Topological string geometry

Matsuo Sato (Hirosaki U.)

- String geometry and non-perturbative formulation of string theory
  M.S.   Int. J. Mod. Phys. A34 (2019)1950126

- Topological string geometry
  M.S. , Yuji Sugimoto (USTC)   arXiv:1903.05775

- String geometric phenomenology
  Masaki Honda (Waseda Univ.), M.S. in preparation

Motivation

String Geometry Theory arXiv:1709.03587 M.S.

- String geometry theory is a candidate of non-perturbative formulation of string theory.

- The theory unifies particles and the space-time.

- We can **derive the all-order perturbative scattering amplitudes that possess the super moduli in IIA, IIB and SO(32) I superstring theories from the single theory** by considering fluctuations around fixed perturbative IIA, IIB, SO(32) I vacua, respectively.

- The theory is background independent. (in preparation, Masaki Honda, M.S.)

Next task is to derive non-perturbative effects from the theory.

↓ Topological twist

Rather easy to derive non-perturbative effects.

- We may derive **non-perturbative corrections** to the partition function conjectured by Lockhart-Vafa, Hatsuda-Marino-Moriyama-Okuyama **from the ``first principle.”**

Review

String manifold

For presentations, bosonic closed only. In general, supersymmetric open and closed.

String manifolds are constructed by patching open sets of a model space E by general coordinate transformations.

model space $E := \bigcup_{\hat{D}} \{[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]\}$

- Σ Riemann surface

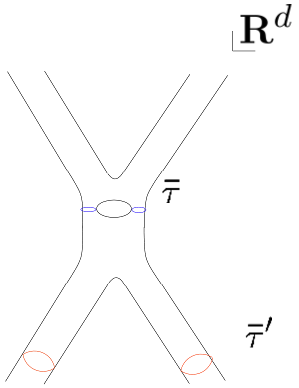
- $\bar{\tau}$: global time (Krichever-Novikov 1987)

  Defined based on Abelian differential that exists uniquely on Riemann surfaces.

  $\Sigma|_{\bar{\tau}} \cong S^1 \cup \cdots \cup S^1$: many body states of closed strings

- $X_{\hat{D}}(\bar{\tau}) : \Sigma|_{\bar{\tau}} \to \mathbf{R}^d$

  $\hat{D}$: backgrounds (B, dilaton) all fixed in a string manifold.



metric

- cotangent vectors

cotangent space of manifolds are spanned by continuous variables:

$$dX^{\mu}_{\hat{D}}(\bar{\sigma},\bar{\tau}) \;\ddot{=}\; dX^{(\mu\bar{\sigma})}_{\hat{D}}$$

Treat $(\mu\bar{\sigma})$ as indices.

$$d\bar{\tau} \;\ddot{=}\; dX^{d}_{\hat{D}}$$

summarize → $dX^{I}_{\hat{D}} \quad (I = d, (\mu\bar{\sigma}))$

- Σ is a discrete variable

Take summation by $\int d\bar{\sigma}\bar{e}(\bar{\sigma},\bar{\tau}) \qquad (\bar{e} := \sqrt{\bar{h}_{\bar{\sigma}\bar{\sigma}}})$

Invariant under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$

(*super transformation in the super summetric case)

- metric

$$ds^2(\bar{h},\bar{\tau},X_{\hat{D}}) = G_{IJ}(\bar{h},\bar{\tau},X_{\hat{D}})dX^{I}_{\hat{D}}dX^{J}_{\hat{D}}$$

Σ ⟷ metric $\bar{h}_{ab}$ up to diffeo x Weyl trans.

equivalent

Non-perturbative formulation of string theory

- $$Z = \int \mathcal{D}G \mathcal{D}A e^{-S}$$

$$S = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}\bar{\tau} \mathcal{D}X_{\hat{D}} \sqrt{G} (-R + \frac{1}{4} G_N G^{I_1 I_2} G^{J_1 J_2} F_{I_1 J_1} F_{I_2 J_2})$$

F_{IJ} : field strength of an u(1) gauge field A_I

Derive the all order perturbative string amplitudes

- Consider fluctuations around *a* **perturbative vacuum solution**.

  generalization of Majumdar-Papapetrou solution (1947, 1948)

The propagators of some modes of the fluctuations

$$\Delta_F(\bar{h}, \bar{\tau}, X(\bar{\tau});\ \bar{h},' \bar{\tau},' X'(\bar{\tau}'))$$

↓ Schwinger representation (1st quantization formalism)

$$\int_{h^{in}}^{h^{out}} \mathcal{D}h \int_{h^{in}}^{h^{out}} \mathcal{D}h' \Delta_F(\bar{h}, \infty, X^{out};\ \bar{h},' -\infty, X^{in}) = \int_{h^{in}, X^{in}}^{h^{out}, X^{out}} \mathcal{D}h \mathcal{D}X e^{-\lambda\chi} e^{-\int d^2\sigma \sqrt{h} \frac{1}{2} (\partial_m X^\mu)^2}$$

all order perturbative string amplitudes including its moduli

Topological Twist

Topological string theory in a superfield formalism

Topological twist

N=(2,2) SCFT → Topological string theory

- Make the fields couple with the 2-dim. gravitons
- Change the spins of the fermions

$$\Phi^I = \phi^I - \mathrm{i}\theta^{z^*}\bar\theta\nabla_{z^*}\phi^I - \mathrm{i}\theta\bar\theta^z\nabla_z\phi^I - \theta^{z^*}\bar\theta\theta\bar\theta^z\nabla_{z^*}\nabla_z\phi^I + \theta^{z^*}\rho^I_{z^*} - \mathrm{i}\theta^{z^*}\theta\bar\theta^z\nabla_z\rho^I_{z^*} + \theta\chi^I - \mathrm{i}\theta\theta^{z^*}\bar\theta\nabla_{z^*}\chi^I + \theta^{z^*}\theta F^I$$

spin: θ^{z^*}: -1, $\rho^I_{z^*}$: 1, θ: 0, χ^I: 0

Φ^I and $\Phi^{\bar I}$ satisfy chirality conditions:

$$\bar{\mathcal{D}}_z\Phi^I = \bar{\mathcal{D}}_{z^*}\Phi^I = \mathcal{D}_z\Phi^{\bar I} = \mathcal{D}_{z^*}\Phi^{\bar I} = 0$$

$$\{Q_A, \bar{\mathcal{D}}_z\}\Phi^I = \{Q_A, \bar{\mathcal{D}}_{z^*}\}\Phi^I = \{Q_A, \mathcal{D}_z\}\Phi^{\bar I} = \{Q_A, \mathcal{D}_{z^*}\}\Phi^{\bar I} = 0$$

Q_A: spin 0 supercharge

action
$$\int \mathrm{d}^2z\mathrm{d}^4\theta\sqrt{h}K(\Phi^I, \Phi^{\bar I})$$
$$= \int \mathrm{d}^2z\sqrt{h}\Big\{h^{ab}G_{I\bar J}\partial_a\phi^I\partial_b\bar\phi^{\bar J} - \mathrm{i}h^{ab}G_{I\bar J}\big(\bar\rho^{\bar J}_a D_b\chi^I + \rho^I_a D_b\bar\chi^{\bar J}\big)$$
$$-R_{I\bar J K\bar L}\rho^I_{z^*}\chi^K\bar\rho^{\bar J}_z\bar\chi^{\bar L} + G_{I\bar J}\big(F^I - \Gamma^I_{JK}\rho^J_{z^*}\chi^K\big)\big(\bar F^{\bar J} - \Gamma^{\bar I}_{\bar J\bar K}\bar\chi^{\bar J}\bar\rho^{\bar K}_z\big)\Big\}.$$

$$G_{I\bar J} = \partial_I\partial_{\bar J}K$$

Q exactness of the action

$$
\begin{aligned}
S &= \int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}\Phi\sqrt{G}G^{\mathbf{IK}}G^{\mathbf{JL}}(-R_{\mathbf{IJKL}} + \frac{1}{4}F_{\mathbf{IJ}}F_{\mathbf{KL}}) \\
&= \int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}\Phi\sqrt{G}\int d\bar{\sigma}d^4\theta\bar{e}G^{(\tilde{I}\bar{\sigma}\theta)\mathbf{K}}H_{(\tilde{I}\bar{\sigma}\theta)\mathbf{K}} \\
&= Q_A V
\end{aligned}
$$

$$
\begin{aligned}
V = \int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}\Phi\sqrt{G}\int d\bar{\sigma}\bar{e}\Big(& (i\nabla_- G_{+}^{\tilde{I}\mathbf{K}} + G_{+-\bar{-}}^{\tilde{I}\mathbf{K}})H_{0\tilde{I}\mathbf{K}} + G_{0}^{\tilde{I}\mathbf{K}}(i\nabla_- H_{+\tilde{I}\mathbf{K}} + H_{+-\bar{-}\tilde{I}\mathbf{K}}) \\
& +(i\nabla_- G_{+}^{\tilde{I}\mathbf{K}} + G_{+-\bar{-}}^{\tilde{I}\mathbf{K}})H_{0\tilde{I}\mathbf{K}} + G_{0}^{\tilde{I}\mathbf{K}}(i\nabla_- H_{+\tilde{I}\mathbf{K}} + H_{+-\bar{-}\tilde{I}\mathbf{K}}) \\
& +G_{\bar{-}}^{\tilde{I}\mathbf{K}}H_{+-\tilde{I}\mathbf{K}} + G_{+-}^{\tilde{I}\mathbf{K}}H_{\bar{-}\tilde{I}\mathbf{K}} - G_{+\bar{-}}^{\tilde{I}\mathbf{K}}H_{-\tilde{I}\mathbf{K}} - G_{-}^{\tilde{I}\mathbf{K}}H_{+\bar{-}\tilde{I}\mathbf{K}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
G(\theta) &= G_0 \\
&+ \theta^+ G_+ + \bar{\theta}^+ G_{\bar{+}} + \theta^- G_- + \bar{\theta}^- G_{\bar{-}} \\
&+ \theta^+\bar{\theta}^+ G_{+\bar{+}} + \theta^-\bar{\theta}^- G_{-\bar{-}} + \theta^+\theta^- G_{+-} + \bar{\theta}^+\theta^- G_{\bar{+}-} + \theta^+\bar{\theta}^- G_{+\bar{-}} + \bar{\theta}^+\bar{\theta}^- G_{\bar{+}\bar{-}} \\
&+ \bar{\theta}^+\theta^-\bar{\theta}^- G_{\bar{+}-\bar{-}} + \theta^+\theta^-\bar{\theta}^- G_{+-\bar{-}} + \theta^+\bar{\theta}^+\bar{\theta}^- G_{+\bar{+}\bar{-}} + \theta^+\bar{\theta}^+\theta^- G_{+\bar{+}-} \\
&+ \theta^+\bar{\theta}^+\theta^-\bar{\theta}^- G_4
\end{aligned}
$$

Results

- string geometry theory $\longrightarrow$ topological string geometry theory
  topological twist

- We have derived the all order perturbative topological string partition function from fluctuations around a perturbative vacuum solution in the theory.

- The action can be written in a **Q-exact** form.

  $\downarrow$

  We can apply **the localization teqniques**
  to obtain non-perturbative corrections to the partition function.

appendix

$$
\begin{aligned}
& \int d\bar{\sigma} d^4\theta G^{(\tilde{I}\bar{\sigma}\theta)\mathbf{K}} H_{(\tilde{I}\bar{\sigma}\theta)\mathbf{K}} \\
= & \int d\bar{\sigma} d^4\theta e^{\theta\cdot Q} G_0^{(\tilde{I}\bar{\sigma})\mathbf{K}} H_{0(\tilde{I}\bar{\sigma})\mathbf{K}} \\
= & \int d\bar{\sigma} Q_A Q_- Q_+ \bar{Q}_- G_0^{(\tilde{I}\bar{\sigma})\mathbf{K}} H_{0(\tilde{I}\bar{\sigma})\mathbf{K}} \\
= & \; Q_A V_0
\end{aligned}
$$

$$
\begin{aligned}
V_0 \; = \; \int \mathcal{D}h \mathcal{D}\bar{\tau} \mathcal{D}\Phi \sqrt{G} \int d\bar{\sigma}\bar{e}\Big(& (i\partial_- G_+^{\tilde{I}\mathbf{K}} + G_{+-\bar{-}}^{\tilde{I}\mathbf{K}}) H_{0\tilde{I}\mathbf{K}} + G_0^{\tilde{I}\mathbf{K}}(i\partial_- H_{+\tilde{I}\mathbf{K}} + H_{+-\bar{-}\tilde{I}\mathbf{K}}) \\
& + (i\partial_- G_+^{\tilde{I}\mathbf{K}} + G_{+-\bar{-}}^{\tilde{I}\mathbf{K}}) H_{0\tilde{I}\mathbf{K}} + G_0^{\tilde{I}\mathbf{K}}(i\partial_- H_{+\tilde{I}\mathbf{K}} + H_{+-\bar{-}\tilde{I}\mathbf{K}}) \\
& + G_{\bar{-}}^{\tilde{I}\mathbf{K}} H_{+-\tilde{I}\mathbf{K}} + G_{+-}^{\tilde{I}\mathbf{K}} H_{\bar{-}\tilde{I}\mathbf{K}} - G_{+\bar{-}}^{\tilde{I}\mathbf{K}} H_{-\tilde{I}\mathbf{K}} - G_{-}^{\tilde{I}\mathbf{K}} H_{+\bar{-}\tilde{I}\mathbf{K}} \Big)
\end{aligned}
$$

Non-perturbative Formulation of Superstring Theory Based on String Geometry

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Contents

0. Introduction

Elementary particle physics

Fundamental problems in elementary particle physics

- Many parameters that cannot be determined by the standard model
- Hierarchy problems
- How to describe the very early universe

⋮

Theory of gravity at Plank scale **= quantum gravity** is necessary

String theory is one of the most strong candidates of **quantum gravity**

String theory

- Is understood well perturbatively.
- Has many perturbatively stable vacua.

Advantages of perturbative string theory In **bottom up** point of view,

Can reproduce almost physics in the experimental region
by choosing a perturbative vacuum (geometry) appropriately.

cf. Calabi-Yau compactification, flux compactification

We can find a new phenomenological mechanism by understanding physics in terms of geometry.

(Chosen by hand)

Disadvantages of perturbative string theory In **top down** point of view,

We might say ``This is just replacing the problem of choosing parameters of physics to choosing geometry."

Cannot determine a vacuum. That is, **NO prediction**.

↓

Non-perturbative formulation of string theory is necessary.

(Chosen automatically)

Motivation

<u>T-duality</u>

- IIA string on a background ⟷ IIB string on another background

Observed values by strings coincide.

↓

"Space observed by strings are the same."

||

Expected to be "geometric principle of string theory."

Q. Is there **spaces T-dual to each other**? $\partial_a X'^9 = i\epsilon_{ab}\partial_b X^9$

A. Yes: **moduli spaces of Riemann surfaces embedded on-shell in the backgrounds**.

$X : \Sigma \to M$ ⟷ $X' : \Sigma \to M'$

↓ Extend off-shell

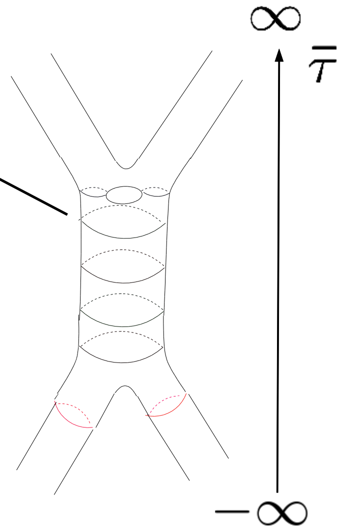
Extend non-perturbatively. (**Consider collections of** $\Sigma|_{\bar{\tau}=const.}$)

- Spaces where *curves parametrized by* $\bar{\tau}(-\infty < \bar{\tau} < \infty)$ *reproduce the right moduli space of the Riemann surfaces in a target manifold.*

||

string geometry

criterion to define string topology



- Construct string theory by regarding string geometry as geometric principle.

1. String geometry

1. 1 String model space

Global time $\bar{\tau}$

- On Σ, there exists an unique Abelian differential dp that has simple poles with residues f_i at Punctures Pi where $\Sigma_i f_i = 0$, if it is normalized to have purely imaginary periods with respect to all contours.

- Global time $\bar{\tau}$ is defined by $\bar{w} = \bar{\tau} + i\bar{\sigma} := \int^P dp$ (Krichever, Novikov 1987)

  $\bar{\tau} = +\infty \ (-\infty)$ on Pi with $f_i > 0 \ (f_i < 0)$

- Determine f_i

  0. $\Sigma_i f_i = 0$: f_i conservation law (if we choose the outgoing direction as positive.)

  1. Divide Pi s to arbitrary incoming and outgoing sets.

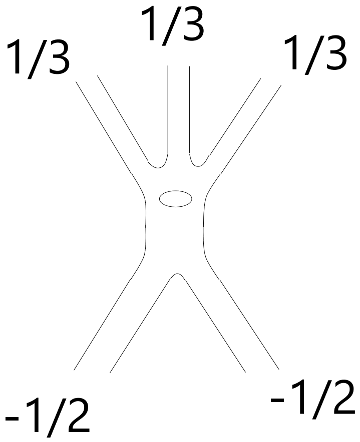
  2. Divide -1 to incoming $f^i \equiv \frac{-1}{N_{in}}$ and 1 to outgoing $f^i \equiv \frac{1}{N_{out}}$

  ↓

  - f_i are determined uniquely on Σ

  ↓

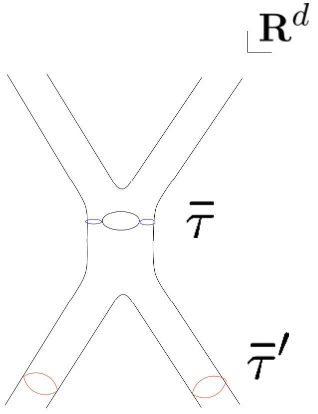
  - $\bar{\tau}$ is uniquely determined.



String model space E

Collection of string states $[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]$

- $\Sigma|_{\bar{\tau}} \cong S^1 \cup \cdots \cup S^1$ many body states of strings



- $X_{\hat{D}}(\bar{\tau}) : \Sigma|_{\bar{\tau}} \to \mathbf{R}^d$

  $\hat{D}$: backgrounds (B, dilaton)

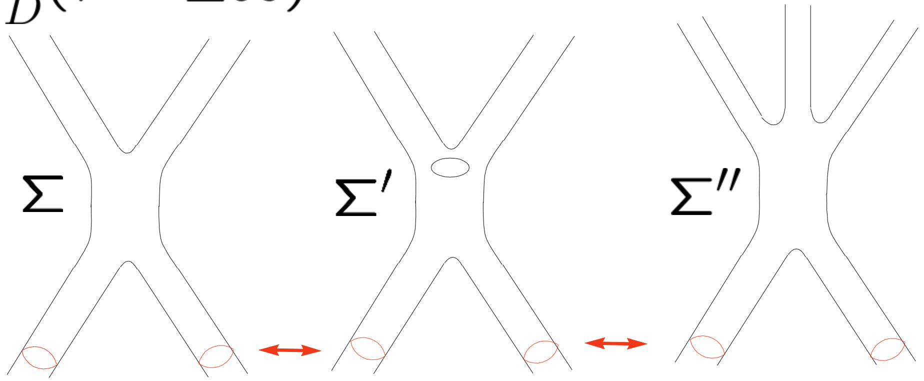
- $[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]$: equivalence class

  at $\bar{\tau} \cong \pm\infty$ $\quad \Sigma \cong C^2 \cup \cdots \cup C^2$

  Here, $\Sigma|_{\bar{\tau}\cong\pm\infty} = \Sigma'|_{\bar{\tau}\cong\pm\infty}, \; X_{\hat{D}}(\bar{\tau} \cong \pm\infty) = X'_{\hat{D}}(\bar{\tau} \cong \pm\infty)$

  ↓

  $$(\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong \pm\infty) \sim (\Sigma', X'_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong \pm\infty)$$



$(\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong -\infty) \sim (\Sigma', X'_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong -\infty) \sim (\Sigma'', X''_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong -\infty)$

- $E := \bigcup_{\hat{D}} \{[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]\}$

1. 2 String toplology

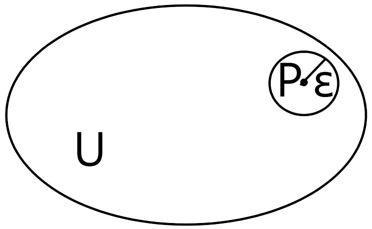
String topology

- ϵ open neighborhood

$$U([\Sigma, X_{0\hat{D}}(\bar{\tau}_0), \bar{\tau}_0], \epsilon) := \left\{ [\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}] \;\middle|\; \sqrt{|\bar{\tau} - \bar{\tau}_0|^2 + \| X_{\hat{D}}(\bar{\tau}) - X_{0\hat{D}}(\bar{\tau}_0) \|^2} < \epsilon \right\}$$

s.t. $$\| X_{\hat{D}}(\bar{\tau}) - X_{\hat{D}0}(\bar{\tau}_0) \|^2 = \int_0^{2\pi} d\bar{\sigma} (X^{\mu}_{\hat{D}}(\bar{\tau}, \bar{\sigma}) - X^{\mu}_{\hat{D}0}(\bar{\tau}_0, \bar{\sigma}))^2$$

- U is defined to be an open set if there exists ϵ such that an ϵ open neighborhood $\subset$ U for an arbitrary point P $\in$ U.



- The open sets satisfies the axiom of topology.

$$(i) \quad \emptyset, E \in \mathcal{U}$$

$$(ii) \quad U_1, U_2 \in \mathcal{U} \Rightarrow U_1 \cap U_2 \in \mathcal{U}$$

$$(iii) \quad U_\lambda \in \mathcal{U} \Rightarrow \cup_{\lambda \in \Lambda} U_\lambda \in \mathcal{U}$$

1. 2 String manifold

General coordinate transformation

- Σ does not transform to $\bar{\tau},\ X_{\hat{D}}$ and vice versa, because Σ is a discrete variable, whereas $\bar{\tau},\ X_{\hat{D}}$ are continuous variables by definition of the neighbourhoods.

- $\bar{\tau}$ and $\bar{\sigma}$ do not transform to each other because the string states are defined by $\bar{\tau}$ constant lines.

- Under these restrictions, the most general coordinate transformation is given by

$$[\bar{h}_{mn}(\bar{\sigma},\bar{\tau}),\bar{\tau},X^{\mu}_{\hat{D}}(\bar{\tau})] \mapsto [\bar{h}'_{mn}(\bar{\sigma}'(\bar{\sigma}),\bar{\tau}'(\bar{\tau},X_{\hat{D}}(\bar{\tau}))),\bar{\tau}'(\bar{\tau},X_{\hat{D}}(\bar{\tau})),X'^{\mu}_{\hat{D}}(\bar{\tau}')(\bar{\tau},X_{\hat{D}}(\bar{\tau}))]$$

$\Sigma \longleftrightarrow \bar{h}_{mn}(\bar{\sigma},\bar{\tau})$ up to diff × Weyl

- String manifolds $\mathfrak{M}$ are constructed by patching open sets of E by general coordinate transformations.

Example of string manifolds $\mathcal{M}_D$

- $\mathcal{M}_D := \{[\Sigma, x_D(\bar{\tau}), \bar{\tau}]\}$

  where $x_D(\bar{\tau}) : \Sigma|_{\bar{\tau}} \to M$ which has target mertic $ds^2 = dx_D^{\mu}(\bar{\tau}, \bar{\sigma}) dx_D^{\nu}(\bar{\tau}, \bar{\sigma}) G_{\mu\nu}(x_D(\bar{\tau}, \bar{\sigma}))$

- D: backgronds including the target metric . D is fixed on a string manifold.

- Open sets of $\mathcal{M}_D \longleftrightarrow$ Open sets of E

  homeomorphic

  diffeo: $[\Sigma, x_D(\bar{\tau}), \bar{\tau}] \mapsto [\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]$

  $X_{\hat{D}}(\bar{\tau})(x_D(\bar{\tau}))$ Is induced by the diffeomorphism transformation of the target space

  $x \mapsto X = X(x)$

  $\downarrow$

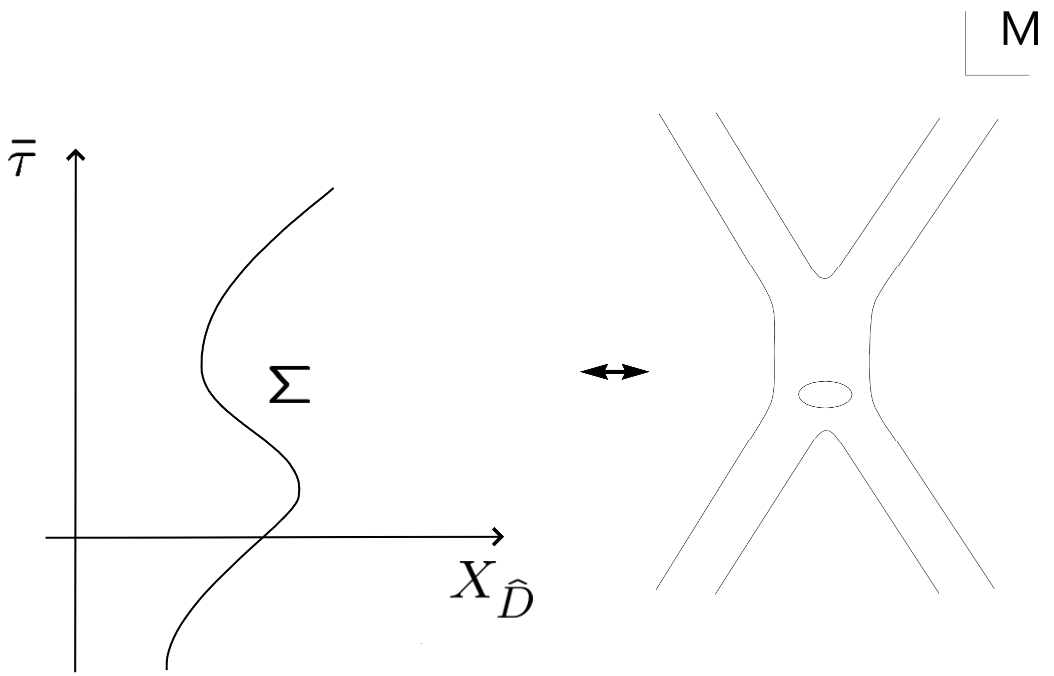
  $X_{\hat{D}}(\bar{\tau}, \bar{\sigma}) = X(x_D(\bar{\tau}, \bar{\sigma}))$

Example of string manifolds $\mathcal{M}_D$ (cont'd)

- Trajectories in asymptotic processes on $\mathcal{M}_D$ represents 2-dim. Riemann surfaces in the target manifold.

↓

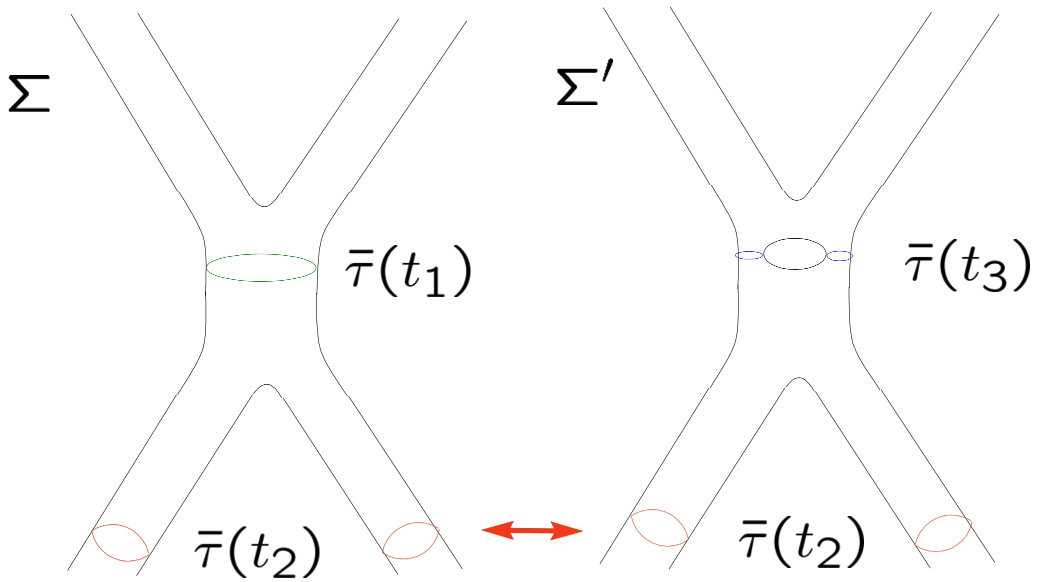
reproduce the right moduli space.



Example of string manifolds $\mathcal{M}_D$ (cont'd)

- By a general trajectory,
  string states on different two-dimensional Riemann surfaces that have **different genus numbers** can be connected continuously.

v.s. the moduli space



1. 4 Riemannian string manifold

Riemannian string manifold

- cotangent vectors

cotangent space of manifolds are spanned by continuous variables:

$$dX^{\mu}_{\hat{D}}(\bar{\sigma},\bar{\tau}) = dX^{(\mu\bar{\sigma})}_{\hat{D}}$$

Treat $(\mu\bar{\sigma})$ as indices.

$$d\bar{\tau} = dX^{d}_{\hat{D}}$$

summarize

$$dX^{I}_{\hat{D}} \quad (I = d, (\mu\bar{\sigma}))$$

Take summation by $\int d\bar{\sigma}\bar{e}(\bar{\sigma},\bar{\tau})$ $\quad (\bar{e} := \sqrt{\bar{h}_{\bar{\sigma}\bar{\sigma}}})$

Invariant under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$

Transformed as scalar under $\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X)$

- metric

$$ds^2(\bar{h},\bar{\tau},X_{\hat{D}}) = G_{IJ}(\bar{h},\bar{\tau},X_{\hat{D}})dX^{I}_{\hat{D}}dX^{J}_{\hat{D}}$$

2 Non-perturbative formulation of string theory

Non-perturbative formulation of superstring theory

- $$Z = \int \mathcal{D}G \mathcal{D}A e^{-S}$$

$$S = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}\bar{\tau} \mathcal{D}X_{\hat{D}} \sqrt{G} (-R + \frac{1}{4} G_N G^{I_1 I_2} G^{J_1 J_2} F_{I_1 J_1} F_{I_2 J_2})$$

$\mathcal{D}h$: the invariant measure of h_{mn} on Σ .

$h_{mn} \quad \longleftrightarrow \quad \bar{h}_{mn}$

diff × Weyl

$F_{\mathbf{IJ}}$: field strength of an u(1) gauge field $A_{\mathbf{I}}$

- **The theory is background independent.**

diffeomorphism invariance

- Under $(\bar{\tau}, X) \mapsto (\bar{\tau}'(\bar{\tau}, X), X'(\bar{\tau}, X))$

  $G_{IJ}(\bar{h}, \bar{\tau}, X)$: symmetric tensor $A_I(\bar{h}, \bar{\tau}, X)$: vector

  Action is manifestly invariant

- Under $\bar{h}_{mn} \to \bar{h}'_{mn}$, $G_{IJ}(\bar{h}, \bar{\tau}, X)$ and $A_I(\bar{h}, \bar{\tau}, X)$ are defined as scalars

  Under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$, the fields that have index $\bar{\sigma}$ transform as scalars.
  $\int d\bar{\sigma}\bar{e}(\bar{\sigma}, \bar{\tau})$ is invariant.

↓

The action is invariant under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$

* In a supersymmetric case, the action is invariant under $(\bar{\sigma}, \bar{\theta}^{\alpha}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'^{\alpha}(\bar{\sigma}, \bar{\theta}))$

3 String geometry solution that represents a perturbative vacuum of string theory

Perturbative vacuum solution (Extension of Majumdar-Papapetrou solution (1947,1948))

- $$\bar{ds}^2 = 2\lambda\bar{\rho}(\bar{h})N^2(X)(dX^d)^2 + \int d\bar{\sigma}\bar{e}\int d\bar{\sigma}'\bar{e}' N^{\frac{2}{2-D}}(X)\frac{\bar{e}^3(\bar{\sigma},\bar{\tau})}{\sqrt{\bar{h}(\bar{\sigma},\bar{\tau})}}\delta_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}dX^{(\mu\bar{\sigma})}dX^{(\mu'\bar{\sigma}')}$$

$$\bar{A}_d = i\sqrt{\frac{2-2D}{2-D}}\frac{\sqrt{2\lambda\bar{\rho}(\bar{h})}}{\sqrt{G_N}}N(X), \qquad \bar{A}_{(\mu\bar{\sigma})} = 0$$

is a solution to the equations of motion. ($\bar{h}_{mn}(\bar{\sigma},\bar{\tau})$, $\bar{\tau}$, $X^\mu(\bar{\sigma},\bar{\tau})$ are all independent.)

where $\bar{\rho}(\bar{h}) := \frac{1}{4\pi}\int d\bar{\sigma}\sqrt{\bar{h}}\bar{R}_{\bar{h}}$ ($\bar{R}_{\bar{h}}$ is the scalar curvature of $\bar{h}_{mn}$)

$$D := \int d\bar{\sigma}\bar{e}\delta_{(\mu\bar{\sigma})(\mu\bar{\sigma})} = d2\pi\delta(0) \quad \text{(index volume)}$$

$$N(X) = \frac{1}{1+v(X)} \quad \left(v(X) = \frac{\alpha}{\sqrt{d-1}}\int d\bar{\sigma}\epsilon_{\mu\nu}X^\mu\partial_{\bar{\sigma}}X^\nu\right)$$

- We derive all the perturbative string amplitudes on flat spacetime from the fluctuations around this solution.
- The solution is defined on $\mathcal{M}_D$ where the target metric is fixed to be flat .
- The equations of motion are differential equations with respect to $\bar{\tau}$, $X^\mu(\bar{\sigma},\bar{\tau})$

↓

The functions of $\bar{h}_{mn}(\bar{\sigma},\bar{\tau})$ are constants in the solution **(determined by the consistency of the fluctuations.)**

4 Derive all order scattering amplitudes of perturbative string

Propagators around the perturbative vacuum

1. Expand the action around the perturbtive vacuum up to 2nd order: $G_{IJ} = \bar{G}_{IJ} + \tilde{G}_{IJ}$, $A_I = \bar{A}_I + \tilde{A}_I$

2. Take $G_N \to 0$. Then, the fluctuations of the gaguge field are suppressed.

3. Take the harmonic gauge to fix diffeo. Then, the gauge fixing term is added.

$$S_{fix} = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}\bar{\tau} \mathcal{D}X \sqrt{\bar{G}} \frac{1}{2} \Big(\bar{\nabla}^J (\tilde{G}_{IJ} - \frac{1}{2} \bar{G}_{IJ} \tilde{G}) \Big)^2$$

4. Take slowly varying field limit:

derivative expansion
$$\begin{cases} \tilde{G}_{IJ} \to \frac{1}{\alpha} \tilde{G}_{IJ} \\ \partial_K \tilde{G}_{IJ} \to \partial_K \tilde{G}_{IJ} \\ \partial_K \partial_L \tilde{G}_{IJ} \to \alpha \partial_K \partial_L \tilde{G}_{IJ} \end{cases}$$
and $\alpha \to 0$

5. Normalize to obtain canonical kinetic term: $\tilde{H}_{IJ} := Z_{IJ} \tilde{G}_{IJ}$

6. Take $D \to \infty$

Propagators around the perturbative vacuum (cont'd)

- $S+S_{fix}=\int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X\frac{1}{4}\tilde{H}H(-i\frac{\partial}{\partial\bar{\tau}},-i\frac{1}{\bar{e}}\frac{\partial}{\partial X},X,\bar{h})\tilde{H}$ +(terms do not mix with $\tilde{H}$)

  $\tilde{H}$ is one of the modes of $\tilde{H}_{d\,(\mu\bar{\sigma})}$

$$H(p_{\bar{\tau}},p_X,X,h)=\frac{1}{2}\frac{1}{2\lambda\bar{\rho}}p_{\bar{\tau}}^2+\int_0^{2\pi}d\bar{\sigma}\left(\sqrt{\bar{h}}\left(\frac{1}{2}(p_X^\mu)^2+\frac{1}{2}\bar{e}^{-2}(\partial_{\bar{\sigma}}X^\mu)^2\right)+i\bar{e}\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X_\mu p_X^\mu\right)$$

ADM decomposition $\bar{h}_{mn}=\begin{pmatrix}\bar{n}^2+\bar{n}_{\bar{\sigma}}\bar{n}^{\bar{\sigma}} & \bar{n}_{\bar{\sigma}}\\ \bar{n}_{\bar{\sigma}} & \bar{e}^2\end{pmatrix}$

- Differential equation for the propagator $\Delta_F(\bar{h},\bar{\tau},X;\ \bar{h}',\bar{\tau}',X')$

$$H(-i\frac{\partial}{\partial\bar{\tau}},-i\frac{1}{\bar{e}}\frac{\partial}{\partial X},X,\bar{h})\Delta_F(\bar{h},\bar{\tau},X;\ \bar{h}',\bar{\tau}',X')=\delta(\bar{h}-\bar{h}')\delta(\bar{\tau}-\bar{\tau}')\delta(X-X')$$

Schwinger representation of the propagator = path integral of the perturbative strings

- In order to compare with perturbative strings,
  Take the Schwinger representation of the propagator by using the first quantization formalism.

  operators $(\hat{\bar{h}}, \hat{\bar{\tau}}, \hat{X})$ conjugate momenta $(\hat{p}_{\bar{h}}, \hat{p}_{\bar{\tau}}, \hat{p}_X)$ eigen states $|\bar{h}, \bar{\tau}, X >$

- $$H(-i\frac{\partial}{\partial\bar{\tau}}, -i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\Delta_F(\bar{h}, \bar{\tau}, X;\ \bar{h}',\bar{\tau}',X') = \delta(\bar{h}-\bar{h}')\delta(\bar{\tau}-\bar{\tau}')\delta(X-X')$$

  $\Updownarrow$

  $$\begin{aligned}\Delta_F(\bar{h}, \bar{\tau}, X;\ \bar{h}',\bar{\tau}',X') &= <\bar{h}, \bar{\tau}, X|\hat{H}^{-1}(\hat{p}_{\bar{\tau}}, \hat{p}_X, \hat{X}, \hat{\bar{h}})|\bar{h}',\bar{\tau}',X'> \\ &= \int_0^\infty dT <\bar{h}, \bar{\tau}, X|e^{-T\hat{H}}|\bar{h}',\bar{\tau}',X'>\end{aligned}$$

- $$\Delta_F(X;\ X') := \int_0^\infty dT < X|_{out} e^{-T\hat{H}}|X'>_{in}$$

  $$< X|_{out} := \int \mathcal{D}h < \bar{h}, \bar{\tau}=\infty, X|$$
  $$|X'>_{in} := \int \mathcal{D}h' |\bar{h}', \bar{\tau}=-\infty, X'>$$

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- path integral representation

$$\begin{aligned}
&\Delta_F(X;\ X') \\
=\ &\int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X \int \mathcal{D}p_T\mathcal{D}p_{\bar{\tau}}\mathcal{D}p_X \\
&\exp\Bigg(-\int_0^1 dt\Big(-ip_T(t)\frac{d}{dt}T(t) - ip_{\bar{\tau}}(t)\frac{d}{dt}\bar{\tau}(t) - ip_X(t)\cdot\frac{d}{dt}X(t) \\
&+T(t)H(p_{\bar{\tau}}(t), p_X(t), X(t), \bar{h})\Big)\Bigg)
\end{aligned}$$

* By introducing $p_T(t)$, constant T → field T(t)

- move onto Lagrange formalism from the canonical formalism by integrating out $p_{\bar{\tau}},\ p_X$.

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- $$\Delta_F(X;\ X')$$

$$= \int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X\mathcal{D}p_T$$

$$\exp\Bigg(-\int_0^1 dt\Big(-ip_T(t)\frac{d}{dt}T(t) + \lambda\bar{\rho}\frac{1}{T(t)}(\frac{d\bar{\tau}(t)}{dt})^2$$

$$+\int d\bar{\sigma}\sqrt{\bar{h}}(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^{\mu}(\bar{\sigma},\bar{\tau},t)\partial_t X_{\mu}(\bar{\sigma},\bar{\tau},t) + \bar{h}^{01}\partial_t X^{\mu}(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_{\mu}(\bar{\sigma},\bar{\tau},t)$$

$$+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}} X^{\mu}(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_{\mu}(\bar{\sigma},\bar{\tau},t))\Big)\Bigg)$$

- This path integral is obtained

  if $F_1(t) := \frac{d}{dt}T(t) = 0$ gauge is chosen in the next covariant form w.r.t. t diffeo:

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- Covariant form w.r.t. t diffeo

$$
\begin{aligned}
&\Delta_F(X;\ X') \\
=\ & Z_1 \int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X \,\text{exp}\Bigg(- \int_0^1 dt \Big(+\lambda\bar{\rho}\frac{1}{T(t)}(\frac{d\bar{\tau}(t)}{dt})^2 \\
& + \int d\bar{\sigma}\sqrt{\bar{h}}(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_t X_\mu(\bar{\sigma},\bar{\tau},t) + \bar{h}^{01}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_\mu(\bar{\sigma},\bar{\tau},t) \\
& + \frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}}X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}}X_\mu(\bar{\sigma},\bar{\tau},t))\Big)\Bigg)
\end{aligned}
$$

* $T(t)$ is transformed as an einbein.

- $T(t)$ disappears under $\dfrac{d\bar{\tau}}{d\bar{\tau}'} = T(t)$:

$$
\begin{aligned}
\bar{h}^{00} &= T^2\bar{h}'^{00} & \sqrt{\bar{h}} &= \frac{1}{T}\sqrt{\bar{h}'} & (\frac{d\bar{\tau}(t)}{dt})^2 &= T^2(\frac{d\bar{\tau}'(t)}{dt})^2 \\
\bar{h}^{01} &= T\bar{h}'^{01} & & & & \\
\bar{h}^{11} &= \bar{h}'^{11} & \bar{\rho} &= \frac{1}{T}\bar{\rho}' & &
\end{aligned}
$$

* This action is still invariant under the diffeomorphism with respect to t if $\bar{\tau}$ transforms in the same way as t.

- Take $\bar{\tau} = t$ gauge.

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- $$\Delta_F(X;\ X')$$

$$= Z\int_{X'}^{X}\mathcal{D}h\mathcal{D}X\exp\Bigg(-\int d\bar{\tau}\int d\bar{\sigma}\sqrt{\bar{h}}(\frac{\lambda}{4\pi}\bar{R}(\bar{\sigma},\bar{\tau})$$

$$+\frac{1}{2}\bar{h}^{00}\partial_{\bar{\tau}}X^{\mu}(\bar{\sigma},\bar{\tau})\partial_{\bar{\tau}}X_{\mu}(\bar{\sigma},\bar{\tau})+\bar{h}^{01}\partial_{\bar{\tau}}X^{\mu}(\bar{\sigma},\bar{\tau})\partial_{\bar{\sigma}}X_{\mu}(\bar{\sigma},\bar{\tau})+\frac{1}{2}\bar{h}^{11}\partial_{\bar{\sigma}}X^{\mu}(\bar{\sigma},\bar{\tau})\partial_{\bar{\sigma}}X_{\mu}(\bar{\sigma},\bar{\tau}))\Bigg)$$

- Diff $\times$ Weyl transformation gives

$$\Delta_F(X;\ X') = Z\int_{X'}^{X}\mathcal{D}h\mathcal{D}X e^{-\lambda\chi}e^{-S_s}$$

$$S_s = \int_{-\infty}^{\infty}d\tau\int d\sigma\sqrt{h(\sigma,\tau)}\left(\frac{1}{2}h^{mn}(\sigma,\tau)\partial_m X^{\mu}(\sigma,\tau)\partial_n X_{\mu}(\sigma,\tau)\right)$$

χ: Euler number

- We obtain the all-order perturbative scattering amplitudes that possess the moduli in the string theory, by inserting asymptotic states.

- The consistency of the fluctuations around the backgrounds → **the critical dimension d=26**. (**d=10** in the supersymmetric cases)

5 General supersymmetric case that includes open strings

Supersymmetric generalization including open strings

So far	General
Riemann surface Σ	super Riemann surface $\mathbf{\Sigma}$ with or without boundaries
$X_{\hat{D}} : \Sigma\|_{\bar{\tau}} \to \mathbf{R}^d$	$\mathbf{X}_{\hat{D}} : \mathbf{\Sigma}\|_{\bar{\tau}} \to \mathbf{R}^d$ Boundaries have CP factors and map to D-branes
$\hat{D}$: background (B, dilaton)	$\hat{D}$: background (B, dilaton , RR, submanifolds of M that represent D-branes and O-planes gauge fields on D-branes)
model space $E := \bigcup_{\hat{D}} \{[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]\}$	$\mathbf{E} := \bigcup_{\hat{D}_T} \{[\mathbf{\Sigma}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau}]\}$ (T= IIA, IIB, I) • For T=I, Ω projected • For T=IIA (T=IIB, I), IIA (IIB) GSO projection is attached on asymptotic states * We can define GSO projection because functions over the model space are functions of ψ^{μ}_{α} $\boldsymbol{X}^{\mu}_{\hat{D}_T} = X^{\mu} + \bar{\theta}^{\alpha}\psi^{\mu}_{\alpha} + \frac{1}{2}\bar{\theta}^2 F^{\mu}$
index $(\mu\bar{\sigma})$	$(\mu\bar{\sigma}\bar{\theta})$

Non-perturbative formulation of superstring theory

- $$Z = \int \mathcal{D}G\mathcal{D}Ae^{-S}$$

  $$S = \int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}_{\hat{D}}\sqrt{G}(-R + \frac{1}{4}G_N G^{\mathbf{I}_1\mathbf{I}_2}G^{\mathbf{J}_1\mathbf{J}_2}F_{\mathbf{I}_1\mathbf{J}_1}F_{\mathbf{I}_2\mathbf{J}_2})$$

- **The theory is background independent.**

Supersymmetry is a part of the diffeomorphisms symmetry

$$(\bar{\sigma}, \bar{\theta}^\alpha) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'^\alpha(\bar{\sigma}, \bar{\theta}))$$

$$\updownarrow$$

$$[\mathbf{E}_M{}^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), \mathbf{X}^\mu_{\hat{D}_T}(\bar{\tau}), \bar{\tau}] \mapsto [\mathbf{E}'_M{}^A(\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\tau}, \bar{\theta}'^\alpha(\bar{\sigma}, \bar{\theta})), \mathbf{X}'^\mu_{\hat{D}_T}(\bar{\tau})(\mathbf{X}_{\hat{D}_T})), \bar{\tau}]$$

- These are dimensional reductions in $\bar{\tau}$ direction of the two-dimensional $\mathcal{N} = (1, 1)$ local susy trans.
- supercharges $\xi^\alpha Q_\alpha = \xi^\alpha (\frac{\partial}{\partial \bar{\theta}^\alpha} + i \gamma^1_{\alpha\beta} \bar{\theta}^\beta \frac{\partial}{\partial \bar{\sigma}})$
- The number of supercharges is the same as of the two-dimensional ones.
- The supersymmetry algebra closes in a field-independent sense as in ordinary supergravities.

Derive the all order perturbative superstring scattering amplitudes

- We obtain the all-order scattering amplitudes that possess the supermoduli in the perturbative type IIA, IIB and SO(32) type I superstring,
  if we consider the fluctuations after fixing IIA, IIB and SO(32) type I charts, respectively.

- **These amplitudes are derived from the single theory**.

- The consistency of the fluctuations around the backgrounds → **d=10**

- We obtain amplitudes of the superstrings with Dirichlet and Neumann boundary conditions in the normal and tangential directions to the D-submanifolds, respectively.

↓

D-submanifolds represent D-brane backgrounds where back reactions from the D-branes are ignored.

6 String geometry and a new type of supersymmetric matrix models

String geometry and a new type of supersymmetric matrix models

Gravity and a matrix moldel (Hanada-Kawai-Kimura 2006)

Equations of motion of $S_e = \frac{1}{G_N}\int d^{10}x\sqrt{g}(-R + \frac{1}{4}G_N F_{\mu\nu}F^{\mu\nu})$

↕equivalent

Equations of motion of $S_m = tr(-[A_\mu, A_\nu][A^\mu, A^\nu])$ where we replace $A_\mu \equiv \nabla_\mu$

String geometry and a matrix model

Equations of motion of $S = \int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}\sqrt{G}(-R + \frac{1}{4}G_N G^{\mathbf{I}_1\mathbf{I}_2}G^{\mathbf{J}_1\mathbf{J}_2}F_{\mathbf{I}_1\mathbf{J}_1}F_{\mathbf{I}_2\mathbf{J}_2})$

↕equivalent

Equations of motion of $S_M = \int \mathcal{D}\mathbf{E} tr(-[A_\mathbf{I}(\mathbf{E}), A_\mathbf{J}(\mathbf{E})][A^\mathbf{I}(\mathbf{E}), A^\mathbf{J}(\mathbf{E})])$ where we replace $A_\mathbf{I} \equiv \nabla_\mathbf{I}$

↑ (extended) large N reduction ?

More simple

$S_{M_0} = tr(-[A_\mathbf{I}, A_\mathbf{J}][A^\mathbf{I}, A^\mathbf{J}])$ (a supersymmetric matrix model that has ∞ indices $\mathbf{I} = (d, (\mu\bar{\sigma}\bar{\theta}))$)

is interesting.

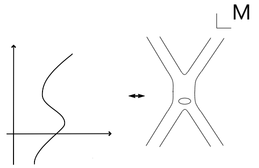
Worldsheets can be derived in general by perturbations of matrix models

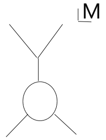
7 Unification of particles and the space-time

Unification of space-time and particles

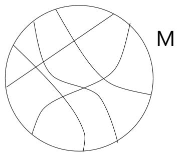
- space-time and string geometry

asymptotic trajectory on $\mathfrak{M}_D$ with target M = string world-sheet in M $\xrightarrow[\text{macro}]{}$ trajectory of a particle in M





Space-time M is identified by: observing all trajectories of a particle in M.



∴ $\mathfrak{M}_D$ is observed as M macroscopically.

Conversely, we see a string, if we microscopically observe a point of the space-time.

- particle and string geometry

A fluctuation of $\mathfrak{M}_D$ = string $\xrightarrow[\text{macro}]{}$ particle

Conversely, we see a string, if we microscopically observe a particle.

- **unification of space-time and particle**

Macroscopically, space-time = string manifold particle = a fluctuation of string manifold