Doubled Aspects of Vaisman Algebroid and Gauge Symmetry in Double Field Theory

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1. Introduction

T-duality: appears in the string mass spectrum (hidden symmetry)

Kaluza-Klein modes $\leftrightarrow$ string winding modes

- It is not a manifest symmetry in the string effective action.
- When the compact space is T^D, the T-duality group is $O(D,D)$.

Double Field Theory (DFT): [Hull-Zwiebach '09]

- an effective theory of strings where T-duality is realized manifestly
- defined on the doubled spacetime $\mathcal{M}$ ($\dim \mathcal{M} = 2D$)
  - $\mathcal{M}$ is characterized by the "doubled" coordinate $x^M = (x^\mu, \tilde{x}_\mu)$.
  - x^μ is the Fourier dual of KK momentum, $\tilde{x}_\mu$ is the dual of winding charges.

The section condition:

- Since DFT has extra d.o.f., it is necessary imposed the physical condition.
- **The "strong constraint"**: $\partial_M * \partial^M * = 0$ ($*$: arbitrary fields & parameters).
- A trivial solution of the strong constraint: $\tilde{\partial}^\mu * = 0$ (independent of $\tilde{x}$).
  $\rightarrow$ Under this condition, DFT is reduced to type II supergravity.

The C-bracket: (doubled vectors $\Xi_i^M = (A_i^\mu, \alpha_{i\mu})$)

$$[\Xi_1, \Xi_2]_C = [A_1, A_2]_L + \mathcal{L}_{A_1}\alpha_2 - \mathcal{L}_{A_2}\alpha_1 - \tfrac{1}{2}\mathrm{d}(\iota_{A_1}\alpha_2 - \iota_{A_2}\alpha_1) + [\alpha_1, \alpha_2]_{\tilde{L}} + \tilde{\mathcal{L}}_{\alpha_1}A_2 - \tilde{\mathcal{L}}_{\alpha_2}A_1 - \tfrac{1}{2}\tilde{\mathrm{d}}(\tilde{\iota}_{\alpha_1}A_2 - \tilde{\iota}_{\alpha_2}A_1) \qquad (\heartsuit)$$

- The C-bracket governs the commutator of the generalized Lie derivative.
- $[\cdot,\cdot]_C$ accommodates the D-dim. diffeo. and the B-field gauge symmetry algebra.
- Under $\tilde{\partial}^\mu * = 0$, the C-bracket reduces to the Courant bracket in gen. geom.
- The algebraic strc. based on $[\cdot,\cdot]_C$ is not a Courant algd. but a Vaisman algd.

Our Results

- The Vaisman algebroid is obtained by the double of two Lie algebroids.
- We find an algebraic origin of the strong constraint in DFT.

2. Drinfel'd Double of Lie Bialgebra

Def: Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$

- a vector space $\mathfrak{g}$
- a skew-symmetric bilinear bracket (the Lie bracket) $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

dual Lie algebra $(\mathfrak{g}^*, [\cdot,\cdot]_*)$

- the dual vector space $\mathfrak{g}^*$
- the dual Lie bracket $[\cdot,\cdot]_* : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$

A natural bilinear inner product $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to K$ is defined (K: a field).

Def: Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$

- Let the Lie bracket $[\cdot,\cdot]$ be a bilinear map $\mu : \wedge^2\mathfrak{g} \to \mathfrak{g}$.
- a co-bracket $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ as an adjoint of the dual bracket $\mu_* : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$
- the adjoint of μ_*, denoted as μ_*^*, is defined by $\langle a, \mu_*(\bar{c})\rangle = \langle \mu_*^*(a), \bar{c}\rangle$ ($\bar{c} \in \wedge^2\mathfrak{g}^*$)
- imposing the 1-cocycle condition: $\delta([a,b]) = \mathrm{ad}_a^{(2)}\delta(b) - \mathrm{ad}_b^{(2)}\delta(a)$

The Drinfel'd double of a Lie bialgebra $\Rightarrow$ a new Lie algebra $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}})$

- a vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$
- a bilinear form: $(a,b) = (\bar{a}, \bar{b}) = 0, \quad (a, \bar{b}) = \langle \bar{b}, a\rangle \qquad (a, b \in \mathfrak{g},\ \bar{a}, \bar{b} \in \mathfrak{g}^*)$
- a skew-symmetric bracket: $[a,b]_{\mathfrak{d}} = [a,b]$, $[\bar{a},\bar{b}]_{\mathfrak{d}} = [\bar{a},\bar{b}]_*$, $[a,\bar{b}]_{\mathfrak{d}} = \mathrm{ad}_a^*\bar{b} - \mathrm{ad}_{\bar{b}}^* a$

3. Lie algebroid and Its Dual

Lie algebroid: a generalization of a Lie algebra defined over a mfd.

Lie algebroid

- defined using **vector fields**
- determined by **structure functions**

Lie algebra

- using left invariant vector fields
- by structure constants

Def: Lie algebroid $(L, [\cdot,\cdot]_L, \rho)$

- $L \xrightarrow{\pi} M$: a vector bundle over a base manifold M
- an anchor map $\rho : L \to TM$
- a skew-symmetric bilinear form $[\cdot,\cdot]_L : L \times L \to L$
  - (Jacobi identity): $[[A_1, A_2]_L, A_3]_L + \text{c.p.} = 0 \qquad (A_i \in L)$
  - (Homomorphic property of ρ): $\rho([A_1, A_2]_L) = [\rho(A_1), \rho(A_2)]_{TM}$
  - (Leibniz rule): $[A_1, fA_2]_L = f[A_1, A_2]_L + (\rho(A_1)\cdot f)A_2 \qquad (f \in C^\infty(M))$

Given a Lie algebroid, we can define the dual Lie algebroid $(L^*, [\cdot,\cdot]_{L^*}, \rho_*)$

The exterior derivative $\mathrm{d} : \wedge^p L^* \to \wedge^{p+1}L^*$ is defined by $\qquad (\alpha \in \wedge^p L^*, A_i \in E)$

$$\mathrm{d}\alpha(A_1, \dots, A_{p+1}) = \sum_{i=1}^{p+1}(-)^{i+1}\rho(A_i)\cdot(\alpha(A_1, \dots, \check{A}_i, \dots, A_{p+1})) + \sum_{i<j}(-)^{i+j}\alpha([A_i, A_j]_L, A_1, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_{p+1})$$

4. Double of Lie Bialgebroid

Def: Lie bialgebroid (L, L^*)

- a pair of a Lie algebroid L and its dual Lie algebroid L^*
- satisfying **the derivation condition**: $\mathrm{d}_*[A,B]_\mathsf{s} = [\mathrm{d}_*A, B]_\mathsf{s} + [A, \mathrm{d}_*B]_\mathsf{s}$

The Schouten bracket $[\cdot,\cdot]_\mathsf{s} : \wedge^p L \times \wedge^q L \to \wedge^{p+q-1}L$ is defined by the properties:

- $[A,B]_\mathsf{s} = -(-)^{(p-1)(q-1)}[B,A]_\mathsf{s} \qquad (A \in \wedge^p L, B \in \wedge^q L)$
- $[A, B\wedge C]_\mathsf{s} = [A,B]_\mathsf{s} \wedge C + (-)^{(p-1)q}B \wedge [A,C]_\mathsf{s} \qquad (C \in \wedge^r L)$
- $(-)^{(p-1)(r-1)}[A,[B,C]_\mathsf{s}]_\mathsf{s} + (-)^{(q-1)(p-1)}[B,[C,A]_\mathsf{s}]_\mathsf{s} + (-)^{(r-1)(q-1)}[C,[A,B]_\mathsf{s}]_\mathsf{s} = 0$

Def: Courant algebroid $(\mathcal{C}, [\cdot,\cdot]_\mathsf{c}, \rho_\mathsf{c}, (\cdot,\cdot))$ [Courant '90, Liu-Weinstein-Xu '97]

- $\mathcal{C} \xrightarrow{\pi} M$: a vector bundle over a manifold M
- a non-degenerate symmetric bilinear form $(\cdot,\cdot) : \mathcal{C} \times \mathcal{C} \to C^\infty(M)$
- an anchor $\rho_\mathsf{c} : \mathcal{C} \to TM$
- a skew-symmetric bracket $[\cdot,\cdot]_\mathsf{c} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
  - (Jacobi identity (up to homotopy)): $[[e_1, e_2]_\mathsf{c}, e_3]_\mathsf{c} + \text{c.p.} = \mathcal{D}T(e_1, e_2, e_3)$
  - (Homomorphic property of ρ_c): $\rho_\mathsf{c}([e_1, e_2]_\mathsf{c}) = [\rho_\mathsf{c}(e_1), \rho_\mathsf{c}(e_2)]$
  - (Leibniz rule): $[e_1, fe_2]_\mathsf{c} = f[e_1, e_2]_\mathsf{c} + (\rho_\mathsf{c}(e_1)\cdot f)e_2 - (e_1, e_2)\mathcal{D}f$
  - (calculation rule of $\mathcal{D}$ and $(\cdot,\cdot)$): $(\mathcal{D}f, \mathcal{D}g) = 0$
  - (Compatibility between $(\cdot,\cdot)$ & ρ_c):
    $\rho_\mathsf{c}(e_1)\cdot(e_2, e_3) = ([e_1, e_2]_\mathsf{c} + \mathcal{D}(e_1, e_2), e_3) + (e_2, [e_1, e_3]_\mathsf{c} + \mathcal{D}(e_1, e_3))$

A Courant algebroid is obtained by the Drinfel'd double of a Lie bialgebroid.

- $(L \oplus L^*, [\cdot,\cdot]_\mathsf{c}, \rho_\mathsf{c} = \rho + \rho_*, (\cdot,\cdot)_+)$ and the morphism $\mathcal{D} = \mathrm{d} + \mathrm{d}_*$
- a vector space $L \oplus L^*$ and its elements $e_i = A_i + \alpha_i$
- Non-degenerate bilinear forms: $(e_1, e_2)_\pm = \frac{1}{2}\big(\langle\alpha_1, A_2\rangle \pm \langle\alpha_2, A_1\rangle\big)$
- a skew-symmetric bracket: same form as the C-bracket ($\heartsuit$) (but $\tilde{\mathrm{d}} \leftrightarrow \mathrm{d}_*$)
- $L \oplus L^*$ satisfies all definitions of the Courant algebroid.

5. "Double" for Vaisman Algebroid

Vaisman Algebroids are algebraic structures based on the C-bracket.

Def: Vaisman algebroid $(\mathcal{V}, [\cdot,\cdot]_\mathsf{v}, \rho_\mathsf{v}, (\cdot,\cdot))$ [Vaisman '12, '13]

- $\mathcal{V} \xrightarrow{\pi} M$: a vector bundle over a manifold M
- a non-degenerate symmetric bilinear form $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to C^\infty(M)$
- an anchor $\rho_\mathsf{v} : \mathcal{V} \to TM$
- a skew-symmetric bracket $[\cdot,\cdot]_\mathsf{v} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
  - (Leibniz rule): $[e_1, fe_2]_\mathsf{v} = f[e_1, e_2]_\mathsf{v} + (\rho_\mathsf{v}(e_1)\cdot f)e_2 - (e_1, e_2)\mathcal{D}f$
  - (Compatibility between $(\cdot,\cdot)$ & ρ_v):
    $\rho_\mathsf{v}(e_1)\cdot(e_2, e_3) = ([e_1, e_2]_\mathsf{v} + \mathcal{D}(e_1, e_2), e_3) + (e_2, [e_1, e_3]_\mathsf{v} + \mathcal{D}(e_1, e_3))$

A Vaisman algebroid is obtained by a "double" of Lie algebroids.

- $(L \oplus L^*, [\cdot,\cdot]_\mathsf{v}, \rho_\mathsf{v} = \rho + \rho_*, (\cdot,\cdot)_+)$ and the morphism $\mathcal{D} = \mathrm{d} + \mathrm{d}_*$
  - (L, L^*) is not a Lie bialgebroid, and the derivation condition is not imposed.
- a skew-symmetric bracket: same form as the C-bracket ($\heartsuit$) (but $\tilde{\mathrm{d}} \leftrightarrow \mathrm{d}_*$)
- We examine the properties that the Courant bracket should satisfy.
  - generalized Jacobi identity: ✗

$$[[e_1, e_2]_\mathsf{v}, e_3]_\mathsf{v} + \text{c.p.} = \mathcal{D}T(e_1, e_2, e_3) - (J_1 + J_2 + \text{c.p.})$$
$$J_1 = \iota_{A_3}\big(\mathrm{d}[\alpha_1, \alpha_2]_\mathsf{v} - \mathcal{L}_{\alpha_1}\mathrm{d}\alpha_2 + \mathcal{L}_{\alpha_2}\mathrm{d}\alpha_1\big) + \iota_{\alpha_3}\big(\mathrm{d}_*[A_1, A_2]_\mathsf{v} - \mathcal{L}_{A_1}\mathrm{d}_*A_2 + \mathcal{L}_{A_2}\mathrm{d}_*A_1\big)$$
$$J_2 = \big(\mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}\alpha_3 + [\mathrm{d}(e_1, e_2)_-, \alpha_3]_\mathsf{v}\big) - \big(\mathcal{L}_{\mathrm{d}(e_1,e_2)_-}A_3 + [\mathrm{d}_*(e_1, e_2)_-, A_3]_\mathsf{v}\big)$$

  - homomorphic property of ρ_v: ✗

$$\rho_\mathsf{v}([e_1, e_2]_\mathsf{v})\cdot f = [\rho_\mathsf{v}(e_1), \rho_\mathsf{v}(e_2)]f + \tfrac{1}{2}(\rho\rho_*^* + \rho_*\rho^*)\mathrm{d}_0\big(\langle\alpha_1, A_2\rangle - \langle\alpha_2, A_1\rangle\big)f - \langle\alpha_1, (\mathcal{L}_{\mathrm{d}f}A_2 - [A_2, \mathrm{d}_*f]_L)\rangle + \langle\alpha_2, (\mathcal{L}_{\mathrm{d}f}A_1 - [A_1, \mathrm{d}_*f]_L)\rangle$$

  - calculation rule of $\mathcal{D}$ in $(\cdot,\cdot)$: ✗ $\qquad (\mathcal{D}f, \mathcal{D}g)_+ = \frac{1}{2}\big(\rho\rho_*^* + \rho_*\rho^*\big)(\mathrm{d}_0 f)g$
  - Leibniz rule and compatibility b/w $(\cdot,\cdot)$ & ρ_v: ✓
- $L \oplus L^*$ **defines a Vaisman algebroid**, but not a Courant algebroid.
- The extra terms vanish by imposing the derivation condition.

Summary of the above discussion:

[Drinfel'd '86]

Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$

↓ 1-cocycle condition

Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$

↓ **Drinfel'd double**

Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$

[Liu-Weinstein-Xu '97]

Lie algebroids L and L^*

↓ derivation condition

Lie bialgebroid (L, L^*)

↓ **Drinfel'd double**

Courant algebroid

[Mori-Sasaki-KS '19]

Lie algebroids L and L^*

↓ **double** (without derivation condition)

Vaisman algebroid

Doubled Aspects of Vaisman Algebroid and Gauge Symmetry in Double Field Theory

6. DFT Geometry

Doubled spacetime: described by a para-Hermitian mfd. [Vaisman '13]

Def: Para-Hermitian manifold $(\mathcal{M}, K, \eta)$

- $\mathcal{M}$: a differential manifold
- $K \in \mathrm{End}(T\mathcal{M})$: a para-complex structure satisfying $K^2 = 1$
- a neutral metric $\eta : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$
  - the compatibility condition: $\eta(K(X), K(Y)) = -\eta(X, Y)$
- imposing the integrability condition: $N_K(X, Y) = 0$
  - N_K is a real analogue of the Nijenhuis tensor

$$N_K(X,Y) = \frac{1}{4}\Big\{[K(X), K(Y)] + [X, Y] - K\big([K(X), Y] + [X, K(Y)]\big)\Big\}$$

By using the para-complex strc., we can decompose $T\mathcal{M} = L \oplus \tilde{L}$.

- L ($\tilde{L}$) is the eigenbundle associated with the eigenvalue $K = +1$ ($K = -1$).
- This decomposition is performed via the projection operator:

$$P = \frac{1}{2}(1 + K), \qquad \tilde{P} = \frac{1}{2}(1 - K).$$

- We can decompose N_K (cf. not-para case, we cannot do it).

$$N_K(X,Y) = N_P(X,Y) + N_{\tilde{P}}(X,Y),$$
$$N_P(X,Y) = \tilde{P}[P(X), P(Y)], \qquad N_{\tilde{P}}(X,Y) = P[\tilde{P}(X), \tilde{P}(Y)].$$

- L and $\tilde{L}$ are distributions of $T\mathcal{M}$ (distribution: a generalization of subbundle)

Integrability of distributions:

- The Frobenius theorem:

  a distribution L is Frobenius integrable iff L is involutive ($[L, L] \subset L$)

- If N_P ($N_{\tilde{P}}$) vanishes, the distribution L ($\tilde{L}$) is involutive.
- The integrability of L and $\tilde{L}$ is independent of each other.

The physical spacetime is identified as a leaf of the foliation:

- An alternative representation of the Frobenius theorem:

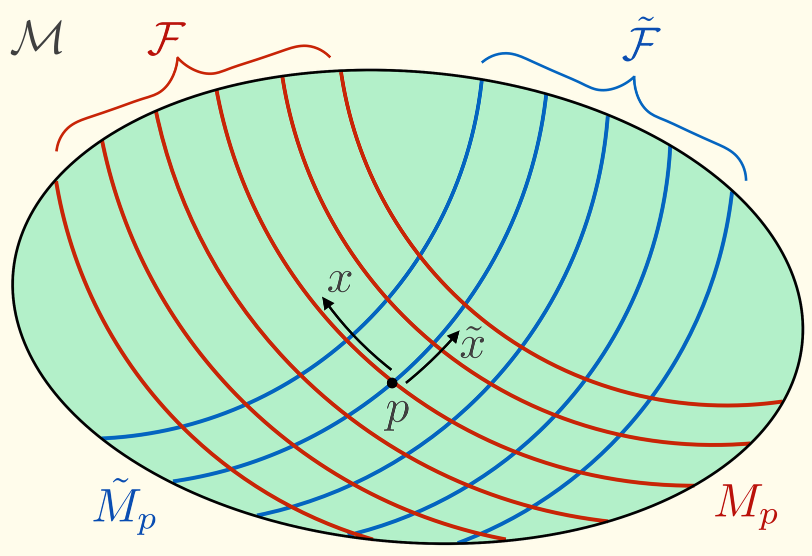
  a subbundle $E \subset T\mathcal{M}$ is integrable iff it is defined by a regular foliation of $\mathcal{M}$

- When L and $\tilde{L}$ are integrable, then they have foliation strc. $L = T\mathcal{F}$, $\tilde{L} = T\tilde{\mathcal{F}}$

Def: Foliation structures

- The foliation $\mathcal{F}$ is given by the union of leaves $\bigsqcup_p M_p$.
- A leaf M_p is a subspace of $\mathcal{F}$ that pass through a point $p \in \mathcal{M}$.
- For $\mathcal{F}$, the local coordinate x^μ is given along a leaf. The one for the transverse directions to leaves is $\tilde{x}_\mu$.

  $\Rightarrow$ $\tilde{x}_\mu$ is a constant on a leaf M_p.



The relations between DFT and the Generalized Geometry:

- η can be seen as a map $\eta : T\mathcal{M} = L \oplus \tilde{L} \to T^*\mathcal{M} = L^* \oplus \tilde{L}^*$.
- By using η, an isomorphism $\phi^+ : \tilde{L} \to L^*$ is defined (vectors $\to$ 1-forms).
- Given ϕ^+, we can define the "natural" isomorphism $\Phi^+ : T\mathcal{M} \to L \oplus L^*$
- Φ^+ is utilized to relate DFT and Hitchin's Generalized Geometry.

7. Exterior Algebras in DFT

Let us define a Lie algebroid structure in DFT.

- Multi-vectors on a mfd. define a Gerstenhaber alg. by the Schouten br. [Tulczyjew '74]
- A Lie algebroid over a vector bundle $V \to M$ and a Gerstenhaber alg. over multi-vectors $\Gamma(\wedge^\bullet V)$ are equivalent. [Vaintrob '97]

$\Rightarrow$ a Lie algebroid is defined by the exterior algebra of multi-vectors in DFT.

We define a natural exterior algebra on $T\mathcal{M}$ and L:

- We introduce a set of doubled multi-vectors $\hat{\mathcal{A}}^k(\mathcal{M}) = \Gamma(\wedge^k T\mathcal{M})$.
- If we define $\mathcal{A}^{r,s}(\mathcal{M})$ as the section of $(\wedge^r L) \wedge (\wedge^s \tilde{L})$, then, we obtain the decomposition $\hat{\mathcal{A}}^k(\mathcal{M}) = \bigoplus_{k=r+s} \mathcal{A}^{r,s}(\mathcal{M})$.
- This decomposition is given by $\pi^{r,s} : \hat{\mathcal{A}}^{r+s}(\mathcal{M}) \to \mathcal{A}^{r,s}(\mathcal{M})$.
  - $\pi^{r,s}$ is called the canonical projection operator and induced by $P, \tilde{P}$.
- The exterior derivatives acting on L and $\tilde{L}$ is defined by

$$\tilde{\mathrm{d}} : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r+1,s}(\mathcal{M}) \qquad \text{(i.e. } \wedge^r L \to \wedge^{r+1} L\text{)},$$
$$\mathrm{d} : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r,s+1}(\mathcal{M}) \qquad \text{(i.e. } \wedge^s \tilde{L} \to \wedge^{s+1} \tilde{L}\text{)}.$$

- $\tilde{\mathrm{d}}$ and d have the following properties: $\tilde{\mathrm{d}}^2 = 0$, $\mathrm{d}^2 = 0$, $\mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d} = 0$.
  - So, they are called the para-Dolbeault operators.
- Also, we define the interior products and the Lie derivatives.

8. Algebroid Structures by DFT Setting

The conventional DFT is described a flat para-Hermitian geometry.

- The flat para-Hermitian mfd. is given by $\left(\mathcal{M}^{2D}, K = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$.
- The tangent space $T\mathcal{M}$ is spanned by ∂_M ($M = 1, \dots, 2D$).
- Vector fields on $T\mathcal{M}$ are decomposed by $P, \tilde{P}$:

$$\Xi^M \partial_M = A^\mu(x, \tilde{x})\partial_\mu + \alpha_\mu(x, \tilde{x})\tilde{\partial}^\mu \qquad (\Xi \in T\mathcal{M}, A \in L, \alpha \in \tilde{L}).$$

Schouten brackets and the action of $\tilde{\mathrm{d}}$ in DFT setting:

- We introduce the "odd coordinate" $\zeta_\mu := \partial_\mu$, then r-vector is

$$A = \frac{1}{r!} A^{\mu_1 \cdots \mu_r} \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_r} = \frac{1}{r!} A^{\mu_1 \cdots \mu_r} \zeta_{\mu_1} \cdots \zeta_{\mu_r}.$$

- ζ_μ can be treated as a Grassmann number $\to$ $\partial / \partial \zeta_\mu$ is the right derivative.
- The Schouten bracket is explicitly given by $(A \in \wedge^r L, B \in \wedge^s L)$

$$[A, B]_{\mathrm{s}} = \left(\frac{\partial}{\partial \zeta_\mu} A\right) \partial_\mu B - (-1)^{(r-1)(s-1)} \left(\frac{\partial}{\partial \zeta_\mu} B\right) \partial_\mu A.$$

- The action of $\tilde{\mathrm{d}}$ on a r-vector A is given by

$$\tilde{\mathrm{d}} A = \frac{1}{r!} \tilde{\partial}^\nu A^{\mu_1 \cdots \mu_r} \partial_\nu \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_r}.$$

- The same discussion holds for $[\cdot, \cdot]_{\mathrm{s}}^*$ and d on $\tilde{L}$ (replaced $\zeta_\mu = \partial_\mu \Leftrightarrow \zeta^{*\mu} = \tilde{\partial}^\mu$).

Lie algebroid structures in DFT:

- The exterior algebras of multi-vectors in DFT is defined.

  $\Rightarrow$ we obtain the Lie algd. $(\wedge^\bullet L, [\cdot, \cdot]_{\mathrm{s}}, \mathrm{d})$ and its dual Lie algd. $(\wedge^\bullet \tilde{L}, [\cdot, \cdot]_{\mathrm{s}}^*, \tilde{\mathrm{d}})$.

- Next, Let us check if the pair of L and $\tilde{L}$ makes a Lie bialgebroid.

We examine the derivation condition in DFT by explicit calculation:

$$\tilde{\mathrm{d}}[A, B]_{\mathrm{s}} = [\tilde{\mathrm{d}} A, B]_{\mathrm{s}} + [A, \tilde{\mathrm{d}} B]_{\mathrm{s}} + (\eta^{MN} \partial_M A^\mu \partial_N B^\nu) \partial_\mu \wedge \partial_\nu.$$

- The last contribution represents the violation of the derivation condition.
- The last term vanishes when the strong constraint is imposed.
- So, L and $\tilde{L}$ are the Lie algebroids in DFT, but $(L, \tilde{L})$ is not a Lie bialgebroid.

  $\to$ the double $L \oplus \tilde{L}$ defines a **Vaisman algebroid in DFT**.

- This completely agrees with the analysis in [Chatzistavrakidis-Jonke-Khoo-Szabo '18]
  - where the pre-DFT algebroid (Vaisman algebroid) becomes a Courant algebroid after imposing the strong constraint.

9. Gauge Symmetry in DFT

- The structure of the C-bracket in DFT naturally arises as a Vaisman bracket on a para-Hermitian geometry. [Vaisman '13, Svoboda '18]
- The geometric realization of the C-bracket (the Vaisman bracket) is NOT necessarily require the section condition.

In DFT, the B-field gauge symmetry is non-Abelian:

- Using Φ^+, $\tilde{L}$ is identified as L^* (i.e. "winding" vectors becomes 1-forms).
- $[\cdot, \cdot]_{\tilde{L}}$ represents the B-field gauge symmetry parametrized by 1-forms.
- Since $[\cdot, \cdot]_{\tilde{L}}$ is generically non-zero, the T-duality covariantized B-field gauge symmetry is effectively enhanced to non-Abelian.

Imposing the strong constraint (SC):

- A trivial sol. of SC is equvalent to imposing the para-holomorphic cond. $\tilde{\mathrm{d}} f = 0$.
  - $[\cdot, \cdot]_{\tilde{L}}$ vanish and C-bracket is reduced to original Courant bracket.
  - The B-field gauge symmetry is realized as Abelian.

10. Conclusion

- We show that the Vaisman algd. is obtained by the "double" of two Lie algd.
- We find that an algebraic origin of SC is traced back the derivation condition.

Future direction

- Finite gauge transformations in DFT is governed by an **"integrated"** version of Vaisman algebroids.
  - This is analogous to relation between Lie algebras and Lie groups.

    $\to$ corresponding group like structures: **groupoids**

  - We expect the existence of groupoids associated with Vaisman algebroids.
- Twisted Vaisman algebroids? $\to$ using the generalized flux in DFT?
- Gauged DFT is not necessary to impose the strong constraint.

  $\to$ Vaisman algebroids would play impotant roles in applications of DFT.

- A geometric origin of DFT gauge symmetry is important to understand the **stringy winding effects** to spacetimes.
- We expect that similar discussions are applied to the exceptional field theories.