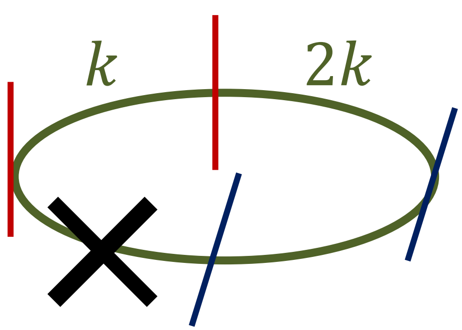
ex)



$$(\widehat{Q}^{1/2}+\widehat{Q}^{-1/2})(\widehat{P}^{1/2}+\widehat{P}^{-1/2})^2(\widehat{Q}^{1/2}+\widehat{Q}^{-1/2})$$

Hanany-Witten

normal ordering



$$\hat{Q}\hat{P}+q^{\frac{1}{2}}\hat{Q}, +q\hat{Q}\hat{P}^{-1}$$
$$+(1+q)\hat{P}+4q^{\frac{1}{2}}+(1+q)\hat{P}^{-1}$$
$$+q\hat{Q}^{-1}\hat{P}+q^{\frac{1}{2}}\hat{Q}^{-1}+\hat{Q}^{-1}\hat{P}^{-1}$$

$$q=e^{2\pi i k}$$

Identification

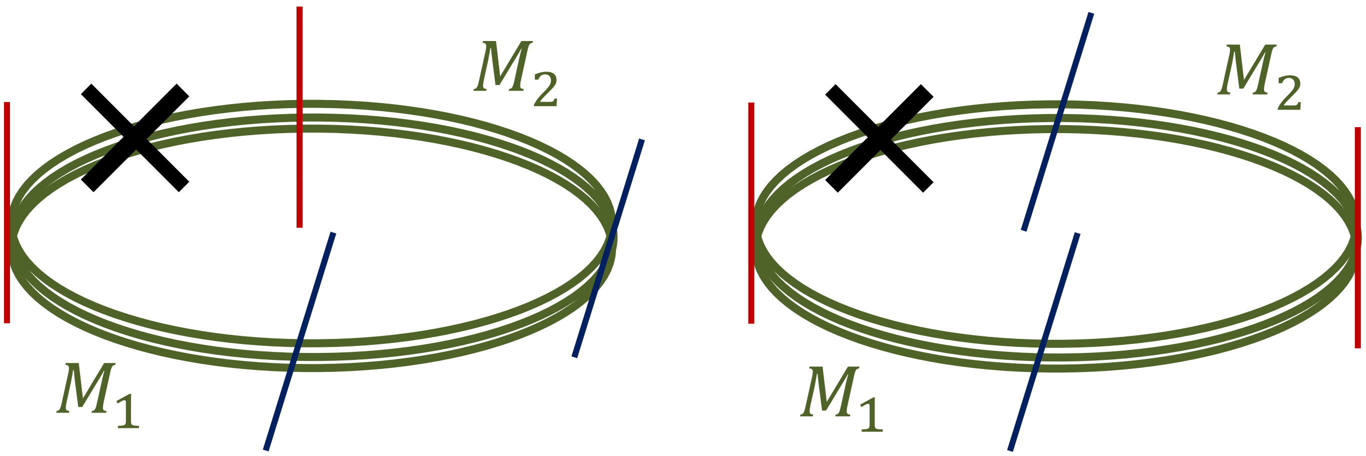
Spectral operator	$\Delta(h,e,f)$	Brane configuration	(M_1,M_2,M_3)
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2$	$(1,1,1)$	$\langle 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{4}{\circ}\rangle$	$(0,0,0)$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(q^{\frac{1}{2}},q^{\frac{1}{2}},1)$	$\langle 0 \overset{2}{\bullet} 0 \overset{3}{\circ} 0 \overset{1}{\bullet} 0 \overset{4}{\circ}\rangle$	$(\frac{k}{2},\frac{k}{2},0)$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2$	$(q^{\frac{1}{2}},1,q^{-\frac{1}{2}})$	$\langle 0 \overset{2}{\bullet} 0 \overset{4}{\circ} 0 \overset{1}{\bullet} 0 \overset{3}{\circ}\rangle$	$(\frac{k}{2},0,-\frac{k}{2})$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(q^{\frac{1}{2}},1,q^{\frac{1}{2}})$	$\langle 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{2}{\bullet} 0 \overset{4}{\circ}\rangle$	$(\frac{k}{2},0,\frac{k}{2})$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1$	$(q^{\frac{1}{2}},q^{-\frac{1}{2}},1)$	$\langle 0 \overset{1}{\bullet} 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{3}{\circ}\rangle$	$(\frac{k}{2},-\frac{k}{2},0)$
$\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(q,q^{\frac{1}{2}},q^{\frac{1}{2}})$	$\langle 0 \overset{3}{\circ} 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} 0 \overset{4}{\circ}\rangle$	$(k,\frac{k}{2},\frac{k}{2})$
$\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q,q^{-\frac{1}{2}},q^{-\frac{1}{2}})$	$\langle 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{1}{\bullet} 0 \overset{3}{\circ}\rangle$	$(k,-\frac{k}{2},-\frac{k}{2})$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2$	$(q,q^{\frac{1}{2}},q^{-\frac{1}{2}})$	$\langle 0 \overset{2}{\bullet} 0 \overset{3}{\circ} 0 \overset{4}{\circ} 0 \overset{1}{\bullet}\rangle$	$(k,\frac{k}{2},-\frac{k}{2})$
$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1$	$(q,q^{-\frac{1}{2}},q^{\frac{1}{2}})$	$\langle 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{4}{\circ} 0 \overset{2}{\bullet}\rangle$	$(k,-\frac{k}{2},\frac{k}{2})$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3$	$(q^{\frac{3}{2}},q^{\frac{1}{2}},1)$	$\langle 0 \overset{3}{\circ} 0 \overset{2}{\bullet} 0 \overset{4}{\circ} 0 \overset{1}{\bullet}\rangle$	$(\frac{3k}{2},\frac{k}{2},0)$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4$	$(q^{\frac{3}{2}},1,q^{-\frac{1}{2}})$	$\langle 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{3}{\circ} 0 \overset{1}{\bullet}\rangle$	$(\frac{3k}{2},0,-\frac{k}{2})$
$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_3$	$(q^{\frac{3}{2}},1,q^{\frac{1}{2}})$	$\langle 0 \overset{3}{\circ} 0 \overset{1}{\bullet} 0 \overset{4}{\circ} 0 \overset{2}{\bullet}\rangle$	$(\frac{3k}{2},0,\frac{k}{2})$
$\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_3\widehat{\mathcal{P}}_1\widehat{\mathcal{Q}}_4$	$(q^{\frac{3}{2}},q^{-\frac{1}{2}},1)$	$\langle 0 \overset{4}{\circ} 0 \overset{1}{\bullet} 0 \overset{3}{\circ} 0 \overset{2}{\bullet}\rangle$	$(\frac{3k}{2},-\frac{k}{2},0)$
$\widehat{\mathcal{P}}_1\widehat{\mathcal{P}}_2\widehat{\mathcal{Q}}_4\widehat{\mathcal{Q}}_3$	$(q^2,1,1)$	$\langle 0 \overset{3}{\circ} 0 \overset{4}{\circ} 0 \overset{2}{\bullet} 0 \overset{1}{\bullet}\rangle$	$(2k,0,0)$

→We conjecture natural parameter identification.

Consistency check

$$\Xi(\vec{M};\kappa) = \mathrm{Det}\left(1+\kappa\widehat{H}^{-1}\right)$$

- We proved the equality for some rank deformations.



The results are consistent with our conjecture.

Large N behavior, revisit

- Only one interval of D3-branes was chosen to be reference frame before our work.

  → We test large N behavior for new reference frames.

$$\Xi_n(\vec{M};\kappa) = \sum_N \kappa^{N+M_n} \left| Z\left(N+M_1, N+M_2, \ldots, N+M_R\right) \right|$$

This factor affects the large N expansion highly non-trivially and changes the symmetry drastically.

The results are completely consistent with quantum curve analysis.

Summary & future work

➢ Summary

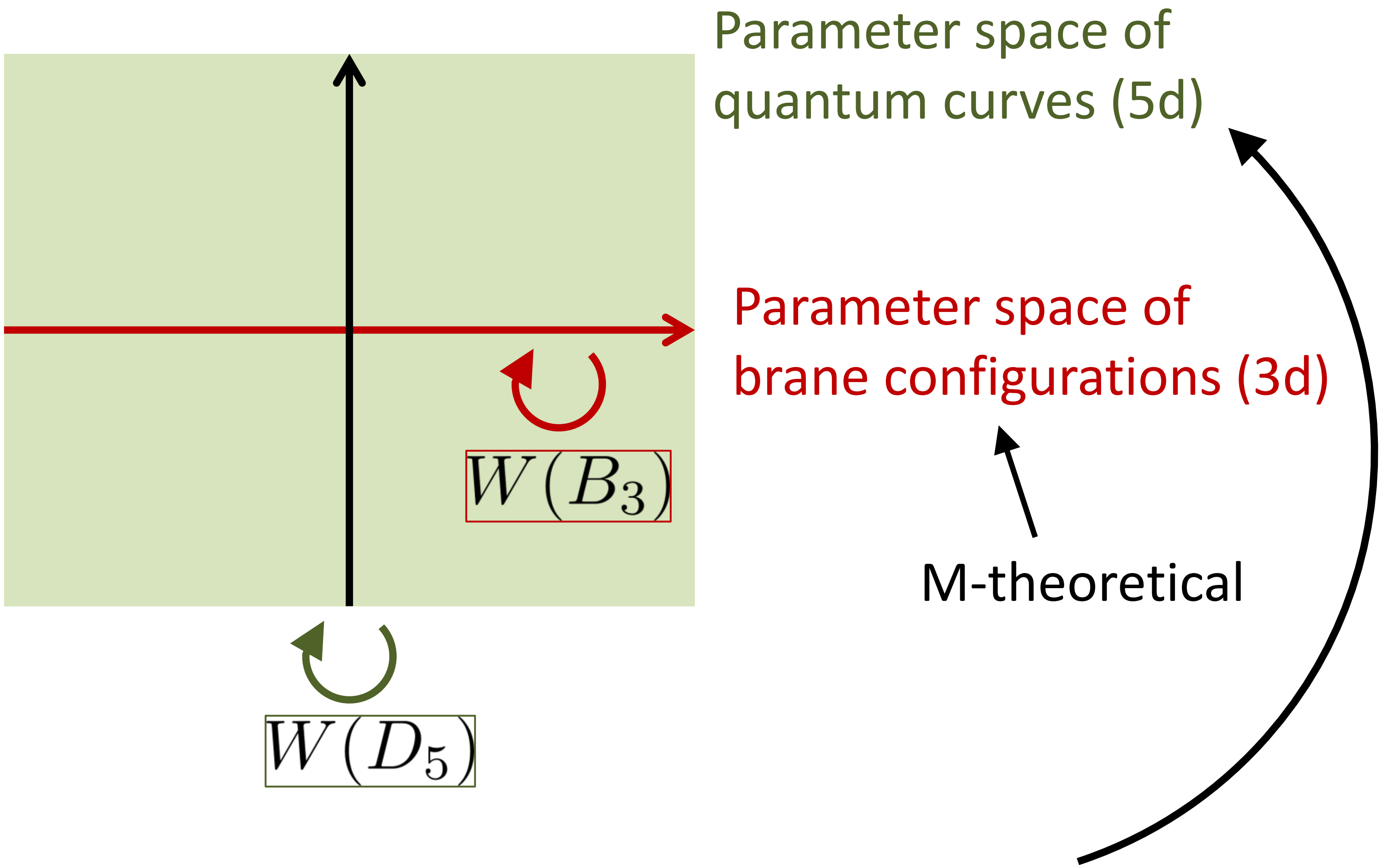
- We conjectured the parameter correspondence between brane configurations and quantum curves.
- We unified Hanany-Witten transitions and $\mathrm{SL}(2,\mathbb{Z})$ transformations into the subset of $W(D_5)$.
- We found new symmetries of brane configurations.

➢ Future work

- Applying the same procedure for quiver gauge theories with exceptional type symmetries and also for higher genus quiver gauge theories.
- Establishing M-theoretical meaning of parameter space of quantum curves.
- Relation to the Painlevé equations through the Weyl group.

Back-up

Rest parameter space?



Does this 5d parameter space have M-theoretical meaning?