# SUSY QM with Conditional Shape Invariance and the Solvability

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## Abstract

Exactly solvable quantum mechanics (or SUSY QM) has extensively been studied for many years. It is known that shape invariance is the sufficient condition for exact solvability and the idea is crucial for finding solutions. Recently, a powerful way of constructing approximate solutions has proposed where the modified concept of shape invariance is applied. We present the prescription and thoroughly discuss the mathematical implications.

## Supersymmetric Quantum Mechanics [1]

We consider the exact solutions for time-independent Schrödinger equations:

$$\mathcal{H}\psi(x) = E\psi(x)\ , \quad \mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\ .$$

The Hamiltonian can be factorized after the subtraction of a constant;

$$\mathcal{H} - E_0 = \mathcal{A}^\dagger\mathcal{A} = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)$$

where $E_0$ corresponds to the ground state energy eigenvalue, and $W(x)$ is called *superpotential*. Note that $V(x) - E_0 = W^2(x) - W'(x)$. The ground state wave function is obtained from

$$\mathcal{A}\psi_0(x) = 0 \quad \therefore \quad W(x) = -\frac{\psi_0'(x)}{\psi_0(x)} = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0(x)$$
$$\Longrightarrow \quad \psi_0(x) = N\exp\left[-\int^x W(y)\mathrm{d}y\right]\ .$$

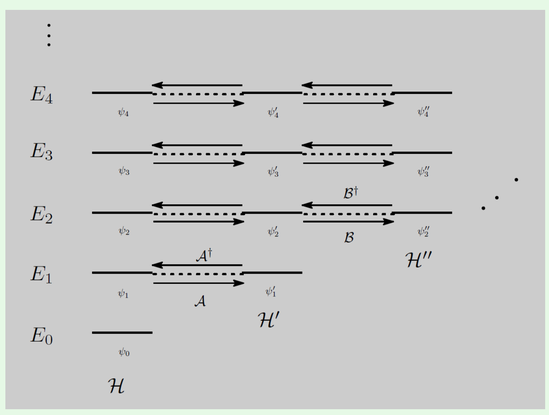
Then, we define the partner Hamiltonian as

$$\mathcal{H}' := \mathcal{A}\mathcal{A}^\dagger = \mathcal{B}^\dagger\mathcal{B} + \varepsilon_1$$

where $\varepsilon_1$ means the energy gap $E_1 - E_0$. Here, one finds the following *intertwining relations*:

$$\mathcal{A}\mathcal{H} = \mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{H}'\mathcal{A}\ ,$$
$$\mathcal{A}^\dagger\mathcal{H}' = \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger = \mathcal{H}\mathcal{A}^\dagger\ ,$$

which guarantee that $\mathcal{H}$ and $\mathcal{H}'$ are iso-spectral and connected via $\mathcal{A}$ or $\mathcal{A}^\dagger$, but the ground state of $\mathcal{H}$. By repeatedly applying the above procedure, one obtains the whole spectra.




The sufficient condition for the exact solvability is

$$\mathcal{H}'(x;\boldsymbol{\lambda}) = \mathcal{H}(x;f(\boldsymbol{\lambda})) + R(\boldsymbol{\lambda}) \qquad : \textit{shape invariance}\ (\mathrm{SI})$$

where $\boldsymbol{\lambda} = \{a_1, a_2, \dots\}$ is the set of potential parameters, and $f(\boldsymbol{\lambda}), R(\boldsymbol{\lambda})$ are functions of $\boldsymbol{\lambda}$. Just a small number of potentials are known as shape invariant potentials. For example:

| Potential | $W(x)$ | $V(x)$ | $f(a,b)$ |
|---|---|---|---|
| H.O. | $ax+b$ | $(ax+b)^2 - a$ | $\{a,b\}$ |
| 3-d H.O. | $ax - b/x$ | $\frac{b(b-1)}{x^2} + a^2x^2 - a(2b+1)$ | $\{a,b+1\}$ |
| Coulomb | $a/b - b/x$ | $\frac{b(b-1)}{x^2} - \frac{2a}{x} + \frac{a^2}{b^2}$ | $\{a+1,b\}$ |
| Rosen–Morse | $a\tanh bx$ | $-\frac{a(a+b)}{\cosh^2 bx} + a^2$ | $\{a-b,b\}$ |
| Pöschl–Teller | $a\tan x - b\cot x$ | $\frac{a(a-1)}{\cos^2 x} + \frac{b(b-1)}{\sin^2 x} - (a+b)^2$ | $\{a+1,b+1\}$ |

In fact, most potentials in physics are not SI. However, they are sometimes the "mixture" of SI potentials. Then, how do we deal with such potentials?

## Conditional Shape Invariance [2]

S. Bera et al. have suggested a way of dealing with a potential:

$$V(x) = \frac{l(l+1)}{x^2} + ax + bx^2 - \frac{c}{x} \qquad : \text{the } \textit{mixed potential}.$$

First, they factorize the Hamiltonian as

$$\mathcal{H} - E_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{l(l+1)}{x^2} + ax + bx^2 + \frac{c}{x} - E_0$$
$$= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + Ax - \frac{B}{x} + C\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + Ax - \frac{B}{x} + C\right)\ .$$

One finds the relations between sets of parameters:

$$A = \sqrt{b}\ , \quad C = \frac{a}{2\sqrt{b}}\ , \quad B = \frac{\sqrt{b}c}{a}\ ,$$

and the ground state energy eigenvalue:

$$E_0 = -\frac{a^2}{4b} - \frac{2bc}{a} + \sqrt{b}\ .$$

However, since the number of the potential parameters is larger than that of the superpotential, one has a parametric constraint condition:

$$\frac{\sqrt{b}c}{a} = l + 1\ .$$

Here, they choose $(l, b, c)$ to be free parameters;

$$\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{l(l+1)}{x^2} + \frac{\sqrt{b}c}{l+1}x + bx^2 - \frac{c}{x}\ .$$

Next, one can define the partner Hamiltonian as

$$\mathcal{H}' := \left(\frac{\mathrm{d}}{\mathrm{d}x} + Ax - \frac{B}{x} + C\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + Ax - \frac{B}{x} + C\right)$$
$$= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(l+1)(l+2)}{x^2} + \frac{\sqrt{b}c}{l+1}x + bx^2 - \frac{c}{x} + R\ .$$

However, one cannot factorize this Hamiltonian into $(-\partial_x + \hat{W})(\partial_x + \hat{W})$ because of the constraint. Therefore, they construct a new Hamiltonian as

$$\hat{\mathcal{H}}' := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(l+1)(l+2)}{x^2} + a_1x + bx^2 - \frac{c}{x} + R$$
$$= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \hat{A}x - \frac{\hat{B}}{x} + \hat{C}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \hat{A}x - \frac{\hat{B}}{x} + \hat{C}\right) + \varepsilon_1 \underset{?}{\approx} \mathcal{H}'$$

with $a_1 := \dfrac{\sqrt{b}c}{l+2}$. Note that $\hat{\mathcal{H}}'$ is not coincident with $\mathcal{H}'$, and there are no mathematical validity in doing so. Now, one can calculate the approximate first excited state energy of the original Hamiltonian as

$$E_1 \underset{?}{\approx} E_0 + \varepsilon_1 = -\frac{a_1^2}{4b} - \frac{2bc}{a_1} + 3\sqrt{b}$$
$$= -\frac{c^2}{4(l+2)^2} + 2\sqrt{b}(l+2) + 3\sqrt{b}\ .$$

Using this prescription again and again, one obtains the $n$th excited state energy:

$$E_n \underset{?}{\approx} -\frac{c^2}{4(l+n+1)^2} + 2\sqrt{b}(l+1) + (4n+1)\sqrt{b}\ .$$

## Logarithmic Perturbation Theory [3,4]

Logarithmic perturbation theory (LPT) requires *only* a knowledge of the initial unperturbed ground state $\psi_0(x)$. We consider a potential

$$V(x) = V_0(x) + \lambda V_1(x)$$

where $\lambda$ is the perturbation parameter. Writing

$$\psi(x) \equiv \mathrm{e}^{S(x)} \quad \text{or} \quad S(x) = \ln\psi(x)$$

and expanding $E$, $S'(x)$ in the perturbation parameter;

$$E \cong E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \cdots\ ,$$
$$S'(x) \cong C^{(0)}(x) + \lambda C^{(1)}(x) + \lambda^2 C^{(2)}(x) + \cdots\ ,$$

we obtain the following formula:

$$E^{(1)} = \int_{-\infty}^{\infty} V_1(x)|\psi_0(x)|^2\mathrm{d}x\ ,$$
$$C^{(1)}(x) = |\psi_0(x)|^{-2}\int_{-\infty}^{x}\left[V_1(y) - E^{(1)}\right]|\psi_0(y)|^2\mathrm{d}y\ ,$$
$$E^{(n)} = -\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n-1} C^{(i)}(x)C^{(n-i)}(x)\right)|\psi_0(x)|^2\mathrm{d}x\ ,$$
$$C^{(n)}(x) = |\psi_0(x)|^{-2}\int_{-\infty}^{x}\left[-E^{(n)} - \sum_{i=1}^{n-1} C^{(i)}(y)C^{(n-i)}(y)\right]|\psi_0(y)|^2\mathrm{d}y$$
$$(n \geqslant 2)\ .$$

# "Exact" Solution for the Mixed Potential

We employ LPT to evaluate S. Bera et al.'s prescription and justify it. Let us consider the correction for the first excited state energy eigenvalue of the original system $\boldsymbol{E_1}$. Our Hamiltonian here is

$$\mathcal{H}' = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(l+1)(l+2)}{x^2} + a_1 x + bx^2 - \frac{c}{x} + R + \lambda\alpha x$$

where we take $\lambda \to 1$, and $\alpha \equiv a - a_1$. The unperturbed Hamiltonian is

$$\begin{aligned}\mathcal{H}'^{(0)} &\equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(l+1)(l+2)}{x^2} + a_1 x + bx^2 - \frac{c}{x} + R \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \hat{A}x - \frac{\hat{B}}{x} + \hat{C}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \hat{A}x - \frac{\hat{B}}{x} + \hat{C}\right) + E^{(0)}\end{aligned}$$

with the parametric relations

$$\hat{A} = \sqrt{b} = A\,, \quad \hat{C} = \frac{a_1}{2\sqrt{b}}\,, \quad \hat{B} = \frac{\sqrt{b}c}{a_1}\,,$$

and

$$\alpha = \frac{2A\hat{C}}{\hat{B}-1} = \frac{\sqrt{b}c}{(l+1)(l+2)}\,.$$

Therefore, the ground state energy for the unperturbed system is

$$E^{(0)} = -\frac{{a_1}^2}{4b} - \frac{2bc}{a_1} + 3\sqrt{b}\,,$$

and the ground state wave function is

$$\psi_0(x) = N x^{\hat{B}} \exp\left[-\frac{A}{2}x^2 - \hat{D}x\right]$$

where the normalization constant $N$ is

$$N = \left\{\frac{1}{2}A^{-\hat{B}-1}\left[\sqrt{A}\,\Gamma\left(\hat{B}+\frac{1}{2}\right){}_1F_1\left(\hat{B}+\frac{1}{2},\frac{1}{2};\frac{\hat{D}^2}{A}\right) - 2\hat{D}\,\Gamma\left(\hat{B}+1\right){}_1F_1\left(\hat{B}+1,\frac{3}{2};\frac{\hat{D}^2}{A}\right)\right]\right\}^{-1/2}.$$

Note that ${}_1F_1(a,c;x)$ is the Kummer's confluent hypergeometric function. Then, the first order correction of the ground state energy is calculated as

$$E^{(1)} = \frac{\alpha\left[\sqrt{A}\,\Gamma\left(\hat{B}+1\right){}_1F_1\left(\hat{B}+1,\frac{1}{2};\frac{\hat{D}^2}{A}\right) - 2\hat{D}\,\Gamma\left(\hat{B}+\frac{3}{2}\right){}_1F_1\left(\hat{B}+\frac{3}{2},\frac{3}{2};\frac{\hat{D}^2}{A}\right)\right]}{\sqrt{A}\left[\sqrt{A}\,\Gamma\left(\hat{B}+\frac{1}{2}\right){}_1F_1\left(\hat{B}+\frac{1}{2},\frac{1}{2};\frac{\hat{D}^2}{A}\right) - 2\hat{D}\,\Gamma\left(\hat{B}+1\right){}_1F_1\left(\hat{B}+1,\frac{3}{2};\frac{\hat{D}^2}{A}\right)\right]}.$$

Similarly, we can obtain the higher order corrections. Computing the quantity $E^{(0)} + E^{(1)} + \cdots$, we obtain the exact value of $E_1$.

Here, we expect $c$ to be some coupling constant, and for now, we fix $c = 1$ for simplicity. Thus, $b$ is the model parameter.

**E.g. The S state ($l = 0$) of the system**

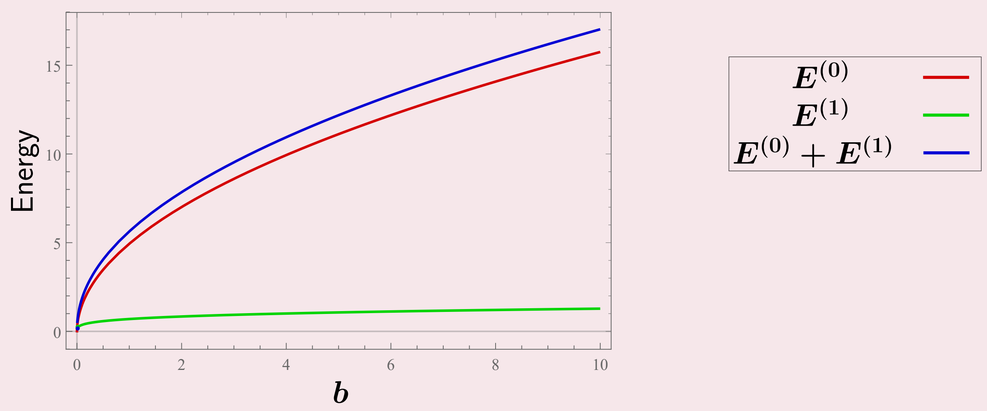
The unperturbed ground state energy is

$$E^{(0)}(b) = -\frac{1}{16} + 7\sqrt{b}\,,$$

and the first order correction is written as

$$\begin{aligned}&E^{(1)}(b) \\ &= \frac{\sqrt{b}\,{}_1F_1\left(3,\frac{1}{2};\frac{1}{16\sqrt{b}}\right) - \frac{\sqrt[4]{b}}{2}\frac{15\sqrt{\pi}}{8}\,{}_1F_1\left(\frac{7}{2},\frac{3}{2};\frac{1}{16\sqrt{b}}\right)}{\sqrt[4]{b}\frac{3\sqrt{\pi}}{4}\,{}_1F_1\left(\frac{5}{2},\frac{1}{2};\frac{1}{16\sqrt{b}}\right) - {}_1F_1\left(3,\frac{3}{2};\frac{1}{16\sqrt{b}}\right)} \\ &= \frac{\sqrt[4]{b}\left(\frac{1}{2}\sqrt{b}\left(-\left(\frac{5e^{\frac{1}{16\sqrt{b}}}}{64b^{3/4}} + \frac{e^{\frac{1}{16\sqrt{b}}}}{1024b^{3/4}} + \frac{15e^{\frac{1}{16\sqrt{b}}}}{16\sqrt[4]{b}}\right)\left(2\sqrt{\pi}\left(1-\mathrm{erf}\left(\frac{1}{4\sqrt[4]{b}}\right)\right) - 8e^{-\frac{1}{16\sqrt{b}}}\sqrt{b} - 2\sqrt{\pi}\right) - 128e^{-\frac{1}{16\sqrt{b}}}\left(\frac{e^{\frac{1}{16\sqrt{b}}}}{1024b^{3/4}} + \frac{5e^{\frac{1}{16\sqrt{b}}}}{128b^{3/4}}\right)b^{3/4} - \frac{e^{\frac{1}{16\sqrt{b}}}\left(-64e^{-\frac{1}{16\sqrt{b}}}b^{3/4} - 1536e^{-\frac{1}{16\sqrt{b}}}b^{3/4}\right)}{1024b^{3/4}}\right) - \frac{15}{16}\sqrt{\pi}\left(\frac{1}{12\sqrt{b}} + \frac{1}{960b} + 1\right)e^{\frac{1}{16\sqrt{b}}}\right)}{2\left(\frac{1}{4}\left(-\left(\frac{e^{\frac{1}{16\sqrt{b}}}}{64b^{3/4}} + 3e^{\frac{1}{16\sqrt{b}}}\sqrt{b} + \frac{3e^{\frac{1}{16\sqrt{b}}}}{4\sqrt[4]{b}}\right)\left(\sqrt{\pi} - \sqrt{\pi}\left(1-\mathrm{erf}\left(\frac{1}{4\sqrt[4]{b}}\right)\right)\right) - 8e^{-\frac{1}{16\sqrt{b}}}\left(\frac{e^{\frac{1}{16\sqrt{b}}}}{64b^{3/4}} + \frac{3e^{\frac{1}{16\sqrt{b}}}}{8\sqrt[4]{b}}\right)\sqrt[4]{b} - \frac{e^{\frac{1}{16\sqrt{b}}}\left(-32e^{-\frac{1}{16\sqrt{b}}}b^{3/4} - 4e^{-\frac{1}{16\sqrt{b}}}\sqrt[4]{b}\right)}{64b^{3/4}}\right) + \frac{3}{4}\sqrt{\pi}e^{\frac{1}{16\sqrt{b}}}\left(\frac{1}{4\sqrt{b}} + \frac{1}{192b} + 1\right)\sqrt[4]{b}\right)}.\end{aligned}$$

We plot $E^{(0)}$, $E^{(1)}$ and $E^{(0)} + E^{(1)}$ as functions of $b$:




Applying the procedure over and over again, we get approximate solutions for higher excited states. We have checked the accuracy of our method for several choices of a set of parameters.

Ⅰ. $l = 1,\ b = 0.5,\ c = 0.01$

| | $E^{(\mathrm{num})}$ | $E^{(0)}$ | $E^{(0)} + E^{(1)}$ |
|---|---|---|---|
| $E_0$ | 3.53 | 3.53 | – |
| $E_1$ | 6.36 | 6.36 | 6.36 |
| $E_2$ | 9.19 | 9.19 | 9.19 |
| $E_3$ | 12.01 | 12.02 | 12.02 |
| $E_4$ | 14.84 | 14.85 | 14.84 |

Ⅱ. $l = 1,\ b = 3.0,\ c = 1.0$

| | $E^{(\mathrm{num})}$ | $E^{(0)}$ | $E^{(0)} + E^{(1)}$ |
|---|---|---|---|
| $E_0$ | 8.59 | 8.59 | – |
| $E_1$ | 15.15 | 15.56 | 15.15 |
| $E_2$ | 21.78 | 22.50 | 21.86 |
| $E_3$ | 28.45 | 29.43 | 28.64 |
| $E_4$ | 35.15 | 36.37 | 35.46 |

# Physical Application: Heavy Baryonic Systems

Our consideration above can be applied for obtaining mass spectra of a singly-heavy baryonic system: $\Sigma_{\mathrm{b}}^{+}$ $(\mathbf{uub})$. We regard the baryon as a three-body system, and employ the Hyperspherical formalism [5]. The Jacobi relative coordinates $(\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3)$ are used to describe quarks' positions. Defining hyper-radius and hyper-angle as

$$x := \sqrt{{\rho_1}^2 + {\rho_2}^2}\,, \quad \xi := \arctan\frac{\rho_1}{\rho_2}\,,$$

one obtains the equations:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{5}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\hat{L}^2}{x^2} + V(x)\right]X(x) = EX(x)$$

$$\hat{L}^2\mathscr{Y}(\Omega) = -\gamma(\gamma+4)\mathscr{Y}(\Omega)$$

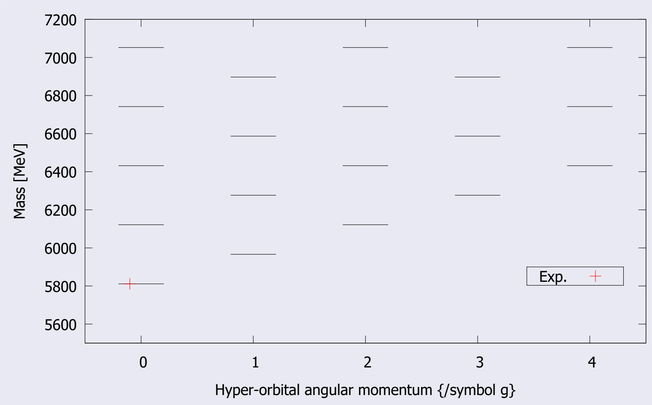
where $\mathscr{Y}(\Omega)$ is the Hypersherical Harmonics, and $\gamma$ is the hyper-orbital angular momentum quantum number. Then, the reduced hyper-radial equation is

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(\gamma+\frac{3}{2})(\gamma+\frac{5}{2})}{x^2} + ax + bx^2 - \frac{c}{x}\right]\chi(x) = E\chi(x)$$

with $V(x) = ax + bx^2 - c/x$. $ax$ means the confinement, $bx^2$ is the correction of two-body forces, and $-c/x$ is the Coulomb-like term. Here, we set $b$ to satisfy experimental data and

$$c = \frac{2}{3}\alpha_s\,, \quad a = \frac{\sqrt{b}c}{\gamma + \frac{5}{2}}$$

where $\alpha_s$ is a strong coupling constant. The result is as follows:




Experimental datum [PDG]:

$\mathbf{5811.3\ MeV}$

# Summary & Future Plans

- The number of exactly solvable potentials is limited. An approximate solution for a "mixture" of SI potentials has been proposed by S. Bera et al.
- By using LPT, we justify their prescription, and give a quantity that evaluates it. We show examples for several choices of a set of parameters.
- We apply our method to a physical system. $\Sigma_{\mathrm{b}}^{+}$ (singly-heavy baryon) is an example. We obtain its mass spectra.
- We are to examine approximate algebras. We expect that, for either exactly or approximately solvable systems, we are able to construct a corresponding algebra.

# References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