Localizing Schur correlation functions

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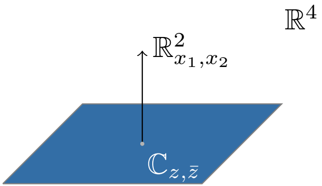
Introduction

Review: SCFT/VOA correspondence

- [Beem, Lemos, Liendo, **Peelaers**, Rastelli, van Rees, '13][Beem, Rastelli, '17]
- Consider $\forall$ 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$ on $\mathbb{R}^4_{x_1,x_2,x_3,x_4}$
  Symmetry generators $P, K, H, M, \tilde{M}, r, R, R^{\pm} | Q^I, \tilde{Q}_I, S_I, \tilde{S}^I$
- $\mathbb{R}^2_{x_3,x_4} = \mathbb{C}_{z,\bar{z}} \subset \mathbb{R}^4$ $(z \equiv x_3 + ix_4)$



Review: SCFT/VOA correspondence

- $\exists$ special supercharge $\mathcal{Q}$: $\mathcal{Q}^2 = 0$
- $\exists$ special operators: **Schur operators $\mathcal{O}(z,\bar{z})$** at $(z,\bar{z}) \in \mathbb{C}_{z,\bar{z}}$

$$\mathcal{Q}\text{-closed:} \qquad [\mathcal{Q}, \mathcal{O}(z,\bar{z})\} = 0$$

- $\mathcal{Q}$-Cohomology class $[\mathcal{O}(z,\bar{z})]$: independent of $\bar{z}$

$$\text{denote } [\mathcal{O}](z) \equiv [\mathcal{O}(z,\bar{z})]$$

- Induced OPE between classes: holomorphic

$$[\mathcal{O}_1(z,\bar{z})][\mathcal{O}_2(0,0)] = [\mathcal{O}_1](z)[\mathcal{O}_2](0) = \sum_k \underbrace{C_{12}^k(z)}_{\text{holomorphic coeff}} [\mathcal{O}_k](0)$$

Review: SCFT/VOA correspondence

- The space of Schur operators forms a vertex operator algebra (VOA) **$\chi_{\mathcal{T}}$ associated to $\mathcal{T}$**
- Basic properties (among many) of $\chi_{\mathcal{T}}$
  - Must $\exists$ Virasoro subalgebra $\mathcal{V}_{c=-12c_{4\mathrm{d}}} \subset \chi_{\mathcal{T}}$, generated by

$$T(z) = \sum_n L_n z^{-n-2} \equiv [(j_{\mathcal{R}})^{11}_{++\dot{+}}](z)$$

  - 4d **Schur index I_{Schur}** = **vacuum character** of $\chi_{\mathcal{T}}$

$$\underbrace{q^{\frac{c_{4\mathrm{d}}}{2}}\,\mathrm{tr}_{\mathcal{H}}(-1)^F q^{E-R} e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}}_{I_{\mathrm{Schur}}(q=e^{2\pi i\tau})} = \underbrace{q^{-\frac{c_{2\mathrm{d}}}{24}}\,\mathrm{tr}_{\chi_{\mathcal{T}}}(-1)^F q^{L_0}}_{ch_0[\chi_{\mathcal{T}}]}$$

Depends on $q = e^{2\pi i\tau}$, not β [Nishinaka's talk]

Review: localization

- Exact method to compute some path integral [Nekrasov, Pestun, Hama, Hosomichi, many others] [Hosseini's talk]
- Crucial for dualities: AGT, AdS/CFT, mirror symmetry
- Input:
  - Spacetime M (curved or flat)
  - Supermultiplets Φ and supersymmetry variations $\delta\Phi$
  - Supersymmetric action $S[\Phi]$ on M: $\delta S[\Phi] = 0$
  - Well chosen functional $V[\Phi]$ ("localizing/deformation term"):

$$(\delta V)|_{\text{bosonic}} \geq 0, \qquad \delta(\delta V[\Phi]) = 0$$

Review: Supersymmetric localization

- "Localization argument": if $\delta\mathcal{O}(x_i) = 0 \Rightarrow$ independence of $\mathfrak{s}$,

$$\int [D\Phi]\mathcal{O}(x_i)\, e^{-S[\Phi]} \;=\; \int [D\Phi]\mathcal{O}(x_i)\, e^{-S[\Phi]-\mathfrak{s}\cdot\delta V}$$

  Send $\mathfrak{s} \to +\infty$: exact WKB approximation, contributions from **BPS solutions** (zeros of $\delta V[\Phi]|_{\text{Bosonic}}$) only,

$$\text{above} = \int_{\text{BPSsoln}} \mathcal{O}|_{\text{BPS}} \cdot e^{-S|_{\text{BPS}}} \cdot Z_{1-\text{loop}} \cdot Z_{\text{non}-\text{pert}}$$

- When $\mathcal{O} = 1$: **partition function Z^M**

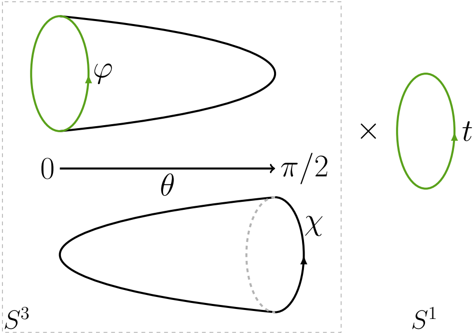
Goal

- Combine the two: **localization** in the context of **SCFT/VOA**
- Using path integral
  - Show $I_{\text{Schur}} = ch_0(\chi_{\mathcal{T}})$, aka, a T^2 partition function
  - Compute correlation functions of Schur operators
  - Explore surface defects in $\mathcal{T}$ and modular differential equations

Preparation

Spacetime background

- Topology: $S^3 \times S^1$: coordinate $(\varphi, \chi, \theta, t)$
  (Reason: I_{Schur} is a partition function $Z^{S^3 \times S^1}$)



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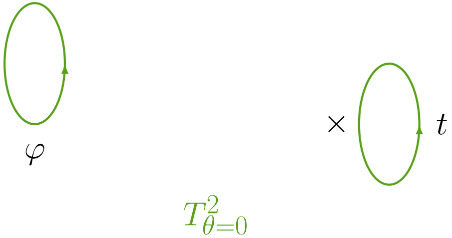

- "Boundary torus" $T^2_{\theta=0} \subset S^3 \times S^1$

Spacetime background

- Geometry: fixed by $\mathcal{N} = 2$ supersymmetry
- Rigid limit of supergravity [Closset, Dumitrescu, Festuccia, Seiberg, ...]
- Solve spinor PDEs (Killing spinor equations) [Hama, Hosomichi]

$$D_\mu \xi_I + T^{\lambda\rho} \sigma_{\lambda\rho} \sigma_\mu \tilde{\xi}_I = - i \sigma_\mu \tilde{\xi}'_I, \ldots$$

- Output:
  - SUSY transformation δ, parameter $\xi_I, \tilde{\xi}_I$
  - background fields $g_{\mu\nu}, T_{\mu\nu}, (V^r)_\mu \ldots$
- Output depend on (τ, β)

Superconformal theories

- Fields: $\mathcal{N} = 2$ vector multiplet and hypermultiplets

$$A_\mu, \underbrace{D_{IJ}}_{\text{hidden VIP}}, \lambda_I, \ldots; \qquad q_{IA}, \psi_A, \ldots$$

- Actions: S_{YM} and S_{HM} [Hama, Hosomichi, '12]

$$S_{\text{YM}} \equiv \int \sqrt{g} d^4 x \frac{1}{g_{\text{YM}}^2} \operatorname{tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \ldots + \frac{1}{2} D_{IJ} D^{IJ} \right)$$
$$S_{\text{HM}} \equiv \int \sqrt{g} d^4 x \left(\frac{1}{2} D_\mu q^{IA} D_\mu q_{IA} + \ldots + \frac{i}{2} q^{IA} (D_{IJ})_A{}^B q_B^J \right)$$

- Crucial: S_{YM} is δ-exact on our geometry, $S_{\text{YM}} = \delta \Xi$
  $\Rightarrow$ good localizing term

Localization

BPS solutions

- Localizing term $(S_{\text{total}} = S_{\text{YM}} + S_{\text{HM}} + \delta V)$

$$\delta V \equiv \underbrace{(\mathfrak{s}-1)S_{\text{YM}}}_{\text{VM sector}} + \underbrace{\mathfrak{s}' \int \sqrt{g} d^4x \, \delta\Big((\delta\psi)^\dagger \psi + (\delta\tilde{\psi})^\dagger \tilde{\psi}\Big)}_{\text{HM sector}}$$

- BPS solutions (zeros of $(\delta V)|_{\text{Bosonic}}$)
  (1) VM fields

$$A = \mathfrak{a} dt \text{ (flat connections)}, \qquad \phi = \tilde{\phi} = D_{IJ} = 0$$

(2) HM fields: q_{IA} solutions to elliptic PDE, one to one correspondence with boundary values on $T^2_{\theta=0}$

Classical and one-loop

- On BPS solutions

$$S_{\mathrm{HM}} \to S_{\beta\gamma}^{T^2} = -\frac{1}{\pi}\int_{T^2_{\theta=0}} \sqrt{g}\, d\varphi\, dt\, \epsilon^{AB} Q_A D_z^{\mathfrak{a}} Q_B, \quad S_{\mathrm{YM}} \to 0$$

$\Rightarrow$ $\beta\gamma$ system on T^2 (where $Q_A \equiv e^{\frac{i\varphi}{2}} q_{1A} - e^{-\frac{i\varphi}{2}} q_{2A}$)

- Quadratic fluctuations

$$Z_{\mathrm{HM-1-loop}} = 1, \quad Z_{\mathrm{VM-1-loop}} = \underbrace{\int [Db][Dc]' e^{-S_{bc}^{T^2}}}_{\text{full path integral}}$$

Schur index

- $I_{\text{Schur}} = Z^{S^3 \times S^1}$ localizes to

$$I_{\text{Schur}} \xrightarrow{\text{localize}} \frac{1}{W} \underbrace{\int d\mathfrak{a}}_{\text{VM BPS}} \left[\underbrace{\int [DQ_A]}_{\text{HM BPS}} e^{-S^{T^2}_{\beta\gamma}} \underbrace{\int [Db][Dc]' e^{-S^{T^2}_{bc}}}_{\text{VM one-loop}} \right]$$

$$= ch_0(\chi_{\mathcal{T}} = bc\beta\gamma \text{ quotient})$$

- Reproduce the known contour integral formula of I_{Schur}, e.g.

$$I_{\text{Schur}} = \frac{1}{W} \int d\mathfrak{a} \underbrace{\prod_\rho \frac{\eta(\tau)}{\theta_4(\rho(\mathfrak{a}))}}_{Z_{\beta\gamma}} \underbrace{\eta(\tau)^{\dim \mathfrak{g} - 3 \operatorname{rank} \mathfrak{g}} \prod_{\alpha \neq 0} \theta_1(\alpha(\mathfrak{a}))}_{Z_{bc}}$$

Naive Schur letters

- Gauge invariant operator ("word") $\mathcal{O}(x \in T^2_{\theta=0})$ from naive "Schur letters" on $S^3 \times S^1$

$$\left.\begin{array}{l} \lambda_z \equiv (\tilde{\xi}^I \tilde{\sigma}_z \lambda_I), \quad \tilde{\lambda}_z \equiv (\xi^I \sigma_z \tilde{\lambda}_I), \quad D_z \\ Q_A \equiv e^{\frac{i\varphi}{2}} q_{1A} - e^{-\frac{i\varphi}{2}} q_{2A} \end{array}\right\} \text{restricted on } T^2_{\theta=0}$$

- Fatal problem: $\delta\mathcal{O} \neq 0$ (**non-BPS**)

$$\delta\lambda_z = \delta\tilde{\lambda}_z \sim D_{IJ}, \quad \delta D_z = -(\lambda_z - \tilde{\lambda}_z) \Rightarrow \delta\mathcal{O} \sim D_{IJ}, \lambda_z, \tilde{\lambda}_z$$

$\Rightarrow$ localization argument doesn't apply!

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- Generalize localization argument

Generalize Localization Argument

- Allow $\mathcal{O}$ to depend on $\mathfrak{s}$

$$\langle \mathcal{O}_{\mathfrak{s}} \rangle_{\mathfrak{s}} \equiv \frac{1}{Z} \int D\Phi \mathcal{O}_{\mathfrak{s}} e^{-S[\Phi] - \mathfrak{s}\delta V}$$

Insist $\mathfrak{s}$-independence

$$\frac{d}{d\mathfrak{s}} \langle \mathcal{O}_{\mathfrak{s}} \rangle_{\mathfrak{s}} = 0 \Rightarrow \left\langle \partial_{\mathfrak{s}} \mathcal{O}_{\mathfrak{s}} - \mathcal{O}_{\mathfrak{s}} \delta V \right\rangle_{\mathfrak{s}} = 0$$

$\Rightarrow$ **Localizability** prior to localization, a full path integral equation needs to be verified

$\xrightarrow{\delta \mathcal{O}_{\mathfrak{s}} = \partial_{\mathfrak{s}} \mathcal{O}_{\mathfrak{s}} = 0}$ usual localization

New Schur letters

- Gauge invariant word $\mathcal{O}_{\mathfrak{s}}(x_i \in T^2_{\theta=0})$ from new letters

$$\underbrace{\sqrt{\mathfrak{s}} \cdot \lambda_z}_{\Lambda_z}, \quad \underbrace{\sqrt{\mathfrak{s}} \cdot \tilde{\lambda}_z}_{\tilde{\Lambda}_z}, \quad D_z, \quad Q_A$$

- Main claim: if $\delta^{\mathrm{os}}\mathcal{O}_{\mathfrak{s}}(x_i) = 0$, then

$$\frac{d}{d\mathfrak{s}}\langle \mathcal{O}_{\mathfrak{s}}(x_i)\rangle_{\mathfrak{s}} = \Big\langle \partial_{\mathfrak{s}}\mathcal{O}_{\mathfrak{s}} - \mathcal{O}_{\mathfrak{s}}\delta V \Big\rangle_{\mathfrak{s}} = 0$$

On-shell supercharge δ^{os} acts on letters by

$$\delta^{\mathrm{os}}\Lambda_z \sim \mu_z, \quad \delta^{\mathrm{os}}\tilde{\Lambda}_z \sim \mu_z, \quad \delta^{\mathrm{os}}D_z = \tilde{\lambda}_z - \lambda_z \quad \delta^{\mathrm{os}}Q_A = 0$$

Short proof

- Integrate out D_{IJ} in the path integral equation

$$\left\langle \frac{\partial}{\partial \mathfrak{s}} \mathcal{O}_{\mathfrak{s}} \right\rangle_{\mathfrak{s}} + (-1)^{F_{\mathcal{O}}} \left\langle \underbrace{\delta\mathcal{O}}_{\sim D_{IJ}} \quad \underbrace{\Xi}_{\sim D_{IJ}, \lambda_z, \tilde{\lambda}_z} \right\rangle_{\mathfrak{s}}$$

Second term splits (using D_{IJ} 1-pt and 2-pt functions)

$$\Rightarrow (-1)^{F_{\mathcal{O}}} \langle \delta\mathcal{O}\Xi \rangle_{\mathfrak{s}} = - \left\langle \frac{\partial}{\partial \mathfrak{s}} \mathcal{O}_{\mathfrak{s}} \right\rangle_{\mathfrak{s}} + (-1)^{F_{\mathcal{O}}} \langle \delta^{\mathrm{os}} \mathcal{O}_{\mathfrak{s}} \Xi \rangle_{\mathfrak{s}}$$

- The localizability condition

$$\frac{d}{d\mathfrak{s}} \langle \mathcal{O}_{\mathfrak{s}}(x_i) \rangle_{\mathfrak{s}} = \left\langle \partial_{\mathfrak{s}} \mathcal{O}_{\mathfrak{s}} - \mathcal{O}_{\mathfrak{s}} \delta V \right\rangle_{\mathfrak{s}} \sim \langle \delta^{\mathrm{os}} \mathcal{O}_{\mathfrak{s}} \rangle_{\mathfrak{s}} = 0$$

Localizing Schur operators

- $\delta^{\mathrm{os}}\mathcal{O}_{\mathfrak{s}}(x) = 0$: algebraic characteristics of Schur operators

  [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees, '13]

Schur operators are localizable

- The localization formula

$$\langle \mathcal{O}_{\mathfrak{s}=1} \rangle_{\mathfrak{s}=1} = \langle \mathcal{O}_{\mathfrak{s}} \rangle_{\mathfrak{s}} \xrightarrow{\mathfrak{s}\to+\infty} \frac{1}{W} \int d\mathfrak{a} \underbrace{\langle \mathcal{O} \rangle^{T^2}_{\beta\gamma bc}}_{\text{2d } \beta\gamma\text{bc VEV}} Z_{bc} Z_{\beta\gamma}$$

Modular Differential Equations

$\mathcal{N} = 4$ SYM

- Example: $\mathcal{N} = 4$ $SU(2)$ SYM
- $\chi_{\mathcal{T}}$ = "2d small $\mathcal{N} = 4$ superconformal algebra" ($c = -9$)
  - Bosonic generators $T(z)$, $J^{ij}(z)$

$$T \equiv \left[\mathrm{tr}\left(-\frac{1}{2\ell}\epsilon_{ij}Q^i D_z^{\mathfrak{a}} Q^j - \tilde{\lambda}_z\lambda_z\right)\right], \quad J^{ij} \equiv \left[-\frac{1}{2}\,\mathrm{tr}\, Q^i Q^j\right]$$

  - Fermionic generators $G^i \equiv \left[\mathrm{tr}\, Q^i\lambda_z\right]$, $\tilde{G}^i \equiv \left[\mathrm{tr}\, Q^i\tilde{\lambda}_z\right]$

$\mathcal{N} = 4$ SYM

- **Expect Sugawara condition: $T + J^{ij}J_{ij} = 0$**
- Verify $\langle (T + J^{ij}J_{ij})(z)X(w_i)\rangle = 0$ (q-series)
- Sample calculation: $X = 1$

$$\langle T \rangle = \frac{1}{I_{\mathcal{N}=4}} \oint d\mathfrak{a} Z_{\beta\gamma bc} \frac{1}{8\pi^2\ell^2} \left[\frac{2\vartheta_1''(2\mathfrak{a})}{\vartheta_1(2\mathfrak{a})} + \frac{\vartheta_1'''(0)}{\vartheta'(0)} - \sum_{n=-1}^{+1} \frac{\vartheta_4''(2n\mathfrak{a}+\mathfrak{b})}{\vartheta_4(2n\mathfrak{a}+\mathfrak{b})} \right]$$

$$\langle J^{ij}J_{ij}\rangle = -\frac{1}{I_{\mathcal{N}=4}} \oint d\mathfrak{a} Z_{\beta\gamma bc} \frac{1}{8\pi^2\ell^2} \left[\frac{3\vartheta_1'''(0)}{\vartheta_1'(0)} + \dots \right]$$

$\mathcal{N} = 4$ SYM

- The Sugawara condition $\Rightarrow$ 2nd order modular differential equation (MDE) [Beem, Peelaers]

$$\left[D_q^{(1)} - \frac{1}{2} D_b^2 + \frac{3}{2} E_2(\tau) + \ldots \right] I_{\text{Schur}} = 0$$

$D_q^{(1)} \sim q\partial_q,\ D_b^{(2)} \sim (b\partial_b)^2$
Covariant under $SL(2|\mathbb{Z})$

- Additional solutions $\sim$ **non-vacuum modules** of $\chi_{\mathcal{T}}$ [Gaberdiel, Keller]

Monodromy defects and extra solution

- Include singular background gauge field $A = \mathfrak{a}_\varphi d\varphi$
  - $F \sim \delta_{T^2_{\theta=\pi/2}}$ ($T^2_{\theta=\pi/2}$ "links" the boundary torus $T^2_{\theta=0}$)
  - gauge monodromy defect
- Recompute the Schur index with defect: new solution to the MDE at special $\mathfrak{a}_\varphi$
  $\Rightarrow$ monodromy defect $\sim$ non-vacuum module
- Related discussions on surface defects $\leftrightarrow$ non-vacuum module
  [Cordova, Gaiotto, Shao; Dedushenko, Fluder ...]

Summary and outlook

Summary

- Apply localization to SCFT/VOA correspondence
- Generalize localization argument
    - Localize Schur index
    - Schur operators are localizable
    - Integral formula for Schur correlation functions
- MDE additional solutions $\sim$ surface defect

Outlook

- Schur correlators on S^4 [Pan, Peelaers]?
- Relation to deformation quantization [Dedushenko, Pufu, Yacoby][Beem, Peelaers, Rastelli]
- Modular transformation of I_{Schur} and surface defects

Thank you!