$O(d,d)$ transformations preserve classical integrability

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Based on 1907.03759 with
Domenico Orlando (INFN, Turin), Susanne Reffert (University of Bern), and Kentaroh Yoshida (Kyoto University)

cf: [Ricci, Tseytlin, Wolf 2007], [Rennecke 2014], and [Hull 2004]

The plan of my talk

1. Motivation
2. Classical integrability of WZW models
3. Doubled formalism and $O(d,d)$ transf.
4. Application
5. Outlook

1. Motivation (Quick!)

1.2 Classical integrability of string theory

- AdS_d/CFT_{d-1}: attractive examples of Gauge/Gravity duality

d=5: [Maldacena-1998]

type IIB string on $AdS_5 \times S_5$ ‘ $\Leftrightarrow$ ’ 4D $\mathcal{N}=4$ SU(N) SYM ($N \to \infty$)

- Intriguing: integrable structures

allows us to determine physical quantities exactly,
even at finite coupling, without relying on supersymmetries.

e.g. scattering amplitudes, conformal dims. of composite ops.
spectrum of strings etc…

→ Many directions of applications of integrability techniques!

A comprehensive review:
[Beisert et al-2010]
An ongoing series of
winter schools
of integrability (=YRISW)

1.2 Classical integrability of string theory

- AdS_d/CFT_{d-1}: attractive examples of Gauge/Gravity duality

d=5:

integrable deformed type IIB string on $AdS_5 \times S_5$ ‘⟺’ integrable deformed 4D $\mathcal{N} \leqq 4$ SU(N) SYM ($N \to \infty$)

- Intriguing: integrable structures

allows us to determine physical quantities exactly,
even at finite coupling, without relying on supersymmetries.

e.g. scattering amplitudes, conformal dims. of composite ops.
spectrum of strings etc…

- Significant: integrable deformations

construct a variety of examples of ↑dualities keeping the integrability

→ Want to follow a systematic approach for such deformations.

→ Yang-Baxter deformation

1.3 A bit about Yang-Baxter deformation

- **The YB deformed σ-model action** [Klimcik 02, 08] **(w/ WZ-term:** [Delduc, Magro, Vicedo, 14] **)**

$$S_\lambda = \int d^2\sigma\, \eta^{ab}\, \mathrm{Tr}\left[J_a \frac{1}{1-\underline{\lambda} R} J_b\right] \qquad g \in G \qquad J_a = g^{-1}\partial_a g \in \mathfrak{g}$$

↑ const. deformation parameter

- **A linear operator** $R: \ \mathfrak{g} \to \mathfrak{g}$

including the classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, $r_{12} = \sum_j a_j \otimes b_j$

$P_\mu \,, M_{\mu\nu} \,, D$

$$R(x) \equiv \langle r_{12}, 1 \otimes x \rangle = \sum_j a_j \langle b_j, x \rangle \qquad \textbf{for } x, a_j, b_j \in \mathfrak{g}$$

- **The classical r-matrix is a solution of classical YB equation (CYBE)**

$$\boxed{[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0}$$

☆ **The deformation simply labelled by the classical r-matrix**

(Given some r, then the deformed background data follows automatically.)

1.3 Yang-Baxter (YB) deformation

[Delduc, Magro, Vicedo], [Matsumoto, Yoshida]
[Kawaguchi, Matsumoto, Yoshida]

In summary, the method consists of

1. Put classical r-matrix into the YB deformed sigma model action:

$$S = -\frac{\sqrt{\lambda_c}}{4}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma (\gamma^{ab} - \epsilon^{ab}) \mathrm{STr}\left[J_a\, d \,\circ \frac{1}{1 - \lambda R \circ d}(J_b)\right]$$

const. deformation parameter ↑ cl. r-matrix inserted.

2. Rewrite the action to read off the deformed background data by comparing with the canonical formula:

$$S = -\frac{\sqrt{\lambda_c}}{4}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \left[\gamma^{ab} G_{MN}\partial_a X^M \partial_b X^N - \epsilon^{ab} B_{MN}\partial_a X^M \partial_b X^N\right]$$
$$- \frac{\sqrt{\lambda_c}}{2} i\bar{\Theta}_I \left[\gamma^{ab}\delta^{IJ} - \epsilon^{ab}\sigma_3^{IJ}\right] e_a^m \Gamma_m D_b^{JK} \Theta_K + \mathcal{O}(\theta^4)$$

with
$$D_a^{IJ} = \delta^{IJ}\left(\partial_a - \frac{1}{4}\omega_a^{mn}\Gamma_{mn}\right) + \frac{1}{8}\sigma_3^{IJ} e_a^m H_{mnp}\Gamma^{np}$$
$$- \frac{e^{\Phi}}{8}\left[\epsilon^{IJ}\Gamma^p F_p + \frac{1}{3!}\sigma_1^{IJ}\Gamma^{pqr} F_{par} + \frac{1}{2\cdot 5!}\epsilon^{IJ}\Gamma^{pqrst} F_{pqrst}\right] e_a^m \Gamma_m$$

[Cvetic-Lu-Pope-Stelle]

1.3 Yang-Baxter (YB) deformation

Important observations: Yang-Baxter → $O(d,d)$

Some YB-deformations related to $O(d,d)$ transformations (or T-duality)

☆ **TsT (T-duality-Shift-T-duality) transformation on T^2** [Alday, Aturyunov, Frolov] etc…

☆ **Some current-current deformations ~ some global $O(d,d)$** [Borsato, Wulff] [Araujo et al]

☆ **T-fold (non-geometric) backgrounds ~ some local $O(d,d)$** [Fernandez-Melgarejo et al]

☆ **Non-abelian T-dual backgrounds** [Borsato, Wulff]

Complimentary: focus on integrability of $O(d,d;\mathbb{R})$ without Yang-Baxter

→ **Use the $O(d,d)$-invariant formalism** [Hull]

→ **To understand classical integrability of non-geometric backgrounds, started to focus on global $O(d,d;\mathbb{R})$ transformations**

Other motivations:

[Ricci, Tseytlin, Wolf], [Beisert]

T-duality vs. conformal symmetry on $\mathrm{AdS}_5 \times S^5$ (dual conformal symmetry), $T\bar{T}$-deformation (see Kentaroh's talk) [Giveon, Itzhaki, Kutasov] etc…

1.4 Upshot

☆ **Motivated by two recent developments in string theory**

$\left\{ \right.$

1. **Classical integrability of string theory**
2. **Duality invariant approach to string theory**

☆ **Take home message:** **a synthetic study of the above topics**

- **Systematically obtained $O(d,d)$-deformed Lax pairs via doubled formalism (or $O(d,d)$-map)**
- **Global $O(d,d;\mathbb{R})$ transformations = integrable**

2. Classical integrability
(=the existence of Lax pairs)
of WZW models

2.1 Basics of WZW model (on S^3 w/ H-flux)

Given the action

$$S[g] = -\frac{1}{4}\int_{\Sigma_2} \mathrm{Tr}\left[j_{\mathrm{L}} \wedge \star j_{\mathrm{L}}\right] + \frac{i}{3!}\int_{\mathcal{V}_3} \mathrm{Tr}\left[j_{\mathrm{L}} \wedge j_{\mathrm{L}} \wedge j_{\mathrm{L}}\right]$$

(Six) Noether currents (for $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$):
(and flat)

$$j_{\mathrm{L}} = +g^{-1}\mathrm{d}g$$
$$j_{\mathrm{R}} = -\mathrm{d}g\, g^{-1}$$

$$J_{\mathrm{L}} = (1 - i\,\star)\, j_{\mathrm{L}}\,, \qquad J_{\mathrm{R}} = (1 + i\,\star)\, j_{\mathrm{R}}$$

→ (Six) Lax pairs given by

$$\mathcal{L}_{\mathrm{L/R}} = a_\lambda J_{\mathrm{L/R}} + b_\lambda \star J_{\mathrm{L/R}}$$

λ : spectral parameter

$$a_\lambda = \frac{1}{2}(1 - \cosh\lambda)$$
$$b_\lambda = \frac{1}{2}\sinh\lambda$$

satisfying $\mathrm{d}\mathcal{L} + \mathcal{L} \wedge \mathcal{L} = 0$ **on-shell for L and R.**
(flatness, zero-curvature condition)

2.2 Lax pair and monodromy matrix

Given Lax pairs, the monodromy matrix

$$\mathcal{T}(\tau;\lambda) = \mathcal{P}\exp\left[-\int_{-\infty}^{+\infty} \mathrm{d}\sigma' \mathcal{L}_\sigma(\sigma')\right]$$

$$= 1 + \sum_{n=0}^{\infty} \lambda^{n+1} \mathcal{Q}^{(n)}(\tau)$$

satisfying the conservation laws:
(using flatness of Lax pairs)

$$\frac{\partial}{\partial\tau}\mathcal{Q}^{(n)} = 0, \qquad n \in \mathbb{Z}_{\geq 0}$$

The ∞-number of conserved charges written in non-local forms.

Intriguing: determine the type of <u>algebra</u> for their commutators.

Note: Lax connections (flat currents) are unique up to gauge transf.

$$\mathcal{L} \to \hat{\mathcal{L}} = h^{-1}\mathcal{L}\, h + h^{-1}\mathrm{d}h \qquad\qquad h \in G$$

2.3 Concrete gauged Lax pairs for S^3 w/ H-flux

Given an $SU(2)$ element

$$g = e^{-Z_+ T_2} e^{Y T_1} e^{+Z_- T_2} = e^{-(Z_1+Z_2)T_2} e^{Y T_1} e^{+(Z_1-Z_2)T_2}$$

$$[T_\alpha, T_\beta] = \epsilon_{\alpha\beta\gamma} T_\gamma, \quad \mathrm{Tr}\left(T_\alpha T_\beta\right) = -\frac{1}{2}\delta_{\alpha\beta}, \quad \epsilon_{123} = 1$$

and $\begin{cases} h_{\mathrm{L}} = e^{-(Z_1-Z_2)T_2} & \text{for L} \\ h_{\mathrm{R}} = e^{-(Z_1+Z_2)T_2} & \text{for R} \end{cases}$ **, the gauged Lax pairs are**

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and $\begin{cases} h_{\mathrm{L}} = e^{-(Z_1-Z_2)T_2} & \textbf{for } \mathrm{L} \\ h_{\mathrm{R}} = e^{-(Z_1+Z_2)T_2} & \textbf{for } \mathrm{R} \end{cases}$ **, the gauged Lax pairs are**

$$\hat{\mathcal{L}}^1_{\mathrm{L}} = +F_1(\lambda)\mathrm{d}Y,$$
$$\hat{\mathcal{L}}^2_{\mathrm{L}} = -F_1(\lambda)\left[\mathrm{d}Z_1 - \mathrm{d}Z_2 - \cos Y\,(\mathrm{d}Z_1 + \mathrm{d}Z_2)\right] - (\mathrm{d}Z_1 - \mathrm{d}Z_2)\,,$$
$$\hat{\mathcal{L}}^3_{\mathrm{L}} = +F_1(\lambda)\sin Y\,(\mathrm{d}Z_1 + \mathrm{d}Z_2)\,,$$

$$\hat{\mathcal{L}}^1_{\mathrm{R}} = +F_2(\lambda)\mathrm{d}Y,$$
$$\hat{\mathcal{L}}^2_{\mathrm{R}} = +F_2(\lambda)\left[\mathrm{d}Z_1 + \mathrm{d}Z_2 - \cos Y\,(\mathrm{d}Z_1 - \mathrm{d}Z_2)\right] - (\mathrm{d}Z_1 + \mathrm{d}Z_2)$$
$$\hat{\mathcal{L}}^3_{\mathrm{R}} = +F_2(\lambda)\sin Y\,(\mathrm{d}Z_1 - \mathrm{d}Z_2)$$

$$F_1(\lambda) = \left[(ib_\lambda - a_\lambda) + (ia_\lambda - b_\lambda)\star\right]$$
$$F_2(\lambda) = \left[(ib_\lambda + a_\lambda) + (ia_\lambda + b_\lambda)\star\right]$$

☆ **Remove the explicit dependence on (Z_1, Z_2) !**

3. Doubled formalism and $O(d,d)$ transformations

3.1 Setting of doubled formalism [Hull]

Assume the following situation:

Doubled torus T^{2d}

Doubled coords: $\mathbb{X}^I = (\, X^i \quad \widetilde{X}_i\,)^t$

Torus fiber T^d

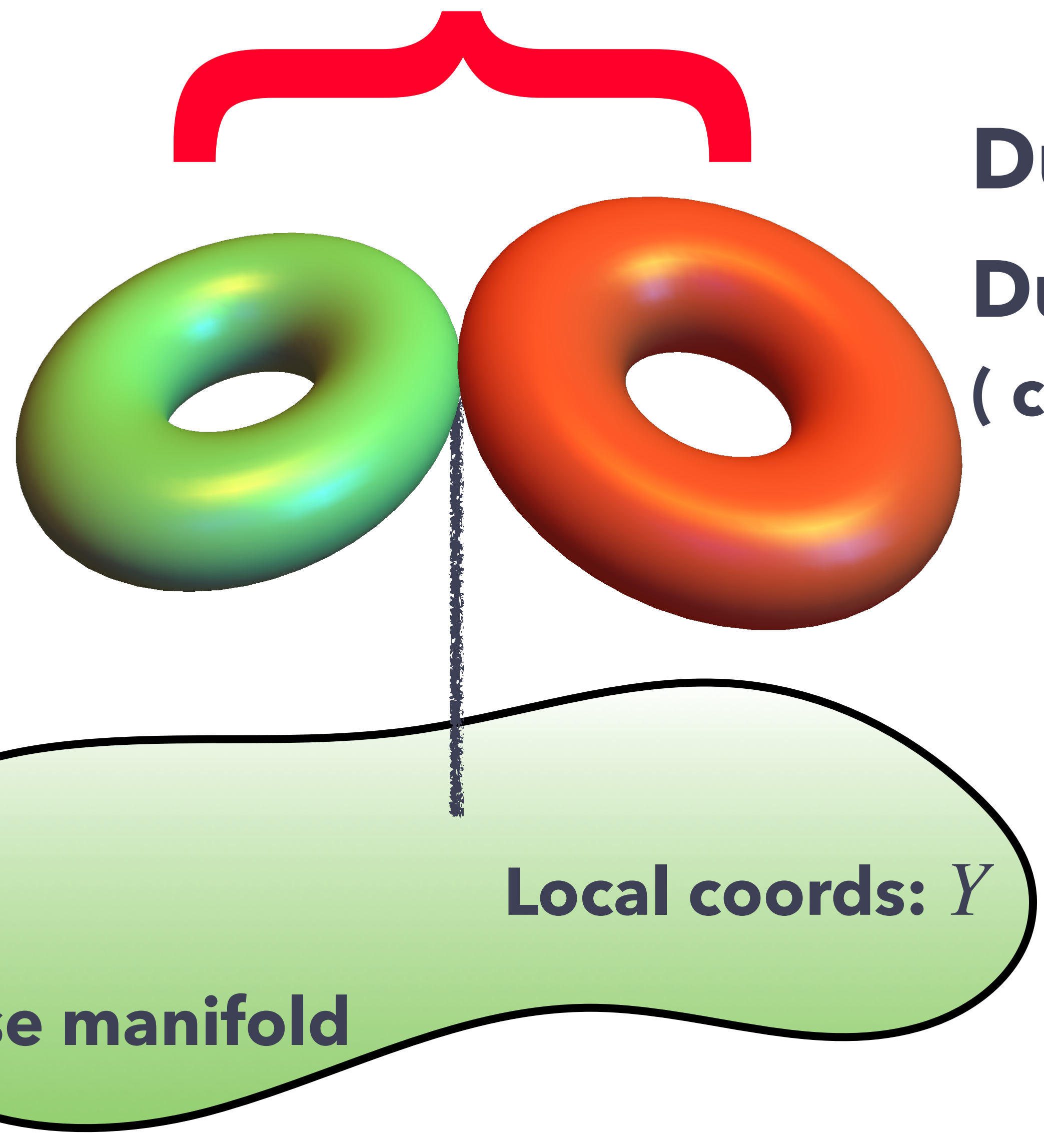
Adapted coords: X^i

(Killing vectors: ∂_{X^i} **)**

Dual torus fiber $\widetilde{T}^d$

Dual coords: $\widetilde{X}_i$

(cf: Buscher rule)



3.2 Action of doubled formalism [Hull]

Doubled sigma model action:

$$S = \int \frac{1}{2}\mathcal{H}_{IJ}\,\mathrm{d}\mathbb{X}^I \wedge \star\mathrm{d}\mathbb{X}^J + \mathrm{d}\mathbb{X}^I \wedge \star\mathcal{J}_I(Y) + \mathcal{L}(Y)$$

No X^i

with the generalized metric

$$\mathcal{H}_{IJ} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}_{IJ}$$

$\in O(d,d)$

Let some $g \in O(d,d)$ act as

$$\mathcal{H} \to g^t \mathcal{H} g, \quad \mathrm{d}\mathbb{X} \to g^{-1}\mathrm{d}\mathbb{X}, \quad \mathcal{J} \to g^t \mathcal{J}$$

, then the action is invariant.

Note!

$\mathbb{X} \to g^{-1}\mathbb{X}, \quad g \in O(d,d;\mathbb{Z})$ **: symmetry**

$g \in O(d,d;\mathbb{R})$ **: deformation (sol. generating tech.)**

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$$S = \int \frac{1}{2}\mathcal{H}_{IJ}\,\mathrm{d}\mathbb{X}^I \wedge \star \mathrm{d}\mathbb{X}^J + \mathrm{d}\mathbb{X}^I \wedge \star \mathcal{J}_I(Y) + \mathcal{L}(Y)$$

☆$O(d,d;\mathbb{R})$ **deformations via field redefinition:**

$$\mathcal{H}(G,B) \to g^t \mathcal{H} g = \mathcal{H}(G', B'), \quad \mathrm{d}\mathbb{X} \to g^{-1}\mathrm{d}\mathbb{X} = \mathrm{d}\mathbb{X}'$$

$$\boxed{\mathcal{H} \to g^t \mathcal{H} g, \quad \mathrm{d}\mathbb{X} \to g^{-1}\mathrm{d}\mathbb{X},} \quad \mathcal{J} \to g^t \mathcal{J}$$

, then the action is invariant.

Note!

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3.3 Constraint on doubled formalism [Hull]

(Suppose that there is no source term)

To get back to the sigma model from the doubled action one condition imposed (self-duality constraint) :

$$\mathrm{d}\mathbb{X}^I = L^{IJ}\mathcal{H}_{JK} \star \mathrm{d}\mathbb{X}^K$$

$$L_{IJ} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

Unpackaging the constraint,

$$\mathrm{d}\widetilde{X}_i = \star\left(G_{ij}\mathrm{d}X^j + B_{ij} \star \mathrm{d}X^j\right) = \star J_i$$

☆ **The winding coordinates turn into $U(1)$-isometry currents.**

☆ $\mathrm{d}^2\widetilde{X}_i = 0$ **leads to the conservation of J_i (EOMs for X^i).**

$$\left\{\begin{array}{l}\text{EOMs for } \mathbb{X}^I \text{ and } Y \\ \text{+ self-duality constraint} \\ \text{of doubled sigma model}\end{array}\right. \longleftrightarrow \left\{\begin{array}{l}\text{EOMs for } X^i \text{ and } Y \\ \text{of physical sigma model}\end{array}\right.$$

3.4 $O(d,d)$ (-duality) map [Rennecke]

The transformation rule for $\mathrm{d}X^i$ under global $O(d,d)$:

Start from $\mathrm{d}\mathbb{X}' = g^{-1}\mathrm{d}\mathbb{X}$ with

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha^i{}_j,\ \beta^{ij},\ \gamma_{ij},\ \delta_i{}^j$$

: $d \times d$ matrices.

Then $\mathrm{d}X^i = \alpha^i{}_j \mathrm{d}X'^j + \beta^{ij}\mathrm{d}\widetilde{X}'_j$.

Using the self-duality constraint,

$$\boxed{\mathrm{d}X^i = \alpha^i{}_j \mathrm{d}X'^j + \beta^{ij} \star J'_j} = \mathfrak{D}_i(\mathrm{d}X')$$

Thus, the $O(d,d)$ action on Lax pairs:

$$\mathcal{L} \longrightarrow \hat{\mathcal{L}}(\mathrm{d}X) \longrightarrow \boxed{\mathcal{L}'(\mathrm{d}X') = \hat{\mathcal{L}}(\mathrm{d}X \to \mathfrak{D}(\mathrm{d}X'))}$$

Gauging

Just applied the $O(d,d)$ map to the gauged Lax pairs!

3.5 Flatness of $O(d,d)$-deformed Lax pairs

Do $O(d,d)$-deformed Lax pairs satisfy zero-curvature condition?

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$$\mathrm{d}\mathcal{L} + \mathcal{L} \wedge \mathcal{L} = 0$$ **(on-shell)**

3.5 Flatness of $O(d,d)$-deformed Lax pairs

Do $O(d,d)$-deformed Lax pairs satisfy zero-curvature condition?

Anyways, start from the curvature of Lax connections

$$\mathrm{d}\hat{\mathcal{L}} + \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} = \underline{(\mathrm{EOMs})}$$

for X^i and Y

of the undeformed model

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☆ **The doubled action $S = S'$ is invariant under $O(d,d)$.**

1. **The EoM for Y of the undeformed model**

   → The EoM for Y of the deformed model.

2. **The EoMs for X^i of the undeformed model**

   →A linear combinations of EoMs for X'^i 's.

   (thanks to the self-duality constraint)

3.5 Flatness of $O(d,d)$-deformed Lax pairs

The transformation rule for $\mathrm{d}\widetilde{X}_i$ **under global** $O(d,d)$ **:**

Start from $\mathrm{d}\mathbb{X}' = g^{-1}\mathrm{d}\mathbb{X}$ **with**

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$\alpha^i{}_j,\ \beta^{ij},\ \gamma_{ij},\ \delta_i{}^j$ **:** $d\times d$ **matrices.**

Then $\mathrm{d}\widetilde{X}_i = \gamma_{ij}\mathrm{d}X'^j + \delta_i{}^k\mathrm{d}\widetilde{X}'_k$ **.**

Using the self-duality constraint,

$$\boxed{\star J_i = \gamma_{ij}\mathrm{d}X'^j + \delta_i{}^k \star J'_k}$$

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: $d \times d$ matrices.

Then $\mathrm{d}\widetilde{X}_i = \gamma_{ij}\mathrm{d}X'^j + \delta_i{}^k\mathrm{d}\widetilde{X}'_k$ **.**

Using the self-duality constraint,

$$\boxed{\mathrm{d}\star J_i = \delta_i{}^k\,\mathrm{d}\star J'_k}$$

Field equations of the original model are mapped to those of the deformed model.

Zero-curvature conditions guaranteed after $O(d,d)$ transformations ✔

4. Application

4.1 $J\bar{J}$ (current-current) deformation [Hassan, Sen]

Marginal deformation by the operator of

$$\boxed{S^{(\alpha+\delta\alpha)} - S^{(\alpha)} \sim \frac{\delta\alpha}{\pi} f(\alpha) \int \mathrm{d}^2 z \underline{J_{(\alpha)} \overline{J}_{(\alpha)}}}$$

Deformed U(1) currents in the Cartan subalgbera

Deformed generalized metric obtained by the $O(2,2)$ matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & \tan\alpha \\ 0 & \frac{1}{1+\tan\alpha} & -\frac{\tan\alpha}{1+\tan\alpha} & 0 \\ 0 & \frac{1}{1+\tan\alpha} & \frac{1}{1+\tan\alpha} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

in the basis of

$$\mathbb{X}^I = \left(Z_1, Z_2, \widetilde{Z}_1, \widetilde{Z}_2\right)^t$$

Deformed background data obtained via field redefinition

$$g^t \mathcal{H}(G, B) g = \mathcal{H}(G', B')$$

4.1 $J\bar{J}$ (current-current) deformation

Given the $O(2,2)$ matrix and deformed background data, apply the $O(2,2)$ map to $(\mathrm{d}Z_1, \mathrm{d}Z_2)$: $\mathrm{d}X^i = \alpha^i{}_j \mathrm{d}X'^j + \beta^{ij} \star J'_j$ **.**

$$g = \begin{pmatrix} 1 & 0 & 0 & \tan\alpha \\ 0 & \frac{1}{1+\tan\alpha} & -\frac{\tan\alpha}{1+\tan\alpha} & 0 \\ 0 & \frac{1}{1+\tan\alpha} & \frac{1}{1+\tan\alpha} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} \mathrm{d}Z_1 = \mathrm{d}Z'_1 + \tan\alpha \star J_2(\alpha) \\ \mathrm{d}Z_2 = \dfrac{1}{1+\tan\alpha}\left[\mathrm{d}Z'_2 - \tan\alpha \star J_1(\alpha)\right] \end{cases}$$

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$$\begin{cases} \mathrm{d}Z_1 = \mathrm{d}Z'_1 + \tan\alpha \star J_2(\alpha) \\ \mathrm{d}Z_2 = \dfrac{1}{1+\tan\alpha}\left[\mathrm{d}Z'_2 - \tan\alpha \star J_1(\alpha)\right] \end{cases}$$

$$\begin{aligned} \hat{\mathcal{L}}^1_{\mathrm{L}} &= +F_1(\lambda)\mathrm{d}Y, \\ \hat{\mathcal{L}}^2_{\mathrm{L}} &= -F_1(\lambda)\left[\mathrm{d}Z_1 - \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right), \\ \hat{\mathcal{L}}^3_{\mathrm{L}} &= +F_1(\lambda)\sin Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right), \end{aligned}$$

$$\begin{aligned} \hat{\mathcal{L}}^1_{\mathrm{R}} &= +F_2(\lambda)\mathrm{d}Y, \\ \hat{\mathcal{L}}^2_{\mathrm{R}} &= +F_2(\lambda)\left[\mathrm{d}Z_1 + \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right) \\ \hat{\mathcal{L}}^3_{\mathrm{R}} &= +F_2(\lambda)\sin Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right) \end{aligned}$$

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$$
g = \begin{pmatrix} 1 & 0 & 0 & \tan\alpha \\ 0 & \frac{1}{1+\tan\alpha} & -\frac{\tan\alpha}{1+\tan\alpha} & 0 \\ 0 & \frac{1}{1+\tan\alpha} & \frac{1}{1+\tan\alpha} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}
\quad
\left\{ \begin{aligned} \mathrm{d}Z_1 &= \mathrm{d}Z'_1 + \tan\alpha \star J_2(\alpha) = \mathfrak{D}_1(\mathrm{d}Z') \\ \mathrm{d}Z_2 &= \frac{1}{1+\tan\alpha}\left[\mathrm{d}Z'_2 - \tan\alpha \star J_1(\alpha)\right] \\ &= \mathfrak{D}_2(\mathrm{d}Z') \end{aligned} \right.
$$

$$
\begin{aligned}
\mathcal{L}'^1_{\mathrm{L}} &= +F_1(\lambda)\mathrm{d}Y, \\
\mathcal{L}'^2_{\mathrm{L}} &= -F_1(\lambda)\left[\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z') - \cos Y\left(\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z')\right)\right] - \left(\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z')\right), \\
\mathcal{L}'^3_{\mathrm{L}} &= +F_1(\lambda)\sin Y\left(\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z')\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}'^1_{\mathrm{R}} &= +F_2(\lambda)\mathrm{d}Y, \\
\mathcal{L}'^2_{\mathrm{R}} &= +F_2(\lambda)\left[\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z') - \cos Y\left(\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z')\right)\right] - \left(\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z')\right), \\
\mathcal{L}'^3_{\mathrm{R}} &= +F_2(\lambda)\sin Y\left(\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z')\right).
\end{aligned}
$$

$\mathrm{d}\mathcal{L}' + \mathcal{L}' \wedge \mathcal{L}' = 0$ **still holds.**

5. Conclusions and Outlook

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This work completed the classical integrability of any global $O(d,d;\mathbb{R})$ transformation using the doubled formalism.

Possible future directions:

☆ **What type of algebra of non-local charges?** [work in progress] [Kawaguchi, Matsumoto, Yoshida] [Kawaguchi, Yoshida]

☆ **The $O(d,d)$ map for the spectral parameter λ ?**

☆ **The classical integrability of doubled formalism itself?**

Beyond global $O(d,d;\mathbb{R})$:

☆ **Extension to local $O(d,d;\mathbb{R})$ transformations?**

→ **classical integrability of non-geometric backgrounds**

(What would be a collective T-duality invariant framework…?)

Thank you!