Quantum Mirror Map for Del Pezzo Geometries

arXiv:190?.?????

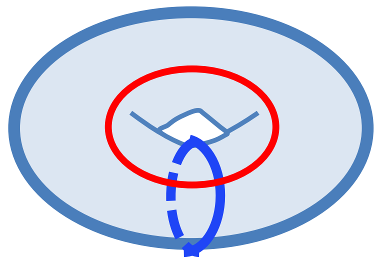
Yuji Sugimoto

(Univ. of Science and Technology of China)

with Sanefumi Moriyama
Tomohiro Furukawa
(Osaka City Univ.)

Susy. Gauge Theory

encoded in algebraic curve



A-period

↓

Mirror Map $\Pi_A(z) \sim c_1 z + c_2 z + \ldots$

redefining the variables

not well studied

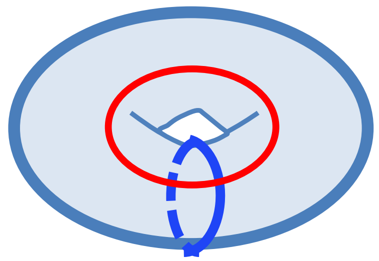
B-period

↓

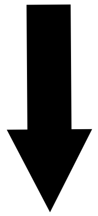
Free energy relating to BPS indices

well studied

Susy. Gauge Theory

encoded in algebraic curve 

A-period



We need this to obtain physical quntities and reach important results.

Mirror Map $\Pi_A(z) \sim c_1 z + c_2 z + \ldots$

redefining the variables

not well studied

ABJM theory ($U(N)_k \times U(N)_{-k}$)

[Aharony-Bergman-Jafferis-Maldacena 2008]

(see also Moriyama, Kubo, Furukawa's slides&Poster)

encoded in local $P_1 \times P_1$ with quantization

$$\widehat{H} = \left(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}\right)\left(\widehat{P}^{1/2} + \widehat{P}^{-1/2}\right) \qquad \widehat{Q}\widehat{P} = q\widehat{P}\widehat{Q}$$

$$\Pi_A(z) \quad \longrightarrow \quad \Pi_A(z, \hbar)$$

$$\left(q = \mathrm{e}^{\mathrm{i}\hbar}\right)$$

Quantum Mirror Map

ABJM theory ($U(N)_k \times U(N)_{-k}$)

[Aharony-Bergman-Jafferis-Maldacena 2008]

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↓ generalization

$U(N)_k \times U(N)_{-k} \times U(N)_k \times U(N)_{-k}$ (1,1,1,1) model

$$\widehat{H} = \left[\left(Q^{1/2} + Q^{-1/2}\right)\left(P^{1/2} + P^{-1/2}\right)\right]^2$$

[Honda, Moriyama, 2014]

ABJM theory ($U(N)_k \times U(N)_{-k}$)

[Aharony-Bergman-Jafferis-Maldacena 2008]

(see also Moriyama, Kubo, Furukawa's slides&Poster)

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$$\widehat{H} = \left(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}\right)\left(\widehat{P}^{1/2} + \widehat{P}^{-1/2}\right) \qquad \widehat{Q}\widehat{P} = q\widehat{P}\widehat{Q}$$

A_1 (SU(2)) sym.

↓ generalization

$U(N)_k \times U(N)_{-k} \times U(N)_k \times U(N)_{-k}$ (1,1,1,1) model

$$\widehat{H} = \left[\left(Q^{1/2} + Q^{-1/2}\right)\left(P^{1/2} + P^{-1/2}\right)\right]^2$$

D_5 (SO(10)) sym. → $[SU(2)]^3$ sym.

We study D_5 Del Pezzo

Result

A-period is similar to B-period

- expressed by Weyl characters
- coef. of characters = Integer
- has same reps. as those in B-period

Plan

① ● expressed by Weyl characters

② [
- ● coef. of characters = Integer
- ● has same reps. as those in B-period
]

D_5 Del Pezzo geometry Kubo-Moriyama-Nosaka(2018)

$$\begin{aligned}\frac{\widehat{H}}{\alpha} = \; & e_3 e_4 \widehat{Q}^{-1}\widehat{P} - (e_3+e_4)\widehat{P} + \widehat{Q}\widehat{P} \\ & - h_2^{-1} e_3 e_4 (e_5+e_6)\widehat{Q}^{-1} + \frac{E}{\alpha} - (e_1^{-1}+e_2^{-1})\widehat{Q} \\ & + h_1^2 (e_1 e_2 e_7 e_8)^{-1}\widehat{Q}^{-1}\widehat{P}^{-1} - h_1 (e_1 e_2)^{-1}(e_7^{-1}+e_8^{-1})\widehat{P}^{-1} + (e_1 e_2)^{-1}\widehat{Q}\widehat{P}^{-1}\end{aligned}$$

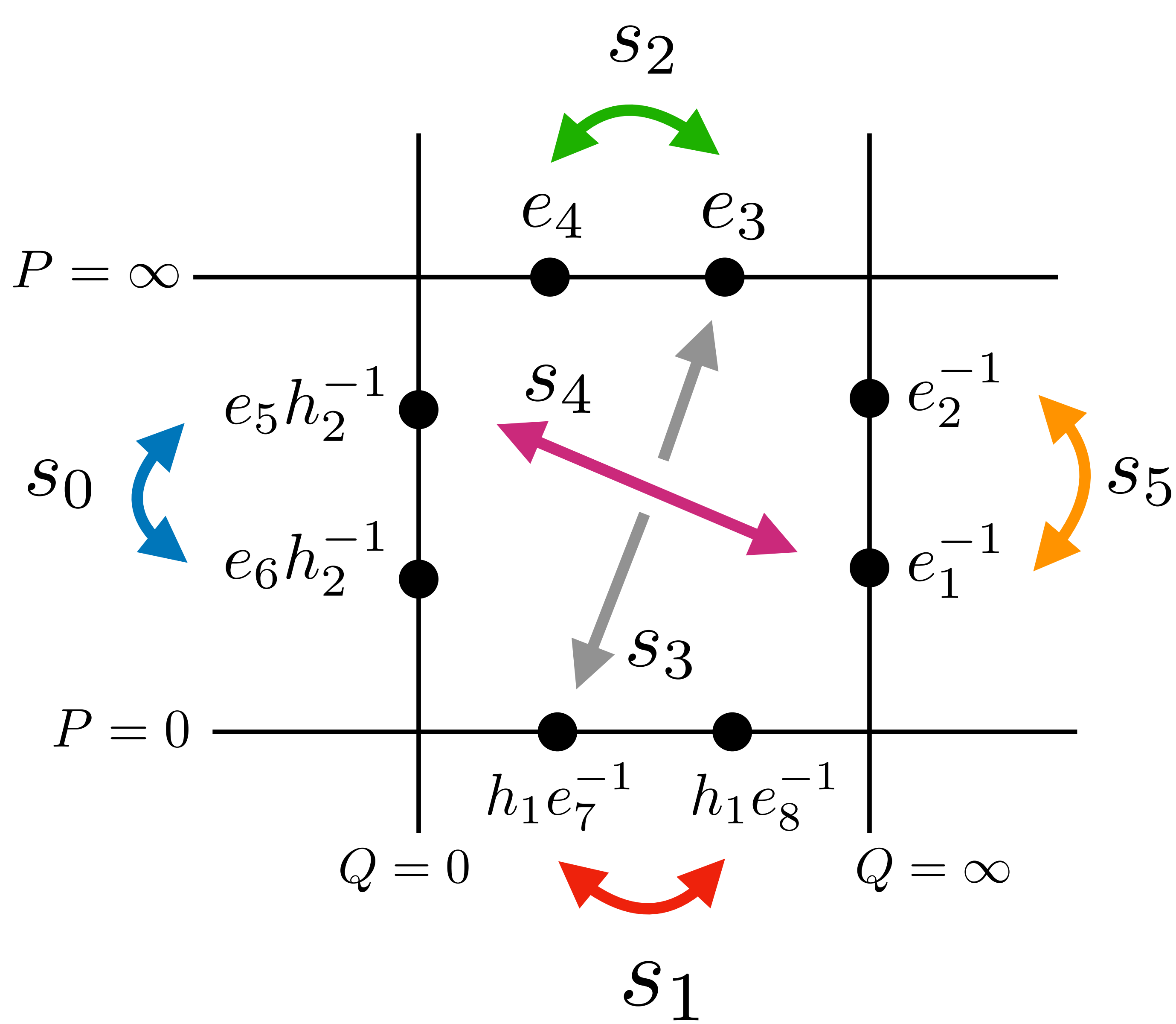
- $h_1^2 h_2^2 = \prod_{i=1}^{8} e_i$

10 parameters

$$(s_1, s_2, s_3, s_4, s_5)$$

↓

D_5 Weyl Group



D_5 Del Pezzo geometry Kubo-Moriyama-Nosaka(2018)

$$\frac{\widehat{H}}{\alpha} = e_3 e_4 \widehat{Q}^{-1}\widehat{P} - (e_3+e_4)\widehat{P} + \widehat{Q}\widehat{P} - h_2^{-1} e_3 e_4 (e_5+e_6)\widehat{Q}^{-1} + \frac{E}{\alpha} - (e_1^{-1}+e_2^{-1})\widehat{Q} + h_1^2 (e_1 e_2 e_7 e_8)^{-1}\widehat{Q}^{-1}\widehat{P}^{-1} - h_1 (e_1 e_2)^{-1}(e_7^{-1}+e_8^{-1})\widehat{P}^{-1} + (e_1 e_2)^{-1}\widehat{Q}\widehat{P}^{-1}$$

10 parameters - (2 + 2 + 1) parameters

- 8 asymptotic values
- $(\widehat{P},\widehat{Q}) \sim (A\widehat{P}, B\widehat{Q})$
- $h_1^2 h_2^2 = \prod_{i=1}^{8} e_i$

➡ 5 parameters $(e_1, e_3, e_5, \bar{h}_1, \bar{h}_2)$

$(\bar{h}_1 = q h_1,\ \bar{h}_2 = q^{-1} h_2)$

Quantum mirror map

$$\Pi_A(z,\hbar) \sim \oint \frac{\log P[X]}{X} \mathrm{d}X = E z^{-1} + \left(-\frac{E^2}{2} - A_2\right) z^{-2} + \cdots$$

$$P[X] = \frac{\Psi[q^{-1}X]}{\Psi[X]},\ \widehat{Q}\Psi[X] = X\Psi[X] \qquad \left[\frac{\widehat{H}}{\alpha} + \frac{z}{\alpha}\right]\Psi[X] = 0$$

solve Schrödinger eq. order by order

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\bar{h}_1}{e_1} + \frac{\bar{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\bar{h}_2^2} + \frac{e_3 e_5}{\bar{h}_1 \bar{h}_2^2} + \frac{e_3^2 e_5}{\bar{h}_1 \bar{h}_2^2} + \frac{e_3}{\bar{h}_2} + \frac{e_3}{\bar{h}_2 e_1} + \frac{e_3 e_5}{\bar{h}_2} + \frac{e_3 e_5}{\bar{h}_2 e_1}$$

10 terms

invariant under Weyl transf. character ??

$$\chi_{\mathbf{10}}(e_i^{-1}, \bar{h}_i^{-1}) = \chi_{\mathbf{10}}(e_i, \bar{h}_i)\ ??$$

Weyl character

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\bar{h}_1}{e_1} + \frac{\bar{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\bar{h}_2^2} + \frac{e_3 e_5}{\bar{h}_1 \bar{h}_2^2} + \frac{e_3^2 e_5}{\bar{h}_1 \bar{h}_2^2} + \frac{e_3}{\bar{h}_2} + \frac{e_3}{\bar{h}_2 e_1} + \frac{e_3 e_5}{\bar{h}_2} + \frac{e_3 e_5}{\bar{h}_2 e_1}$$

$(s_1, s_2, s_3, s_4, s_5)$ act on $(e_1, e_3, e_5, \bar{h}_1, \bar{h}_2, \alpha)$ Kubo-Moriyama-Nosaka(2018)

E.g.)

$$s_2 : (e_1, e_3, e_5, \bar{h}_1, \bar{h}_2, \alpha) \mapsto \left(e_1, \frac{1}{e_3}, e_5, \frac{\bar{h}_1}{e_3}, \bar{h}_2, e_3 \alpha \right)$$

Weyl character

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\overline{h}_1}{e_1} + \frac{\overline{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\overline{h}_2^2} + \frac{e_3 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3^2 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3}{\overline{h}_2} + \frac{e_3}{\overline{h}_2 e_1} + \frac{e_3 e_5}{\overline{h}_2} + \frac{e_3 e_5}{\overline{h}_2 e_1}$$

$(s_1, s_2, s_3, s_4, s_5)$ act on $(e_1, e_3, e_5, \bar{h}_1, \bar{h}_2, \alpha)$ Kubo-Moriyama-Nosaka(2018)

acting only 5 parameters

generate D_5 Weyl group

role of α ??

Weyl character

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\bar{h}_1}{e_1} + \frac{\bar{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\bar{h}_2^2} + \frac{e_3 e_5}{\bar{h}_1 \bar{h}_2^2} + \frac{e_3^2 e_5}{\bar{h}_1 \bar{h}_2^2} + \frac{e_3}{\bar{h}_2} + \frac{e_3}{\bar{h}_2 e_1} + \frac{e_3 e_5}{\bar{h}_2} + \frac{e_3 e_5}{\bar{h}_2 e_1}$$

$(s_1, s_2, s_3, s_4, s_5)$ act on $(e_1, e_3, e_5, \bar{h}_1, \bar{h}_2, \alpha)$ Kubo-Moriyama-Nosaka(2018)

redundant ➡ $\alpha = q^{1/2} \bar{h}_2^{1/2} e_1^{1/4} e_3^{-1/2} e_5^{-1/4}$

➡ $$A_2 = \frac{\sqrt{e_1}\sqrt{e_5}}{\bar{h}_2} + \frac{\bar{h}_2}{\sqrt{e_1}\sqrt{e_5}} + \frac{\sqrt{e_1}\sqrt{e_5}}{\bar{h}_1 \bar{h}_2} + \frac{\bar{h}_1 \bar{h}_2}{\sqrt{e_1}\sqrt{e_5}} \frac{\sqrt{e_1} e_3 \sqrt{e_5}}{\bar{h}_1 \bar{h}_2} + \frac{\bar{h}_1 \bar{h}_2}{\sqrt{e_1} e_3 \sqrt{e_5}} + \sqrt{e_1}\sqrt{e_5} + \frac{1}{\sqrt{e_1}\sqrt{e_5}} + \frac{\sqrt{e_1}}{\sqrt{e_5}} + \frac{\sqrt{e_5}}{\sqrt{e_1}}$$

$$\chi_{\mathbf{10}}$$

Plan

① ● expressed by Weyl characters

② [
- ● coef. of characters = Integer
- ● has same reps. as those in B-period
]

Multi-covering & BPS

$$\Pi_A(z,\hbar) \sim \sum_{l=1}^{\infty} (-1)^{l+1} A_l z^{-l}$$

$$A_4 \ni \frac{3}{2}\chi_{\mathbf{54}} + \frac{5}{2}\chi_{\mathbf{45}} + \frac{11}{2}\chi_{\mathbf{1}}$$ ➡ simpler structure ?

fractional

A-period for A_1 geometry has multi-covering structure

[Hatsuda-Marino-Moriyama-Okuyama(2013)]

multi-covering structure

$$(\text{coef. of n-th order}) = \sum_{j \leqq n} (\text{coef. of j-th order})$$

Multi-covering & BPS

$$\Pi_A(z,\hbar) =: \log z_{\text{eff}} - \log z$$

$$\Rightarrow \quad \log z \sim -A_1 z_{\text{eff}}^{-1} + \left(A_2 - A_1^2\right) z_{\text{eff}}^{-2} - \left(A_3 - 3A_1A_2 + \frac{3}{2}A_1^3\right) z_{\text{eff}}^{-3}$$
$$+ \left(A_4 - 2A_2^2 - 4A_1A_3 - 8A_1^2A_2 - \frac{8}{3}A_1^4\right) z_{\text{eff}}^{-4} + \ldots$$

$$\left[\begin{aligned} &A_1 = 0 \\ &A_2 = \chi_{\mathbf{10}} \\ &A_3 = (q^{1/2} + q^{-1/2})\chi_{\mathbf{16}} \\ &A_4 = \left(q^2 + q^{-2}\right)\chi_{\mathbf{1}} + \left(q^{3/2} + q^{-3/2}\right)(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + \underline{\frac{3\chi_{\mathbf{54}} + 5\chi_{\mathbf{45}} + 11\chi_{\mathbf{1}}}{2}} \end{aligned}\right.$$

still fractional

Multi-covering & BPS

$$\Pi_A(z,\hbar) =: \log z_{\mathrm{eff}} - \log z$$

$$\Rightarrow \log z \sim -A_1 z_{\mathrm{eff}}^{-1} + \left(A_2 - A_1^2\right) z_{\mathrm{eff}}^{-2} - \left(A_3 - 3A_1A_2 + \frac{3}{2}A_1^3\right) z_{\mathrm{eff}}^{-3}$$

$$+ \left(A_4 - 2A_2^2 - 4A_1A_3 - 8A_1^2A_2 - \frac{8}{3}A_1^4\right) z_{\mathrm{eff}}^{-4} + \ldots$$

$$= \epsilon_1(q, e_i, \bar{h}_i)$$

$$= -\epsilon_2(q, e_i, \bar{h}_i) + \frac{\epsilon_1(q^2, e_i^2, \bar{h}_i^2)}{2}$$

$$= \epsilon_3(q, e_i, \bar{h}_i) + \frac{\epsilon_1(q^3, e_i^3, \bar{h}_i^3)}{3}$$

$$= -\epsilon_4(q, e_i, \bar{h}_i) + \frac{\epsilon_2(q^2, e_i^2, \bar{h}_i^2)}{2} + \frac{\epsilon_1(q^4, e_i^4, \bar{h}_i^4)}{4}$$

multi-covering structure

$$\epsilon_1 = 0, \quad -\epsilon_2 = \chi_{\mathbf{10}},$$

$$\epsilon_3 = \left(q^{1/2} + q^{-1/2}\right)\chi_{\mathbf{16}}\ , -\epsilon_4 = \left(q^2 + q^{-2}\right)\chi_{\mathbf{1}} + \left(q + q^{-1}\right)\left(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}\right) + 4\chi_{\mathbf{1}}$$

integer !

Multi-covering & BPS

$$\Pi_A(z,\hbar) =: \log z_{\text{eff}} - \log z$$

$$\Rightarrow \quad \log z \sim \sum_{l=1}^{\infty}\left(\sum_{n|l}\frac{(-1)^n \epsilon_{\frac{l}{n}}(q^n, e_i^n, \bar{h}_i^n)}{n}\right) z_{\text{eff}}^{-l}$$

$$\left[\begin{aligned} \epsilon_1 &= 0, \\ -\epsilon_2 &= \chi_{\mathbf{10}}, \\ \epsilon_3 &= (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\chi_{\mathbf{16}}, \\ -\epsilon_4 &= (q^2 + q^{-2})\chi_{\mathbf{1}} + (q + q^{-1})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}}, \end{aligned}\right.$$

multi-covering structure

→ (1,1,1,1) has multi covering structure

Multi-covering & BPS

[Moriyama-Nosaka-Yano(2017)]

d	$(j_{\mathrm{L}}, j_{\mathrm{R}})$	BPS	$(-1)^{d-1}\sum_{\lvert d\rvert=1}(N^{d}_{j_{\mathrm{L}},j_{\mathrm{R}}})_{d'} \quad d$	representations
1	$(0;0)$	16	$8_{+1} + 8_{-1}$	**16**
2	$(0, \frac{1}{2})$	10	$1_{+2} + 8_0 + 1_{-2}$	**10**
3	$(0, 1)$	16	$8_{+1} + 8_{-1}$	**16**
4	$(0, \frac{1}{2})$	1	1_0	**1**
	$(0, \frac{3}{2})$	45	$8_{+2} + 29_0 + 8_{-2}$	**45**
	$(\frac{1}{2}, 2)$	1	1_0	**1**

$$
\begin{cases}
\epsilon_1 = 0, \\
-\epsilon_2 = \chi_{\mathbf{10}}, \\
\epsilon_3 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\chi_{\mathbf{16}}, \\
-\epsilon_4 = (q^2 + q^{-2})\chi_{\mathbf{1}} + (q + q^{-1})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}},
\end{cases}
$$

B-period

same reps. as B-period in same degree

Summary

A-period is similar to B-period

- expressed by Weyl characters
- coef. of characters = Integer
- has same reps. as those in B-period

Future work

Q. why same structure ?

Q. what happen when use SU(2) character ?

Q. what meaning of α ?

before using SU(2) character

$$\epsilon_7 = (q^{\frac{11}{2}} + q^{-\frac{11}{2}})\chi_{\mathbf{16}} + (q^{\frac{9}{2}} + q^{-\frac{9}{2}})(\chi_{\mathbf{144}} + 4\chi_{\mathbf{16}}) + (q^{\frac{7}{2}} + q^{-\frac{7}{2}})(\chi_{\mathbf{560}} + 4\chi_{\mathbf{144}} + 13\chi_{\mathbf{16}})$$
$$+ (q^{\frac{5}{2}} + q^{-\frac{5}{2}})(\chi_{\mathbf{720}} + 4\chi_{\mathbf{560}} + 9\chi_{\mathbf{144}} + 25\chi_{\mathbf{16}}) + (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(3\chi_{\mathbf{560}} + 8\chi_{\mathbf{144}} + 27\chi_{\mathbf{16}})$$
$$+ (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(\chi_{\mathbf{720}} + 3\chi_{\mathbf{560}} + 9\chi_{\mathbf{144}} + 27\chi_{\mathbf{16}}),$$

---

after using SU(2) character

$$\epsilon_7 = \chi_{\frac{11}{2}}\chi_{\mathbf{16}} + \chi_{\frac{9}{2}}(\chi_{\mathbf{144}} + 3\chi_{\mathbf{16}}) + \chi_{\frac{7}{2}}(\chi_{\mathbf{560}} + 3\chi_{\mathbf{144}} + 9\chi_{\mathbf{16}})$$
$$+ \chi_{\frac{5}{2}}(\chi_{\mathbf{720}} + 3\chi_{\mathbf{560}} + 5\chi_{\mathbf{144}} + 12\chi_{\mathbf{16}}) + \chi_{\frac{3}{2}}(-\chi_{\mathbf{720}} - \chi_{\mathbf{560}} - \chi_{\mathbf{144}} + 2\chi_{\mathbf{16}})$$
$$+ \chi_{\frac{1}{2}}(\chi_{\mathbf{720}} + \chi_{\mathbf{144}}),$$

χ_n SU(2) character

much simpler for q-dependent term & coef.

but negative coef,

Future work

Q. why same structure ?

Q. what about SU(2) character ?

Q. what meaning of α ?

Summary

A-period is similar to B-period

- expressed by Weyl characters

- coef. of characters = Integer

- has same reps. as those in B-period

without overall minus sign do not appear

$$\epsilon_4 = (q^2 + q^{-2})\,\chi_{\mathbf{1}} + (q + q^{-1})\,(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + (-\chi_{\mathbf{54}} + \chi_{\mathbf{45}} + 3\chi_{\mathbf{1}})$$

with overall minus sign

$$-\epsilon_4 = (q^2 + q^{-2})\chi_{\mathbf{1}} + (q + q^{-1})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}}$$

4	$(0, \frac{1}{2})$	1	1_0		$\mathbf{1}$	
	$(0, \frac{3}{2})$	45	$8_{+2} + 29_0 + 8_{-2}$		$\mathbf{45}$	
	$(\frac{1}{2}, 2)$	1	1_0		$\mathbf{1}$	

multi-covering structure

(coef. of n-th order) = $\sum_{j \leqq n}$ (coef. of j-th order)

- $(e_1, e_3, e_5, \bar{h}_1, \bar{h}_2) = (1, 1, 1, q, q^{-1})$ (2,2) model

  $= (q^{-1/2}, q^{1/2}, q^{-1/2}, q, q^{-1})$ (1,1,1,1) model

$$\mu = \log z, \quad \mu_{\text{eff}} = \log z_{\text{eff}}$$

(effective) chemical pot.

= log [(effective) complex modulus]

$$\Xi_k(z) = \sum_{N=0}^{\infty} z^N \underline{Z_k(N)} = \text{Det}\left[1 + z\widehat{H}^{-1}\right],$$

Partition fn. of N M2-branes

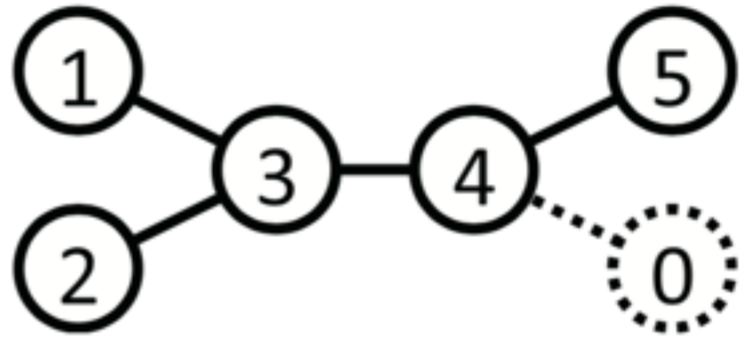


Figure 3: The Dynkin diagram of the D_5 algebra.

from [Kubo,Moriyama,Nosaka(2018)]