Entanglement and Renyi Entropy of Multiple Intervals in $T\overline{T}$-Deformed CFT and Holography

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Finite cutoff holography

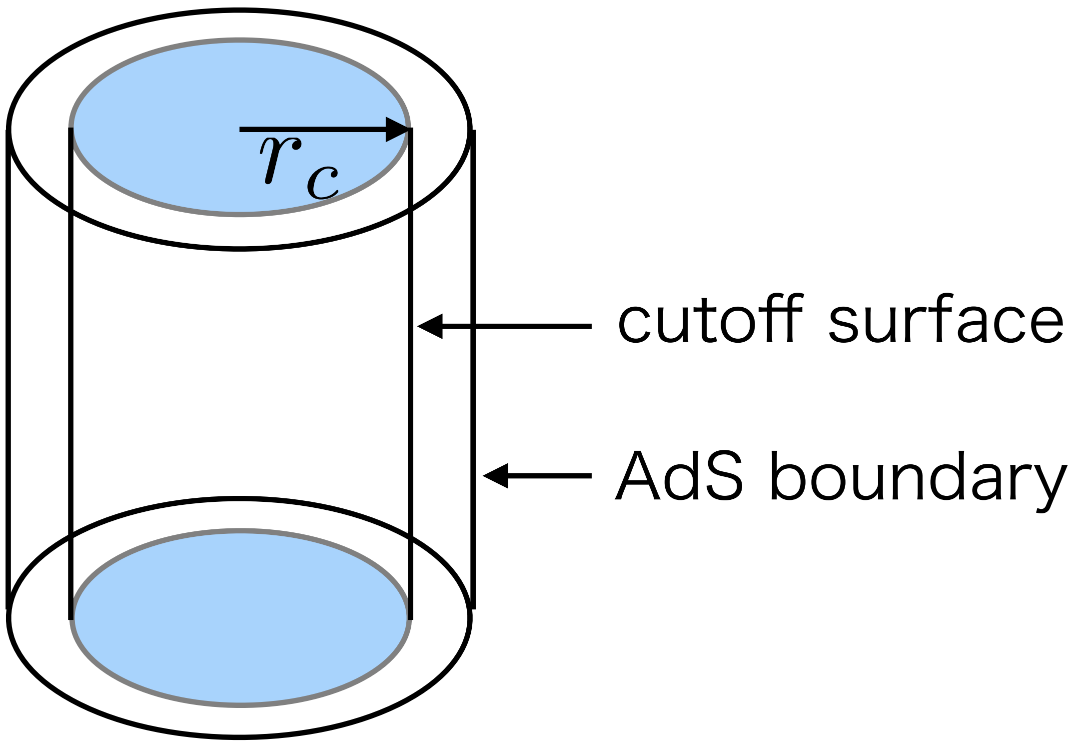
[L. McGough, M. Mezei, H. Verlinde, 2016]

$T\overline{T}$-deformed
2d holographic CFT

$$\frac{dS^{(\mu)}_{\rm QFT}}{d\mu} = \int \mathrm{d}^2 x\, (T\overline{T})_\mu$$

$$S^{(\mu)}_{\rm QFT}\Big|_{\mu=0} = S_{\rm CFT}$$

Finite cutoff asymptotic AdS_3
with $r < r_c$



The energy spectrum in the deformed CFT
and the quasi local energy inside the cutoff surface
are **matched** with $\mu = \dfrac{4G}{\pi r_c^2}$.

Entanglement entropy in finite cutoff holography

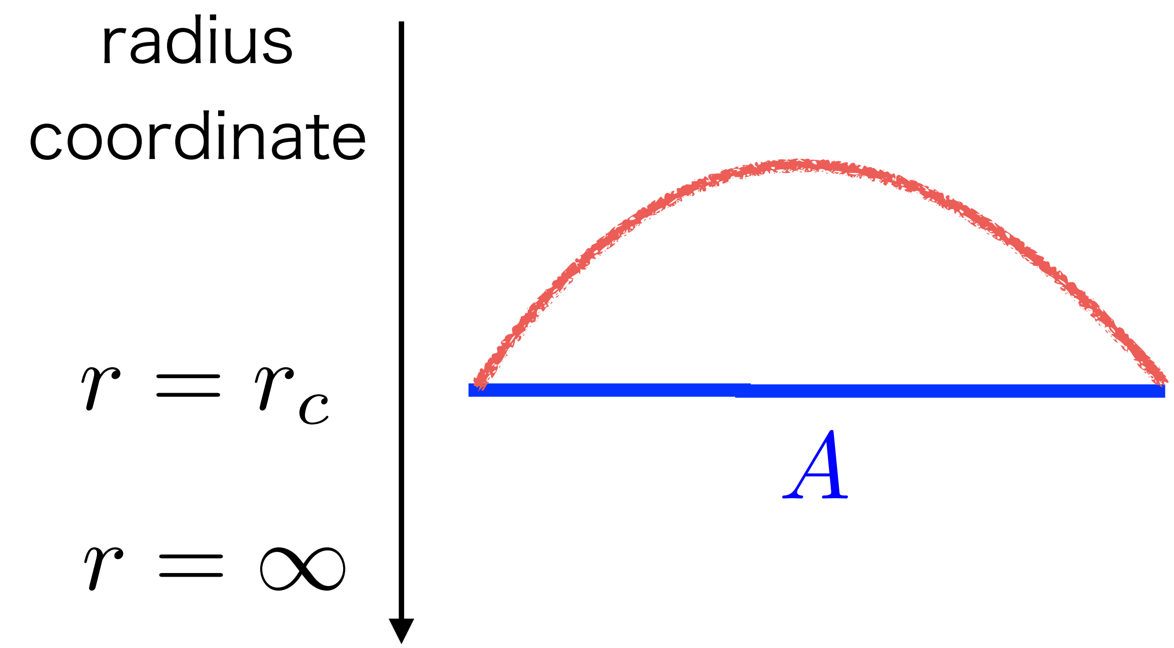
[W. Donnelly, V. Shyam, 2018] [B. Chen, L. Chen, P.-X. Hao, 2018]⋯

Entanglement entropy in the $T\overline{T}$ -deformed CFT

$$S(A) = \lim_{n\to 1} \frac{1}{1-n} \log \frac{Z_n(A)}{Z_1^n}$$

$$Z_n(A) = \int_{\mathcal{M}^n} D\phi \, e^{-S_{\mathrm{QFT}}^{(\mu)}}$$

Holographic entanglement entropy with finite cutoff



Entanglement entropy with the $T\overline{T}$ -deformation are studied by QFT and holographic calculation.

Our target:

Entanglement entropy (EE) of multiple intervals

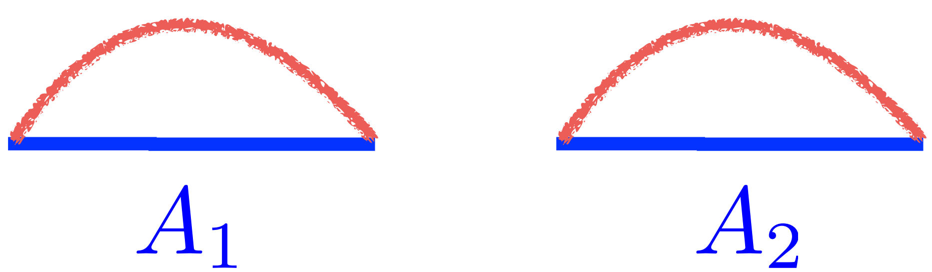
- Entanglement entropy for a single interval of the ground state in 2d CFTs is universal, but EE for multiple intervals is **not** universal.

$$S(A) = \frac{c}{3} \log \frac{l}{\epsilon}$$

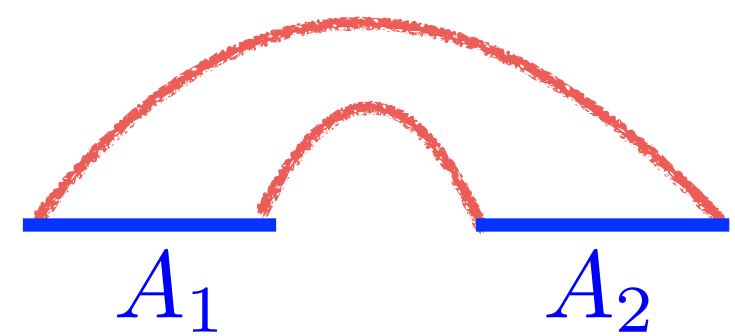
[C. Holzhey, F. Larsen, F. Wilczek, 1994]

- Additive property of EE in the 2d holographic CFTs (EE of multiple intervals = summation of EE of single interval)

[T. Hartman, 2013]



disconnected surface



connected surface

Are they still valid in the $T\overline{T}$-deformed CFTs?

Our result 1

Formula of the first order perturbation of Renyi entanglement entropy

(2d CFT deformed by μ, m intervals, period β of Euclidean time)

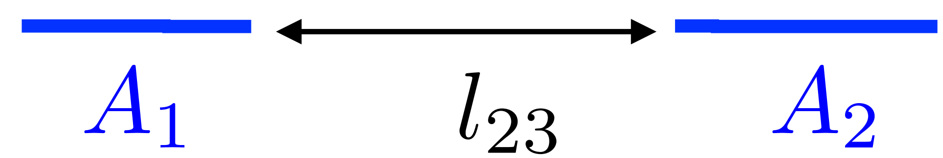
$$\delta S_n(A) = -\frac{\mu c}{12(n-1)}\frac{8\pi^4}{\beta^4}$$
$$\times \int_{\mathcal{M}} \Bigg[z^2 \sum_{j=1}^{2m} \left(\frac{c(n-1/n)}{24(z-z_j)^2} + \frac{\partial_{z_j} \log \langle \prod_{i=1}^m \sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}}{(z-z_j)} \right)$$
$$+ \bar{z}^2 \sum_{j=1}^{2m} \left(\frac{c(n-1/n)}{24(\bar{z}-\bar{z}_j)^2} + \frac{\partial_{\bar{z}_j} \log \langle \prod_{i=1}^m \sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}}{(\bar{z}-\bar{z}_j)} \right) \Bigg]$$
$$+ \frac{\mu}{n(n-1)}\frac{16\pi^4}{\beta^4}\frac{1}{\langle \prod_{i=1}^m \sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}}$$
$$\times \int_{\mathcal{M}} z^2 \left(\sum_{j=1}^{2m} \left(\frac{c(n-1/n)}{24(z-z_j)^2} + \frac{\partial_{z_j}}{(z-z_j)} \right) \right) \bar{z}^2 \left(\sum_{j=1}^{2m} \left(\frac{c(n-1/n)}{24(\bar{z}-\bar{z}_j)^2} + \frac{\partial_{\bar{z}_j}}{(\bar{z}-\bar{z}_j)} \right) \right)$$
$$\times \langle \prod_{i=1}^{m} \sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}$$

Our result 2

Check of the additive property

- Entanglement entropy in the 2d deformed holographic CFTs

$$\delta S_{\text{s-ch}}(A_1 \cup A_2) \sim \delta S(A_1) + \delta S(A_2)$$



- Renyi entanglement entropy of two well-separated intervals

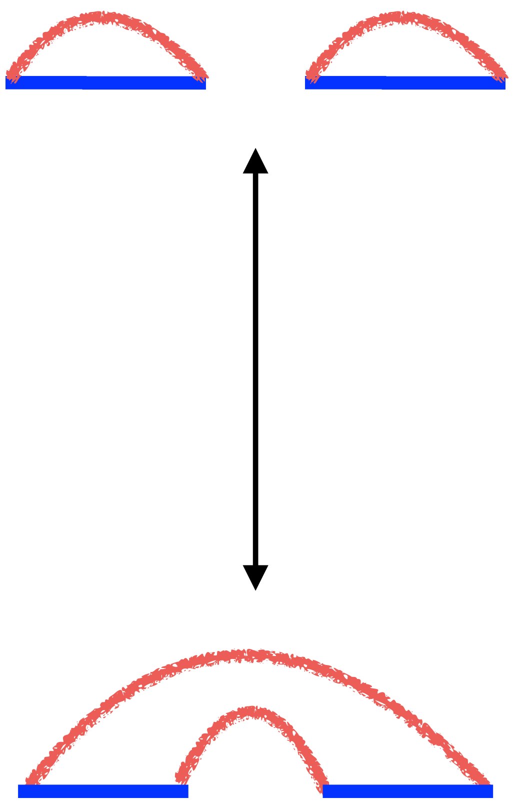
$$\delta S_n(A)|_{l_{23}\to\infty} \sim \delta S_n^{\text{single}}(A_1) + \delta S_n^{\text{single}}(A_2) + \delta S_n^{\text{mixing}}(A)$$

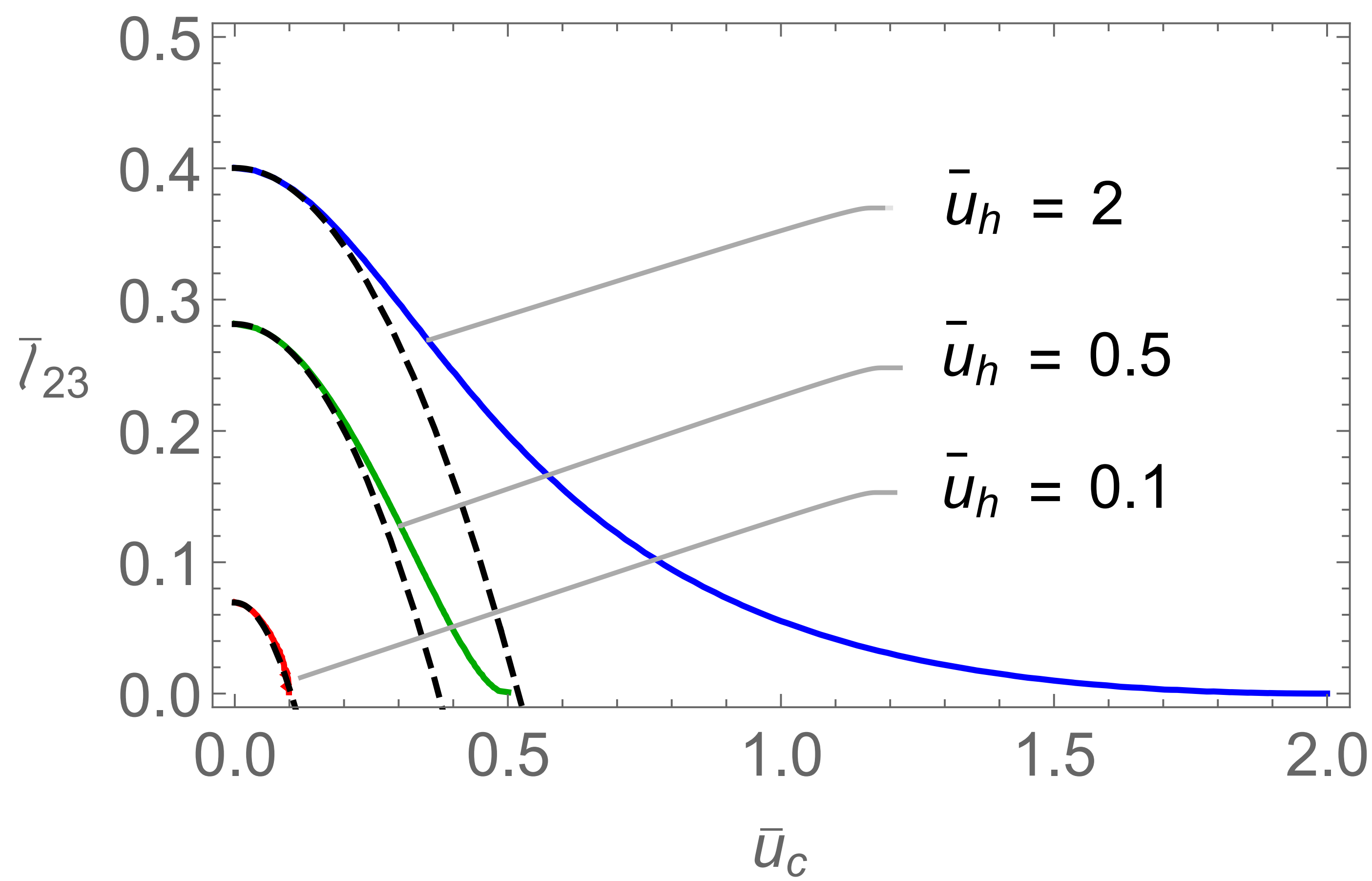
The mixing term goes to zero
when the distance between the intervals are large enough.

Our result 3

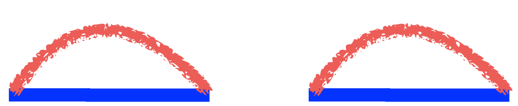
Phase diagram of S^{HEE} with finite cutoff

BTZ black hole with inverse temperature β





In the high temperature $l_{ij} \gg \beta$,

 is always dominant

even with finite cutoff.

Outline

1. Formula of Renyi entanglement entropy

2. Explicit computation in QFT

3. Holographic computation

Replica method
for a perturbative computation

Renyi entanglement entropy

$$S_n(A) = \frac{1}{1-n} \log \frac{Z_n(A)}{Z_1^n}$$

Z_n :partition function on $\mathcal{M}^n$

Perturbative action

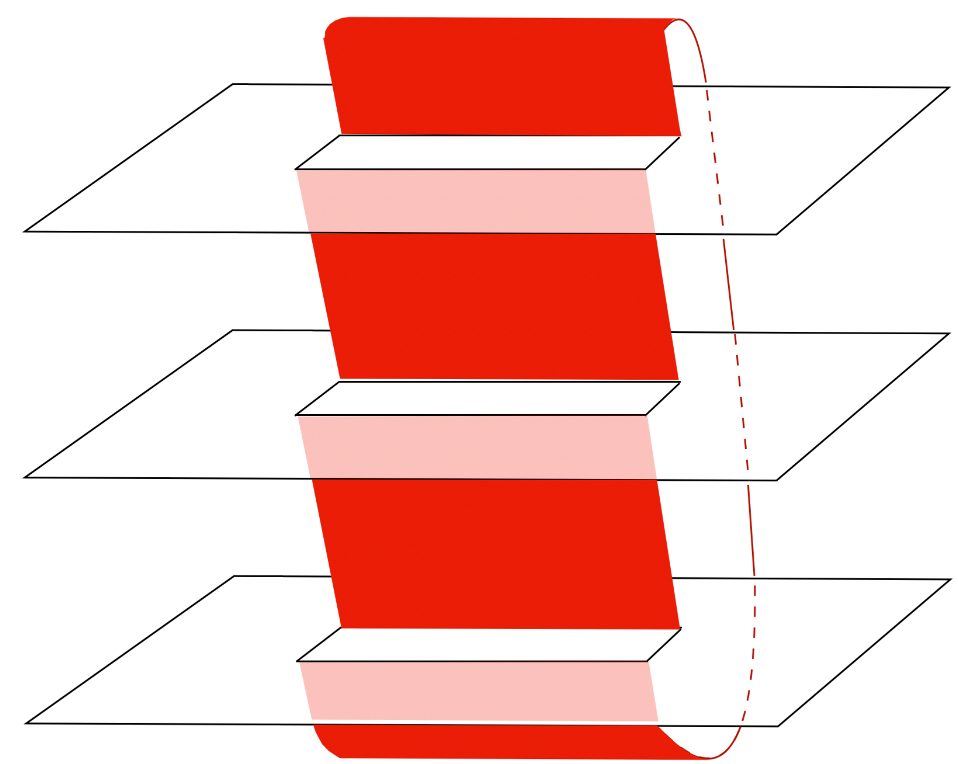
$$S_{\text{QFT}} = S_{\text{CFT}} + \mu \int_{\mathcal{M}^n} T\overline{T}$$

First order perturbation of $S_n(A)$

[B. Chen, L. Chen, P.-X. Hao, 2018]

$$\delta S_n(A) = \frac{\mu}{n-1} \left(\int_{\mathcal{M}^n} \langle T\overline{T} \rangle_{\mathcal{M}^n} - n \int_{\mathcal{M}} \langle T\overline{T} \rangle_{\mathcal{M}} \right)$$

Replica manifold $\mathcal{M}^3$ for a flat plane $\mathcal{M}$



Twist operator method

$$\int_{\mathcal{M}^n} \langle T\overline{T}\rangle_{\mathcal{M}^n} = \int_{\mathcal{M}} \frac{1}{n} \frac{\langle T^{(n)}(w)\overline{T}^{(n)}(\bar{w}) \prod_{i=1}^{m} \sigma_n(w_{2i-1},\bar{w}_{2i-1})\bar{\sigma}_n(w_{2i},\bar{w}_{2i})\rangle_{\mathcal{M}}}{\langle \prod_{i=1}^{m} \sigma_n(w_{2i-1},\bar{w}_{2i-1})\bar{\sigma}_n(w_{2i},\bar{w}_{2i})\rangle_{\mathcal{M}}}$$

$\sigma_n,\ \bar{\sigma}_n$: twist operators

$T^{(n)},\ \overline{T}^{(n)}$: total stress tensor of n replica fields

To compute it, we use

- Transformation of $T^{(n)}$ from a cylinder $\mathcal{M}$ to a plane $\mathcal{C}$

$$T^{(n)}(w) = \left(\frac{2\pi}{\beta}z\right)^2 T^{(n)}(z) - \frac{\pi^2 nc}{6\beta^2} \qquad z = e^{2\pi w/\beta}$$

- Ward identity

$$\langle T^{(n)}(z)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_{2m}(z_{2m},\bar{z}_{2m})\rangle_{\mathcal{C}}$$

$$= \sum_{j=1}^{2m} \left(\frac{h_j}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\right) \langle \mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_{2m}(z_{2m},\bar{z}_{2m})\rangle_{\mathcal{C}}$$

Formula of the first order perturbation of Renyi entanglement entropy

(2d CFT deformed by μ, m intervals, period β of Euclidean time)

$$
\begin{aligned}
\delta S_n(A) &= \frac{\mu}{n-1}\left(\int_{\mathcal{M}^n} \langle T\overline{T}\rangle_{\mathcal{M}^n} - n\int_{\mathcal{M}} \langle T\overline{T}\rangle_{\mathcal{M}}\right) \\
&= -\frac{\mu c}{12(n-1)}\frac{8\pi^4}{\beta^4} \\
&\times \int_{\mathcal{M}} \Bigg[z^2 \sum_{j=1}^{2m}\left(\frac{c(n-1/n)}{24(z-z_j)^2} + \frac{\partial_{z_j}\log\langle\prod_{i=1}^{m}\sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}}{(z-z_j)}\right) \\
&\qquad + \bar{z}^2 \sum_{j=1}^{2m}\left(\frac{c(n-1/n)}{24(\bar{z}-\bar{z}_j)^2} + \frac{\partial_{\bar{z}_j}\log\langle\prod_{i=1}^{m}\sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}}{(\bar{z}-\bar{z}_j)}\right)\Bigg] \\
&+ \frac{\mu}{n(n-1)}\frac{16\pi^4}{\beta^4}\frac{1}{\langle\prod_{i=1}^{m}\sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}} \\
&\times \int_{\mathcal{M}} z^2\left(\sum_{j=1}^{2m}\left(\frac{c(n-1/n)}{24(z-z_j)^2} + \frac{\partial_{z_j}}{(z-z_j)}\right)\right)\bar{z}^2\left(\sum_{j=1}^{2m}\left(\frac{c(n-1/n)}{24(\bar{z}-\bar{z}_j)^2} + \frac{\partial_{\bar{z}_j}}{(\bar{z}-\bar{z}_j)}\right)\right) \\
&\times \langle\prod_{i=1}^{m}\sigma_n(z_{2i-1},\bar{z}_{2i-1})\bar{\sigma}_n(z_{2i},\bar{z}_{2i})\rangle_{\mathcal{C}}
\end{aligned}
$$

Outline

1.Formula of Renyi entanglement entropy

2. Explicit computation in QFT

3. Holographic computation

Entanglement entropy of a single interval

$$\delta S(A) = -\mu \left(\frac{c}{12}\right)^2 \frac{8\pi^4}{\beta^4} \int_{\mathcal{M}} \left(\frac{z^2(z_1-z_2)^2}{(z-z_1)^2(z-z_2)^2} + \text{h.c.} \right)$$
$$= -\mu \frac{\pi^4 c^2 (x_2 - x_1)}{9\beta^3} \coth\left(\frac{\pi(x_2-x_1)}{\beta}\right)$$

The two point function for a single interval

x_1 x_2

$$\langle \sigma_n(z_1,\bar{z}_1)\bar{\sigma}_n(z_2,\bar{z}_2)\rangle_{\mathcal{C}} = \frac{c_n}{|z_1-z_2|^{\frac{c}{6}\left(n-\frac{1}{n}\right)}}$$

$$z_i = e^{2\pi x_i/\beta}$$

This computation is matched with the result in [B. Chen, L. Chen, P.-X. Hao, 2018] .

Entanglement entropy of two intervals in the deformed holographic CFTs

The four point function for the two intervals
in the holographic CFTs is factorized.
[T. Hartman, 2013]

$$\langle \sigma_n(z_1,\bar{z}_1)\bar{\sigma}_n(z_2,\bar{z}_2)\sigma_n(z_3,\bar{z}_3)\bar{\sigma}_n(z_4,\bar{z}_4)\rangle_{\mathcal{C}}$$

$$\sim \frac{1}{|z_1 - z_2|^{2(h_{\sigma_n}+\bar{h}_{\sigma_n})}|z_3 - z_4|^{2(h_{\sigma_n}+\bar{h}_{\sigma_n})}}$$

($n \to 1$, s-channel)

EE of the two intervals in the deformed holographic CFTs becomes summation of ones of the single interval.

$$\delta S_{\text{s-ch}}(A) \sim -\mu \frac{\pi^4 c^2 (x_2 - x_1)}{9\beta^3} \coth\left(\frac{\pi(x_2 - x_1)}{\beta}\right) - \mu \frac{\pi^4 c^2 (x_4 - x_3)}{9\beta^3} \coth\left(\frac{\pi(x_4 - x_3)}{\beta}\right)$$

Renyi entanglement entropy of a single interval

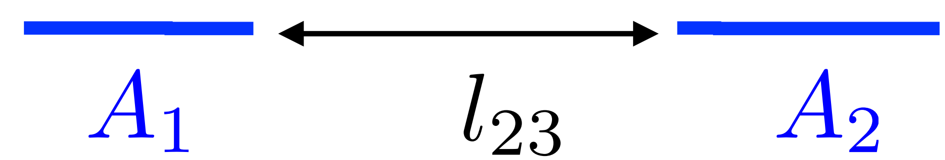
$$\delta S_n^{\mathrm{single}}(A) = -\frac{(n+1)\mu}{2n}\left(\frac{c}{12}\right)^2 \frac{8\pi^4}{\beta^4}\int_{\mathcal{M}}\left(\frac{z^2(z_1-z_2)^2}{(z-z_1)^2(z-z_2)^2}+\mathrm{h.c.}\right)$$
$$+\frac{(n+1)^2(n-1)\mu}{2n^3}\left(\frac{c}{12}\right)^2 \frac{8\pi^4}{\beta^4}\int_{\mathcal{M}}\frac{z^2(z_1-z_2)^2}{(z-z_1)^2(z-z_2)^2}\frac{\bar{z}^2(\bar{z}_1-\bar{z}_2)^2}{(\bar{z}-\bar{z}_1)^2(\bar{z}-\bar{z}_2)^2}$$

The second term becomes zero at $n \to 1$,
and the second integral diverges.

The similar divergence happens
in a perturbative computation
of the correlation function on a flat plane.
[P. Kraus, J. Liu, D. Marolf, 2018]

Renyi Entanglement entropy of two well-separated intervals

When the two intervals are well-separated,
the four point function is factorized.
(cluster decomposition)

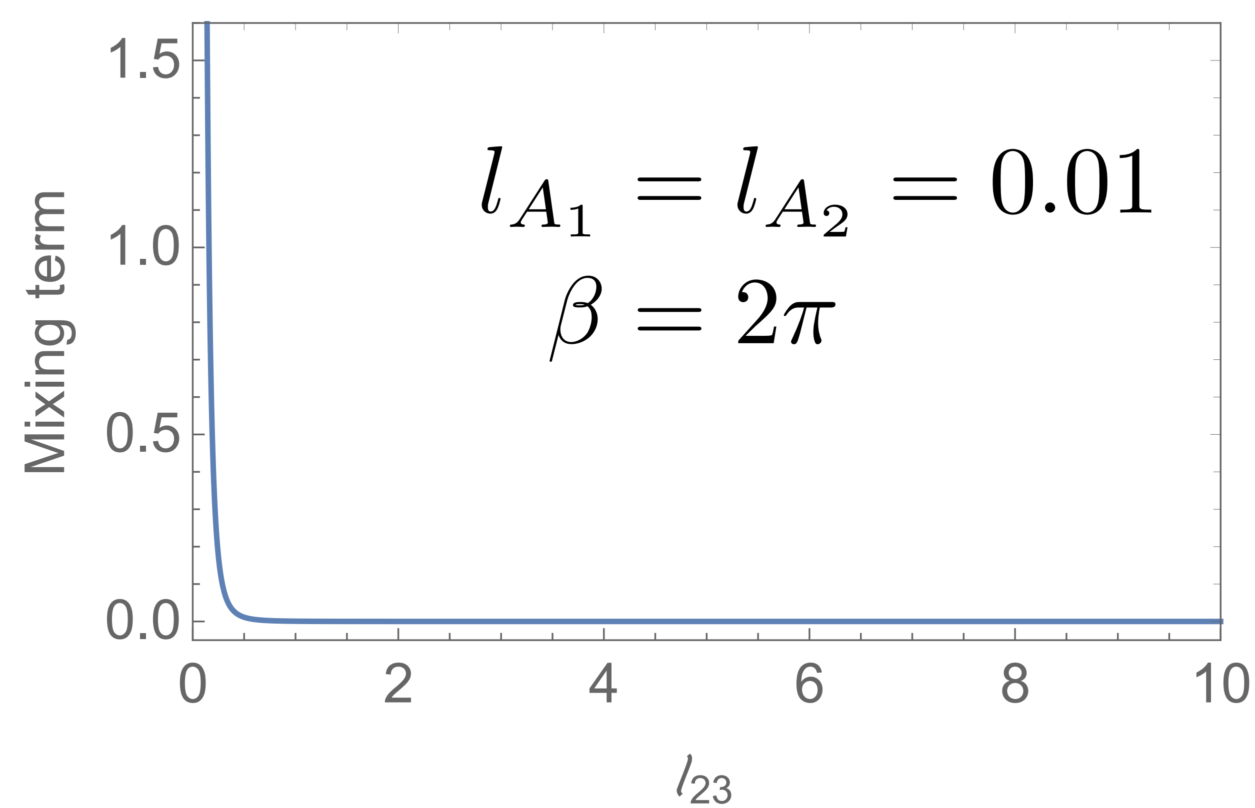


Even though the four point function is factorized,
REE includes a mixing term.

$$\delta S_n(A)|_{l_{23}\to\infty} \sim \delta S_n^{\text{single}}(A_1) + \delta S_n^{\text{single}}(A_2) + \delta S_n^{\text{mixing}}(A)$$

The mixing term goes to zero
when $l_{23} \to \infty$.

$\delta S_n(A)|_{l_{23}\to\infty}$ **has the additive property.**



Outline

1. Formula of Renyi entanglement entropy

2. Explicit computation in QFT

3. Holographic computation

Holographic prescription

in [B. Chen, L. Chen, P.-X. Hao, 2018]

BTZ black hole metric

$$\mathrm{d}s^2 = (r^2 - r_h^2)\mathrm{d}t^2 + \frac{1}{r^2 - r_h^2}\mathrm{d}r^2 + r^2\mathrm{d}\tilde{x}^2$$

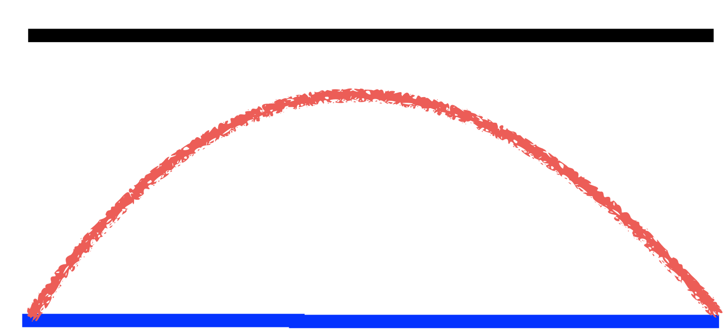
Boundary metric at $r = r_c$ (up to normalization)

$$\mathrm{d}s^2 \sim \mathrm{d}t^2 + \frac{1}{1 - r_h^2/r_c^2}\mathrm{d}\tilde{x}^2 \quad = \mathrm{d}t^2 + \mathrm{d}x^2$$

Bulk's length $\tilde{l}$ and QFT's length l

$$\frac{1}{\sqrt{1 - r_h^2/r_c^2}} \cdot \tilde{l} = l$$

Holographic entanglement entropy of a single interval with finite cutoff



HEE with perturbation by r_h^2/r_c^2

$$S^{HEE} \sim \underbrace{\frac{1}{2G}\log\left(\frac{2r_c \sinh\left(\frac{lr_h}{2}\right)}{r_h}\right)} - \frac{r_h^3}{8G\, r_c^2}\left(\underbrace{l\coth\left(\frac{lr_h}{2}\right)} - \underbrace{\frac{1}{r_h \sinh^2\left(\frac{lr_h}{2}\right)}}\right)$$

usual HEE

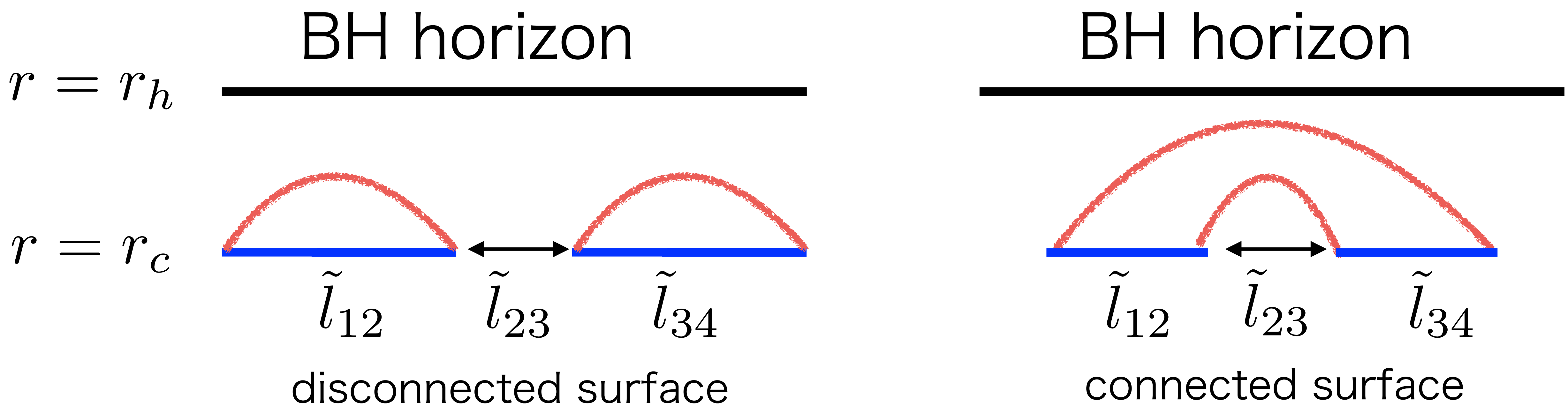
matches with the QFT computation

negligible at high temperature $lr_h \gg 1$

Holographic prescription in [B. Chen, L. Chen, P.-X. Hao, 2018] **at the high temperature matches the QFT computation.**

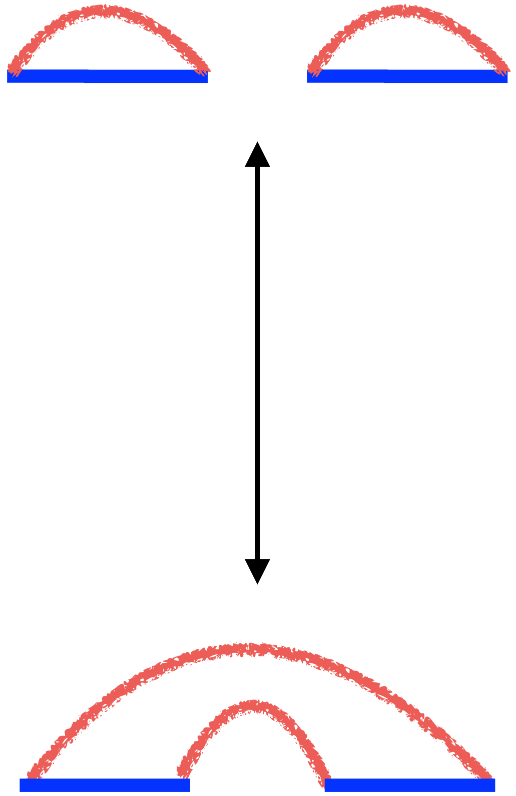
Our holographic computation:

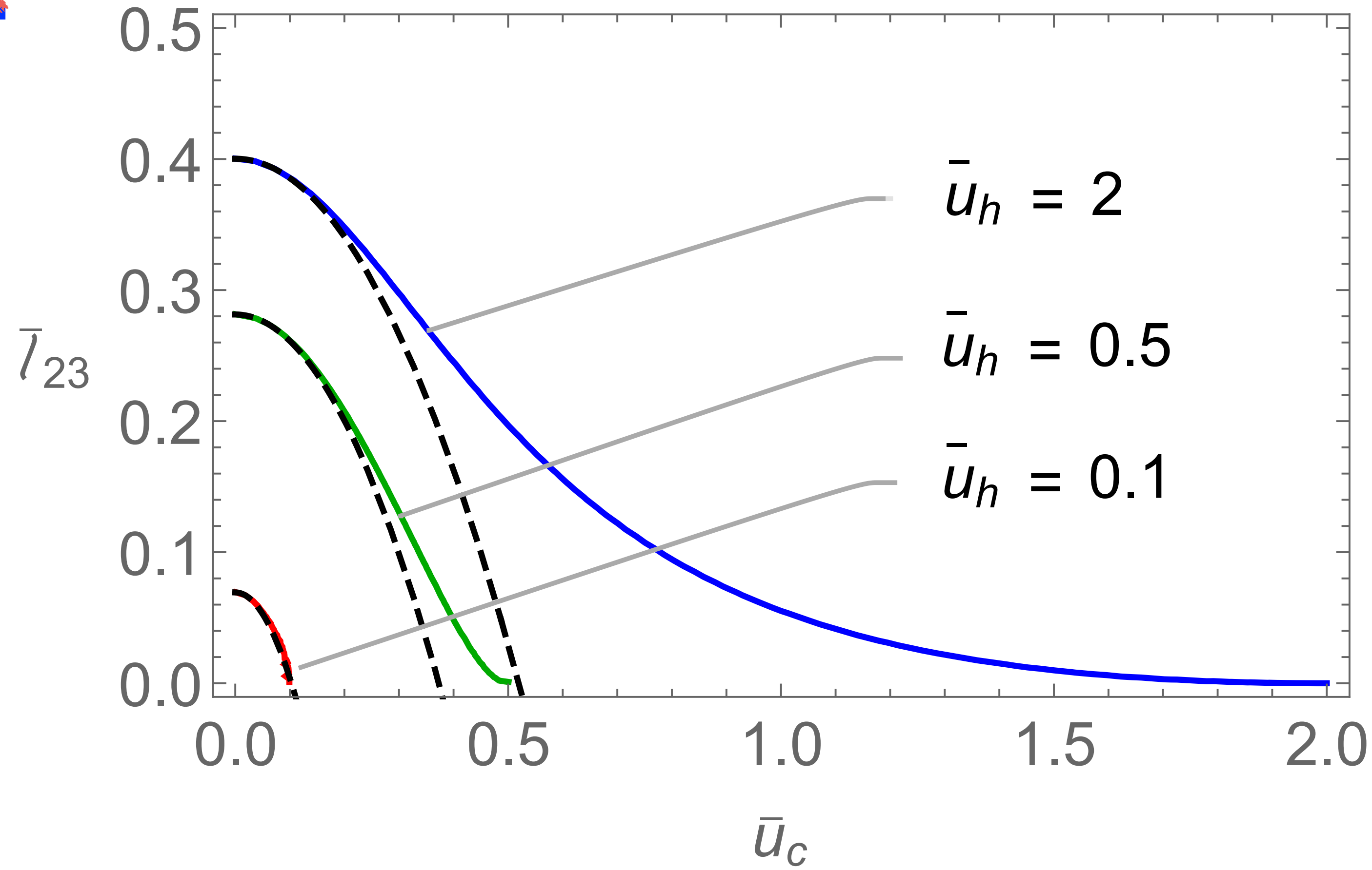
Holographic entanglement entropy
of two intervals with finite cutoff



We study phase transition of S^{HEE}
with finite cut off $r = r_c$
based on the prescription
in [B. Chen, L. Chen, P.-X. Hao, 2018].

Phase diagram of S^{HEE} with $l_{12}=l_{34}$



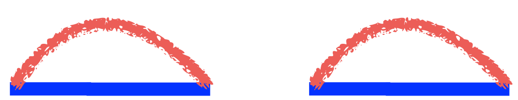


$$\bar{u}_c := l_{12}/r_c$$

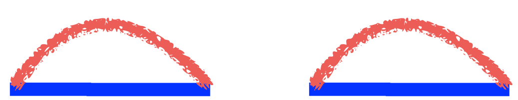
$$\bar{u}_h := l_{12}/r_h$$

$$\bar{l}_{23} := l_{23}/l_{12}$$

- When $\bar{u}_h$ decreases (high temperature with l_{12})

   becomes dominant.

- When $\bar{u}_c$ increases (large cutoff),

   becomes dominant.

- In the high temperature $l_{ij} \gg u_h$,

   is always dominant.

Summary

- We study Renyi and entanglement entropy of multiple intervals in the $T\overline{T}$-deformed CFTs on 2d cylinder.

- In the QFT side, we develop a formula of the Renyi entanglement entropy at the first order and check the additive property of some examples.

- In the holographic side, we study a phase transition of the holographic entanglement entropy of two intervals with a finite cutoff.