国立大学法人 静岡大学

Hermitian Matrix Model with Cusp Potential

Kento Sugiyama (Shizuoka Univ.)

in collaboration with Takeshi Morita

(Ongoing work)

* Summary

* Phase transitions of matrix models at large-N are related to various aspects of gauge theories and string theories.

* In this study, we investigate 0 and 1 dimensional Hermitian matrix models (HMMs) with cusp potentials at large-N.

Ex.) One-HMMs in 0dim.

$$Z = \int dM \exp\left[-N \mathrm{Tr} V(M)\right]$$ (M:N×N Hermitian matrix, V(M):Potential)

In the case of ordinary smooth potentials, the general analysis are well known.

[Brezin,Itzykson,Parisi,Zuber'78]et.al

But in the case of singular ones, the results have not been analyzed yet.

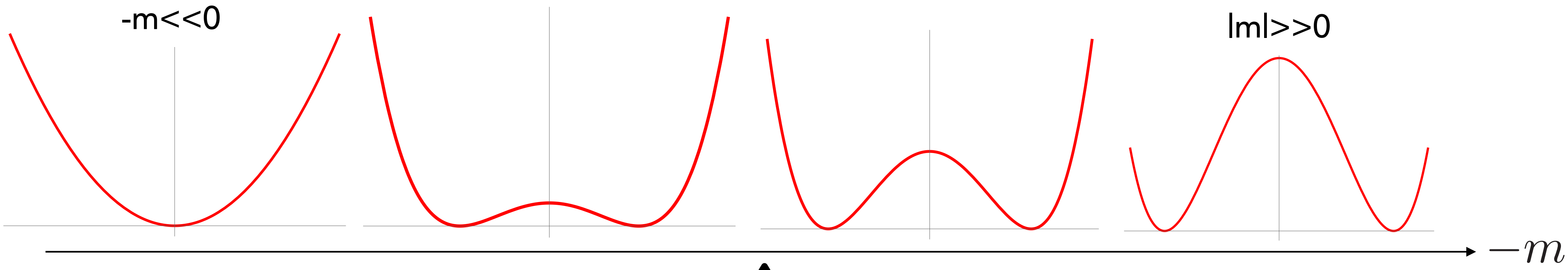
So as a trial, we consider the following singular potential.

$$V(x) = \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases} = \underbrace{\frac{x^2}{2}}_{\text{Gaussian}} - \underbrace{g|x|}_{\text{Singular term (Non-polynomial)}} + \frac{g^2}{2}$$

* Summary

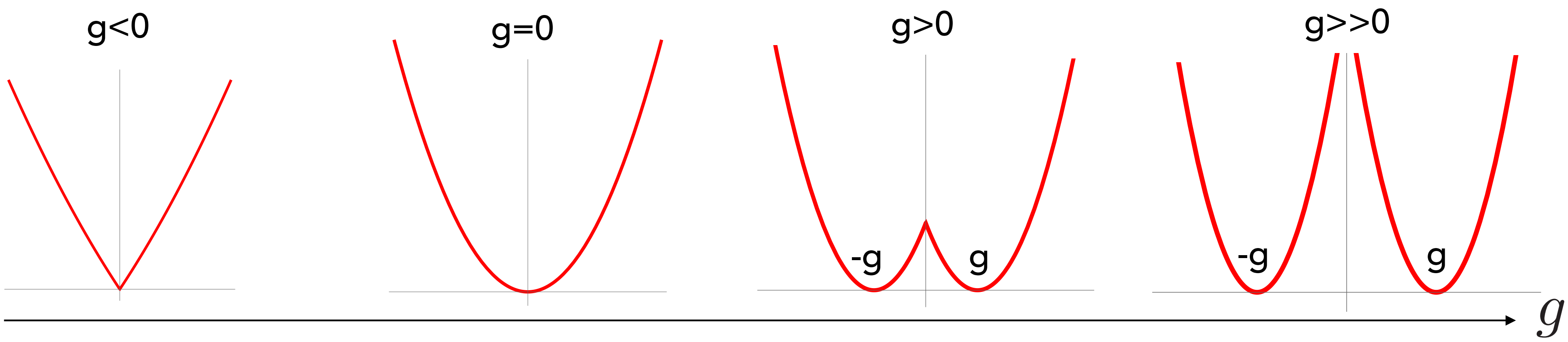
Let us compare a ordinary quartic anharmonic potential and our cusp potential.

Ordinary potential case : $V(x)=mx^2+g_4x^4$ ($g_4>0$:fixed) [Brezin,Itzykson,Parisi,Zuber'78] et.al [Gross,Witten'80],[Wadia'80] et.al



Although this cusp potential is singular at x=0,
this looks similar to a quartic anharmonic potential.

Our cusp potential case : $V(x)=x^2/2-g|x|+g^2/2$

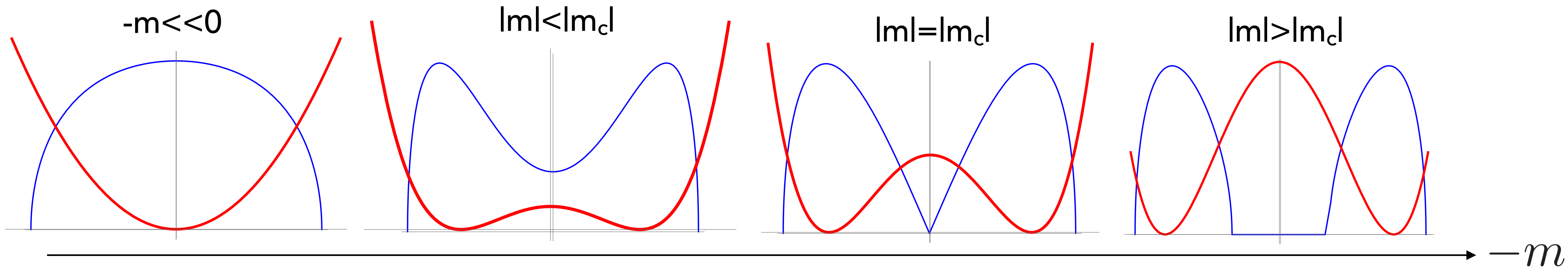


* Summary

In general, the phase structures are characterized by the eigenvalue density $\rho(x)$.

Ordinary potential case : $V(x)=mx^2+g_4x^4$ (g_4>0:fixed)
$\rho(x)$: eigenvalue density

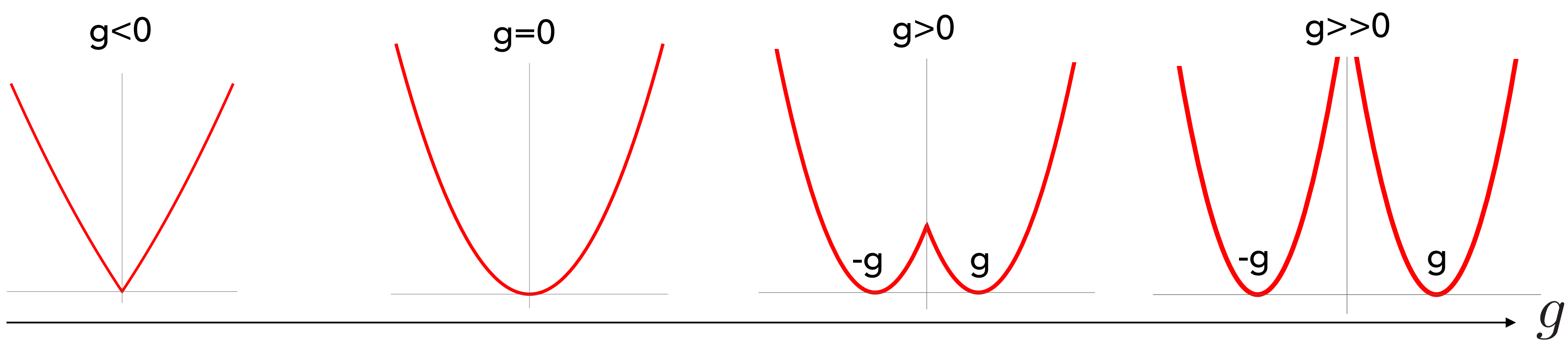
[Brezin,Itzykson,Parisi,Zuber'78] et.al
[Gross,Witten'80],[Wadia'80] et.al



one-cut — 3rd order transition at m=m$_c$: finite — two-cut

A Similar phase structure is anticipated in our cusp potential case.

Our cusp potential case : $V(x)=x^2/2-g|x|+g^2/2$

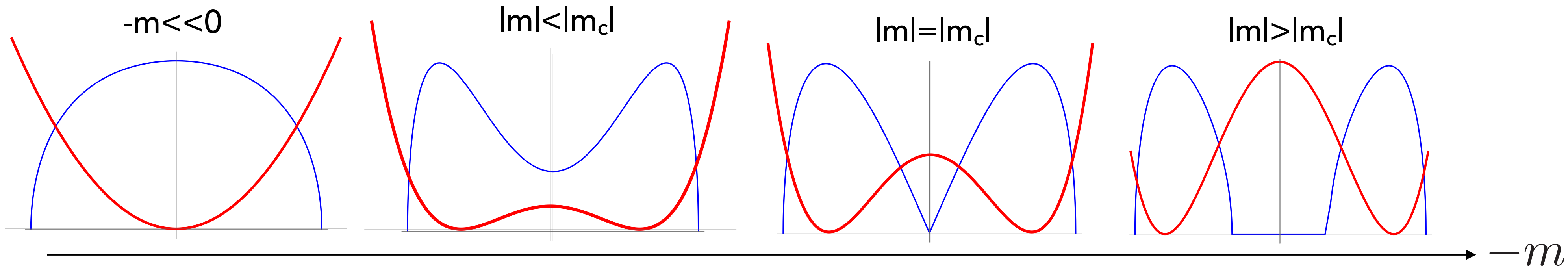


＊ Summary

In general, the phase structures are characterized by the eigenvalue density $\rho(x)$.

Ordinary potential case ： $V(x)=mx^2+g_4x^4$ ($g_4>0$:fixed)　[Brezin,Itzykson,Parisi,Zuber'78] et.al [Gross,Witten'80],[Wadia'80] et.al

$\rho(x)$ ： eigenvalue density



← one-cut →　　3rd order transition at $m=m_c$: finite　　← two-cut →

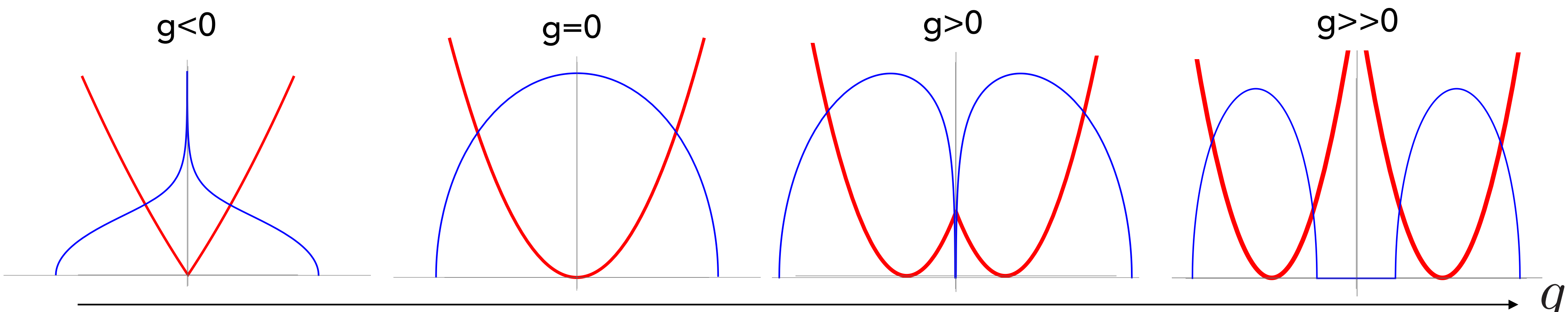
However this expectation is NOT true.

We will show that large N phase transitions in these models are quite different from the 3rd order phase transitions.

Our cusp potential case ： $V(x)=x^2/2-g|x|+g^2/2$

$\rho(x)$ ： eigenvalue density

Difference :
g>0 is always in two-cut phase.



← one-cut →　　NOT GWW-type transition at g=0　　← two-cut →

* Summary

* Table of our results

We investigate the 0dim. cases and 1dim. cases with cusp potentials.

⇒ Amazingly, we find that in the case of cusp potentials, these phase structures are also different.

	Ordinary Potentials	Cusp Potentials
0dim.HMMs	GWW-type 3rd order phase transition at $g=g_c$ [Brezin,Itzykson,Parisi,Zuber'78] et.al [Gross,Witten'80],[Wadia'80] et.al	**No transition in $g>0$**
1dim.HMMs		**2nd order phase transition at $g=g_c$ (NOT GWW-type)**

<u>This talk:</u> We show these results!

Plan of my talk

* ~~Summary~~

* 0dim. HMM with cusp potential

* 1dim. HMM with cusp potential

* Conclusions

* 0dim. HMM with cusp potential

<table>
<tr><th></th><th>Ordinary Potentials</th><th>Cusp Potentials</th></tr>
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3rd order phase transition
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</table>

✱ 0dim. HMM with cusp potential

✱ Review on 0dim. HMMs at large-N [Brezin,Itzykson,Parisi,Zuber'78]

- **Partition function** (M: N×N Hermitian matrix) $(F = -\log Z)$

$$Z = \int dM \exp\left[-N \mathrm{Tr} V(M)\right]$$

$$\propto \int \prod_{i=1}^{N} dx_i \, \Delta^2(x) \exp\left[-N \sum_i V(x_i)\right]$$

gauge fixing $M = U \Lambda U^\dagger$

U: N×N unitary matrix
Λ=diag(x_1,x_2,…,x_N)

$$\left(\Delta(x) = \left| \frac{\partial M}{\partial U} \right| = \prod_{i<j}^{N} (x_i - x_j) \right)$$

- **Saddle point equation at large-N**

potential　　repulsive force between eigenvalues

$$V'(x_i) = \frac{2}{N} \sum_{j \neq i}^{N} \frac{1}{x_i - x_j} \qquad i = 1, \cdots N$$

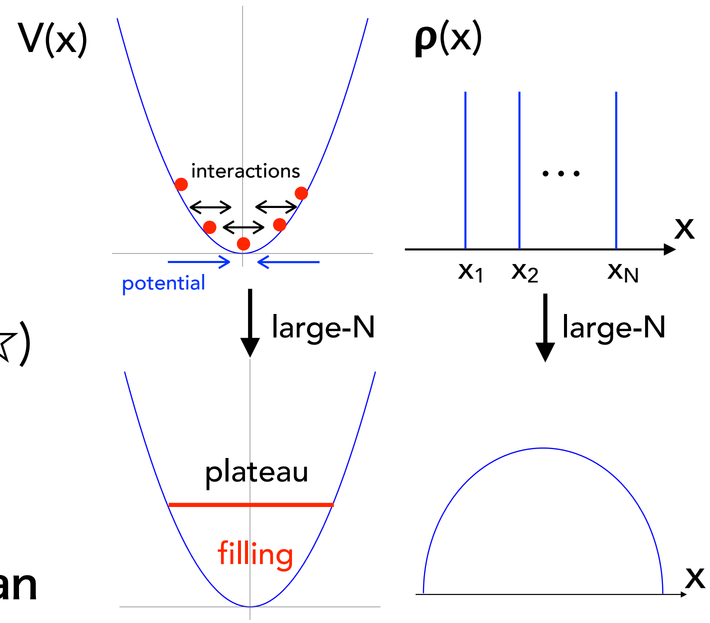
large-N

$$V'(x) = 2 \, ⨍_C dw \frac{\rho(w)}{x - w} \quad \cdots (☆)$$

def.) eigenvalue density

$$\rho(x) := \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \geq 0$$

C: support of ρ(w)



When ρ(x) is obtained at large-N, the free energy can be calculated by using the saddle point approximation. So we solve the equation (☆) in order to evaluate ρ(x) at large-N.

* 0dim. HMM with cusp potential

$$V(x) = \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases}$$

* Consider the cusp potential case.

- One-cut solution

$$\rho(x) = \underbrace{\frac{1}{2\pi}\sqrt{b^2-x^2}}_{\text{The ordinary semi-circle term}} - \underbrace{\frac{g}{2\pi}\log\left(\frac{b+\sqrt{b^2-x^2}}{b-\sqrt{b^2-x^2}}\right)}_{\text{New term}}$$

(b: the end points given by pareameter g)

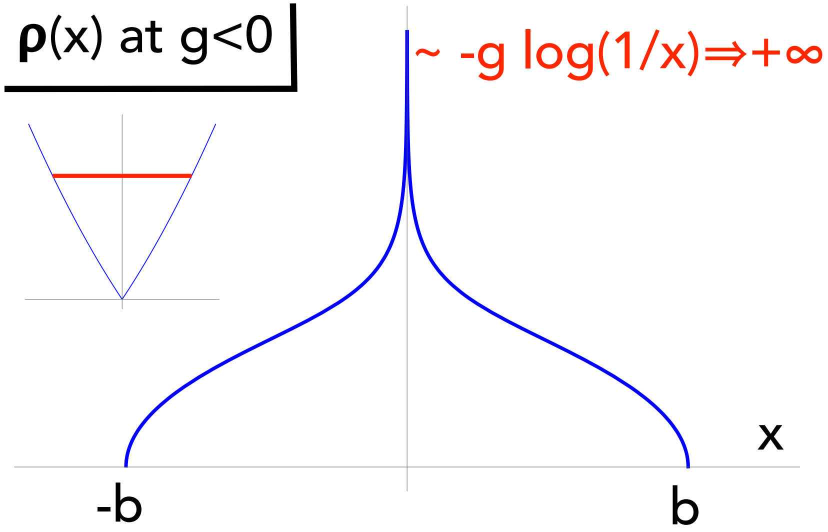
The ordinary semi-circle term

New term
It has a logarithmic singularity at x=0.

$$\rho(x)|_{x=0} \to -\frac{g}{2\pi}\log(\infty)$$

i) g<0

ρ(x) at g<0

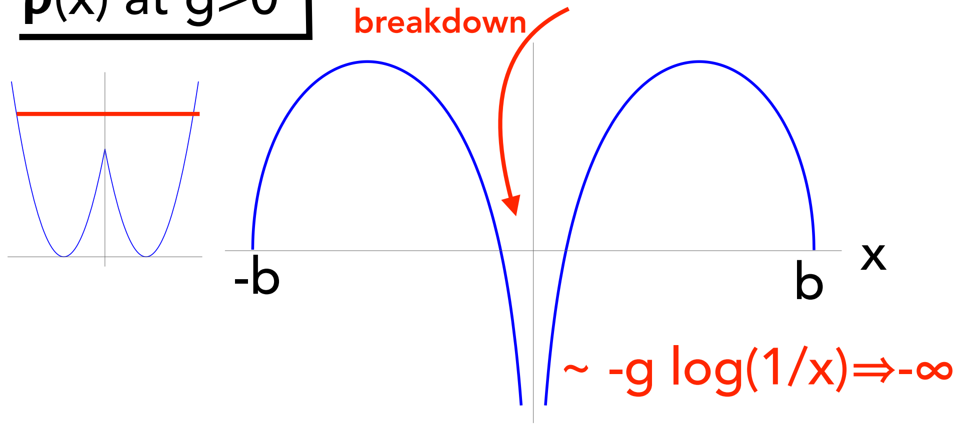


It is consistent on the cut [-b,b].

ii) g>0

$$\rho(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i) \geq 0$$

ρ(x) at g>0



Actually, it is always "wrong" in g>0.
∵ ρ(x) must be positive by definition.
When g>0, the logarithmic divergence makes negativity of ρ(x) near x=0.

＊ 0dim. HMM with cusp potential

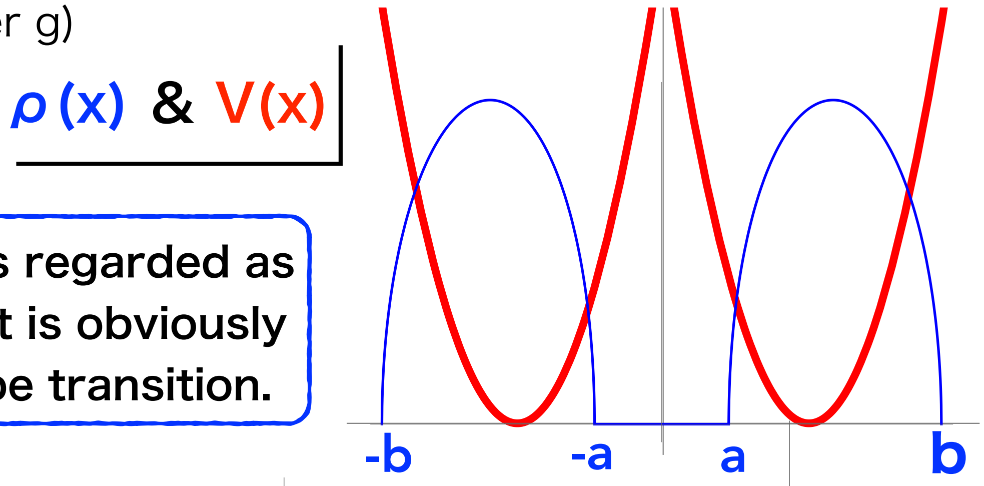
$$V(x) = \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases}$$

＊ Consider the cusp potential case.

- Two-cut solution at g>0

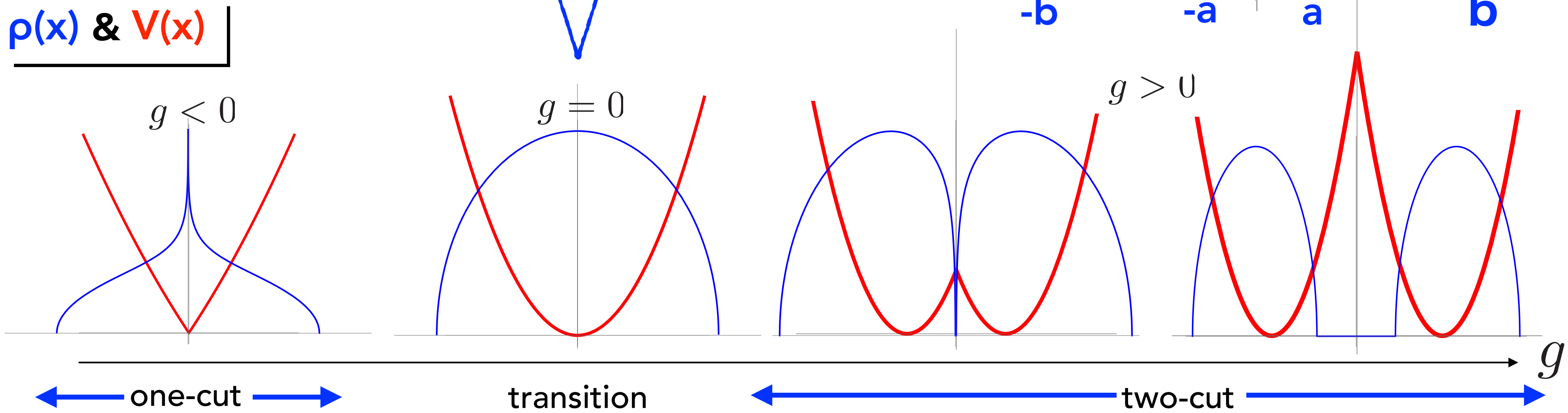
$$\rho(x) = \frac{g}{\pi^2 b x}\sqrt{(x^2-a^2)(b^2-x^2)}\,\mathrm{Im}\left[\Pi\left(\frac{a^2}{x^2},\frac{a^2}{b^2}\right) - \Pi\left(\frac{a^2}{x^2},\sin^{-1}\frac{b}{a},\frac{a^2}{b^2}\right)\right]$$

(a,b: the end points given by pareameter g)



＊ Phase structure

If the phenomenon at g=0 is regarded as a large-N phase transition, it is obviously different from the GWW-type transition.



* 0dim. HMM with cusp potential

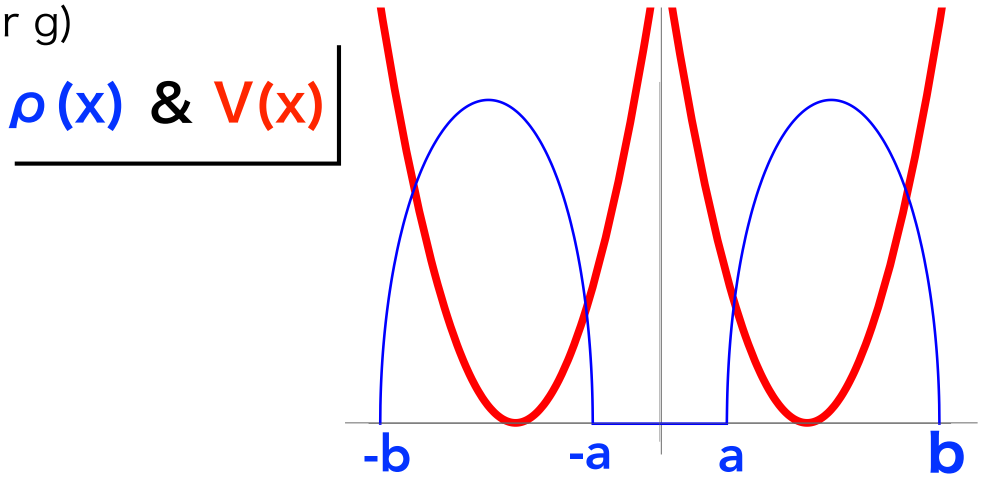
$$V(x) = \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases}$$

* Consider the cusp potential case.

- Two-cut solution at g>0

$$\rho(x) = \frac{g}{\pi^2 bx}\sqrt{(x^2-a^2)(b^2-x^2)}\,\mathrm{Im}\left[\Pi\left(\frac{a^2}{x^2},\frac{a^2}{b^2}\right) - \Pi\left(\frac{a^2}{x^2},\sin^{-1}\frac{b}{a},\frac{a^2}{b^2}\right)\right]$$

(a,b: the end points given by pareameter g)

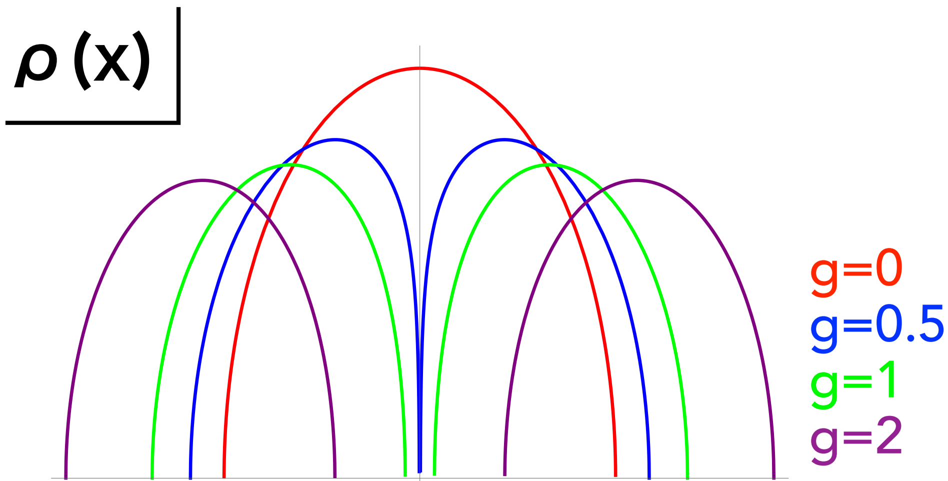


The two-cut solution is consistent in g>0.
We investigate the end points near g=0.

$$\rightarrow \begin{cases} a = 4e^{-\pi/g} + \mathcal{O}\left(e^{-2\pi/g}\right) \\ b = 2 + \mathcal{O}\left(e^{-2\pi/g}\right) \end{cases}$$

- Normally, behaviors of closing to cuts near critical points are order $(g-g_c)^{\#}$.
- In this case, the gap of each cuts is exponentially small $e^{-\pi/g}$ near g=0.

⇒The strange behaviors of the end point suggest that this transition may be different from the ordinary ones.



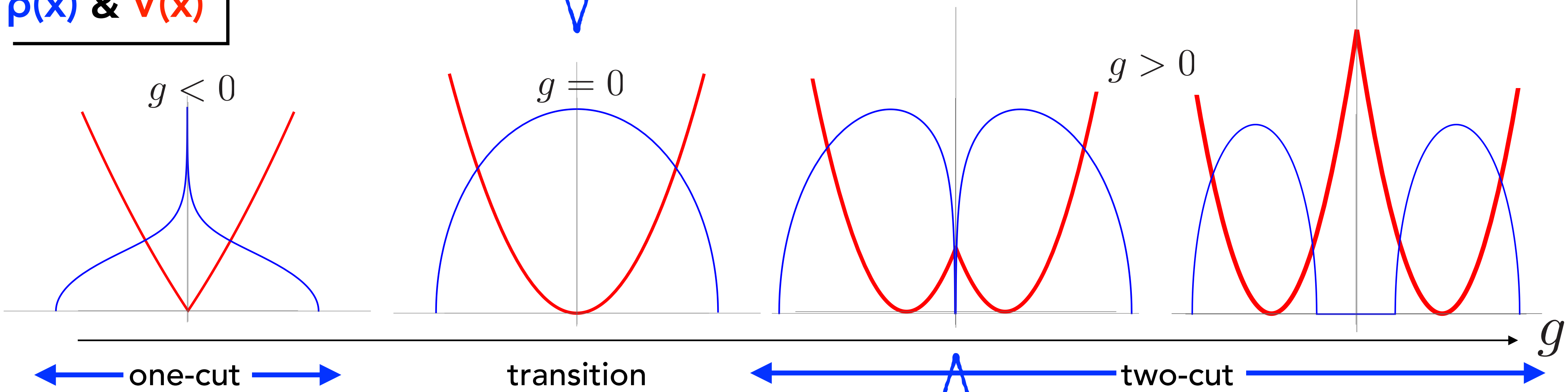
* 0dim. HMM with cusp potential

$$V(x) = \frac{1}{2}\begin{cases}(x-g)^2 & (x>0)\\(x+g)^2 & (x<0)\end{cases}$$

* Phase structure

If the phenomenon at g=0 is regarded as a large-N phase transition, it is obviously different from the GWW-type transition.

ρ(x) & V(x)



If potentials have cusp singularities, Eigenvalues cannot be located at singular points.

⇒ <u>Our Claim</u> : 0dim.HMMs with cusp potentials might have NO large-N phase transition at finite couplings.

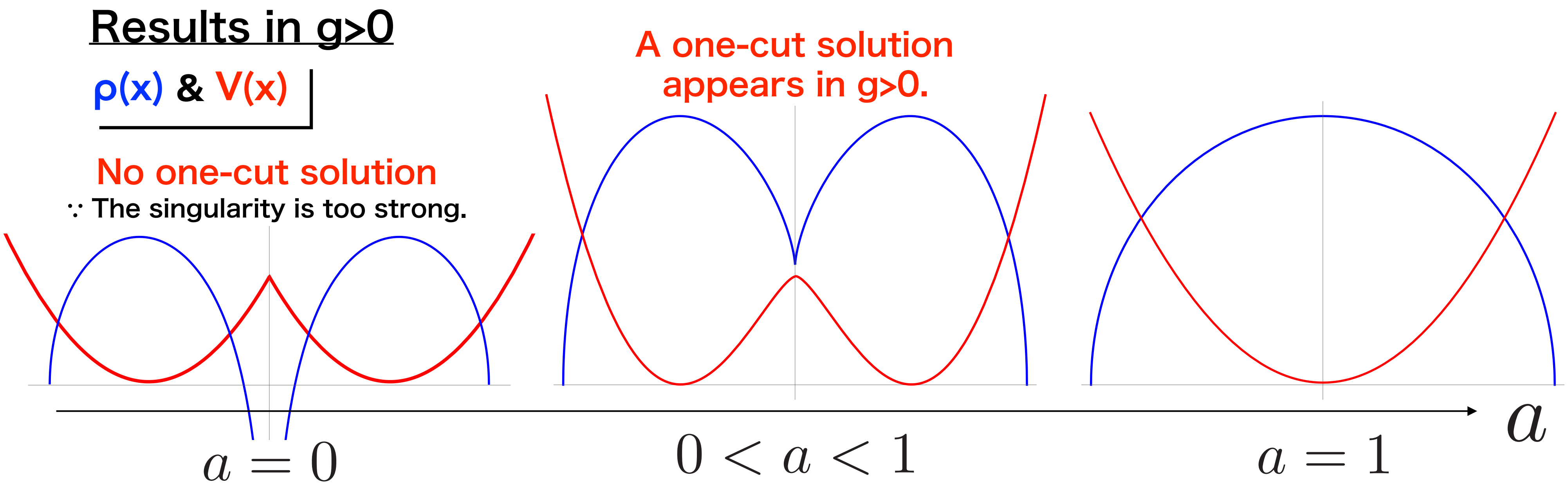
* 0dim. HMM with cusp potential

* Comment

To investigate how to change the phase structure by changing the singularity, we consider the following non-polynomial potential.

$$V(x) = \frac{x^2}{2} - g|x|^{1+a} \qquad (0 \leq a \leq 1)$$

※a=0: Our cusp potential ※a=1: The ordinary Gaussian potential



We can find a consistent one-cut solution in g>0 and o<a<1.
In this mild singularity case 0<a<1, the GWW-type transition may occur.

* 1dim. HMM with cusp potential

	Ordinary Potentials	Cusp Potentials
0dim.HMMs	GWW-type 3rd order phase transition at $g=g_c$ [Brezin,Itzykson,Parisi,Zuber'78] et.al [Gross,Witten'80],[Wadia'80] et.al	**No transition in g>0**
1dim.HMMs		**2nd order phase transition at $g=g_c$ (NOT GWW-type)**

* 1dim. HMM with cusp potential

* Review on 1dim. HMMs at large-N [Brezin,Itzykson,Parisi,Zuber'78]

- Partition function (M(t): N×N Hermitian matrix field, A_0(t): gauge field, t: time)

$$Z = \int \mathcal{D}M\mathcal{D}A_0 \exp\left[-N\int dt \mathrm{Tr}\left[\frac{1}{2}(D_t M)^2 + V(M)\right]\right] \quad (F = -\log Z)$$

- Hamiltonian

$$H = -\frac{1}{2N^2}\mathrm{Tr}\left[\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right] + \mathrm{Tr}V(M)$$

↓ gauge fixing $M = U\Lambda U^\dagger$

U: N×N unitary matrix
Λ=diag(x_1,x_2,…,x_N)

$$\left(\Delta(x) = \left|\frac{\partial M}{\partial U}\right| = \prod_{i<j}^{N}(x_i - x_j)\right)$$

$$H \to \Delta^{-1}\tilde{H}\Delta$$

$$\tilde{H} = \sum_{n=1}^{N}\tilde{H}_n, \quad \left(\tilde{H}_n = \frac{p_n^2}{2} + V(x_n), \; p_n = -\frac{i}{N}\frac{\partial}{\partial x_n}\right)$$

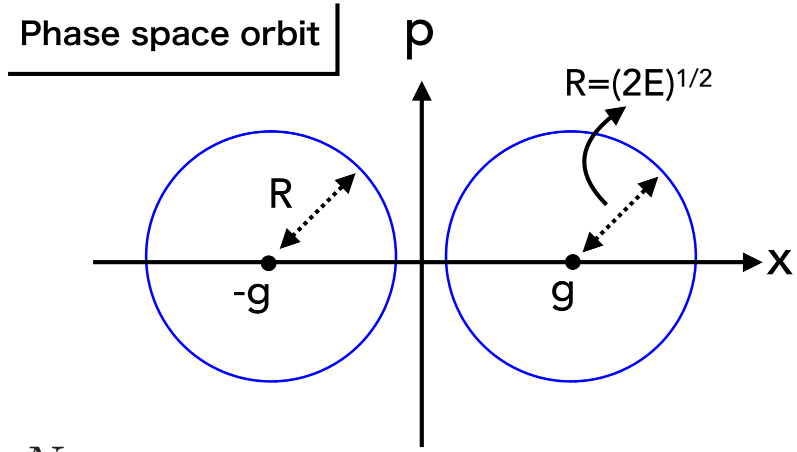
- Schrödinger equation

$$\Delta^{-1}\tilde{H}_n\Delta\phi_n(x) = E_n\phi_n(x) \quad \to \quad \tilde{H}_n\psi_n(x) = E_n\psi_n(x) \quad n = 1,\cdots N$$

$$\psi_n := \Delta\phi_n$$

It is equivalent to a N-body free fermion system. So the free energy at large-N can be obtained by using the WKB approximation.

＊ 1dim. HMM with cusp potential



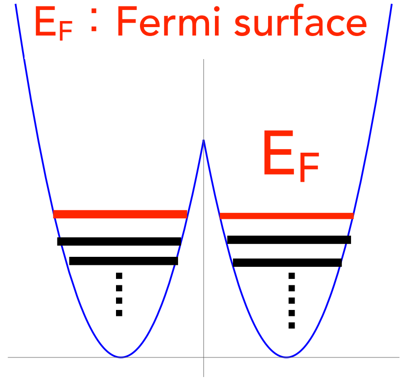
＊ Consider the cusp potential case.

- g>>0 case (two-cut phase)

$$E = \frac{p^2}{2} + \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases}$$

⇒ Bohr-Sommerfeld quantization condition

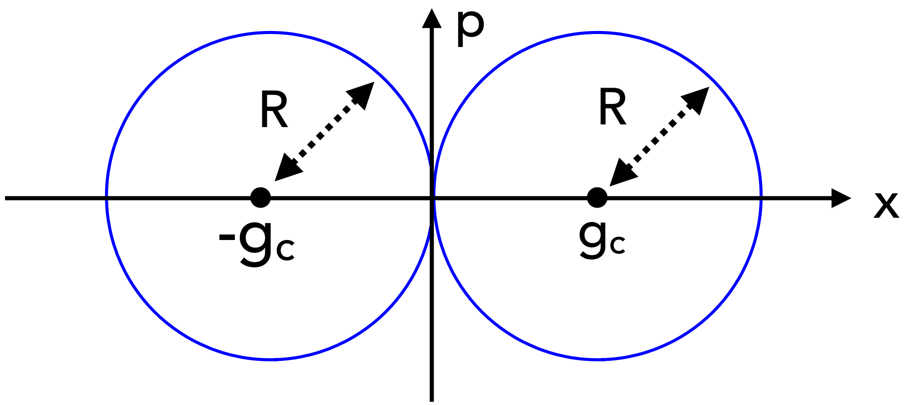
$$\text{Area} = \pi R^2 \times 2 = 2\pi n,\ (n = 1, 2, \cdots) \ \rightarrow E_F = \frac{N}{2}$$

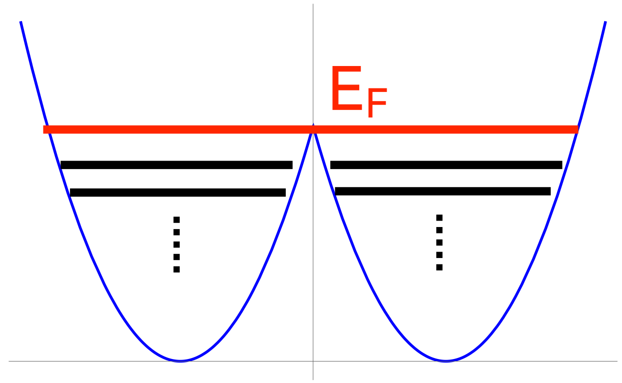


⇒ Free energy at large-N

$$F = \sum_{n=1}^{N} E_n \approx \int_0^{E_F} dE \left(\frac{\partial n}{\partial E}\right) E \ \ = E_F^2 = \frac{N^2}{4}$$

＊ Here, we find a critical point by changing g.





$g_c = R = (2E_F)^{1/2}$ ⇒ $g_c = N^{1/2}$

* 1dim. HMM with cusp potential

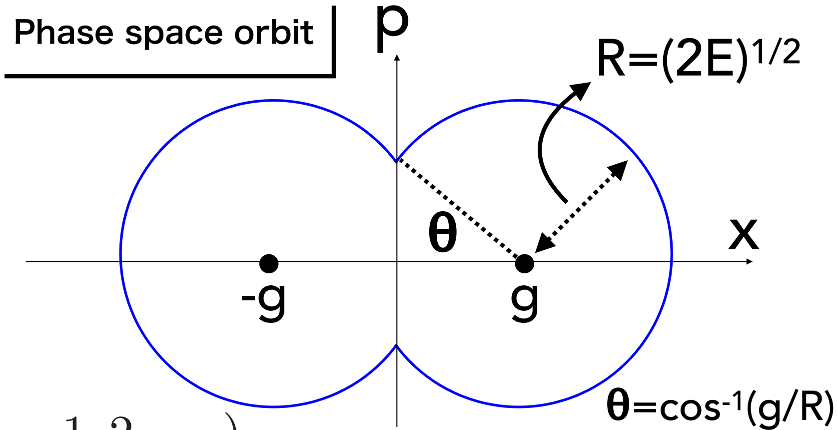
* Consider the cusp potential case.
  - g<gc case (one-cut phase)

$$E = \frac{p^2}{2} + \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases}$$

⇒ Bohr-Sommerfeld quantization condition

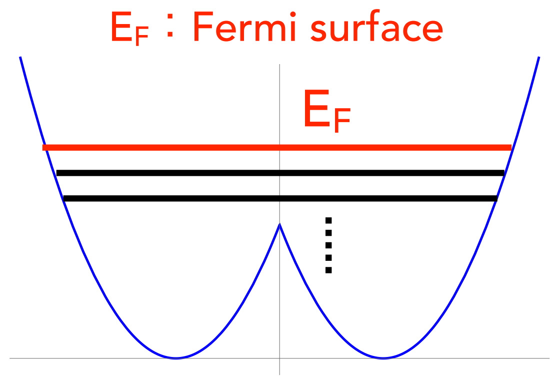
$$\mathrm{Area} = \left(\pi R^2 - \theta R^2 + g\sqrt{R^2 - g^2}\right) \times 2 = 2\pi n, \ (n = 1, 2, \cdots)$$

$$\rightarrow E_F = E_F(g, N)$$



⇒ Free energy at large-N

$$F \approx \int_0^{E_F} dE \left(\frac{\partial n}{\partial E}\right) E = F(g, E_F)$$



Near g=gc

Here we investigate the free energy F(g,EF) near g=gc.

$$\delta F = F_{\mathrm{one}}|_{g\to g_c} - F_{\mathrm{two}}|_{g\to g_c}$$

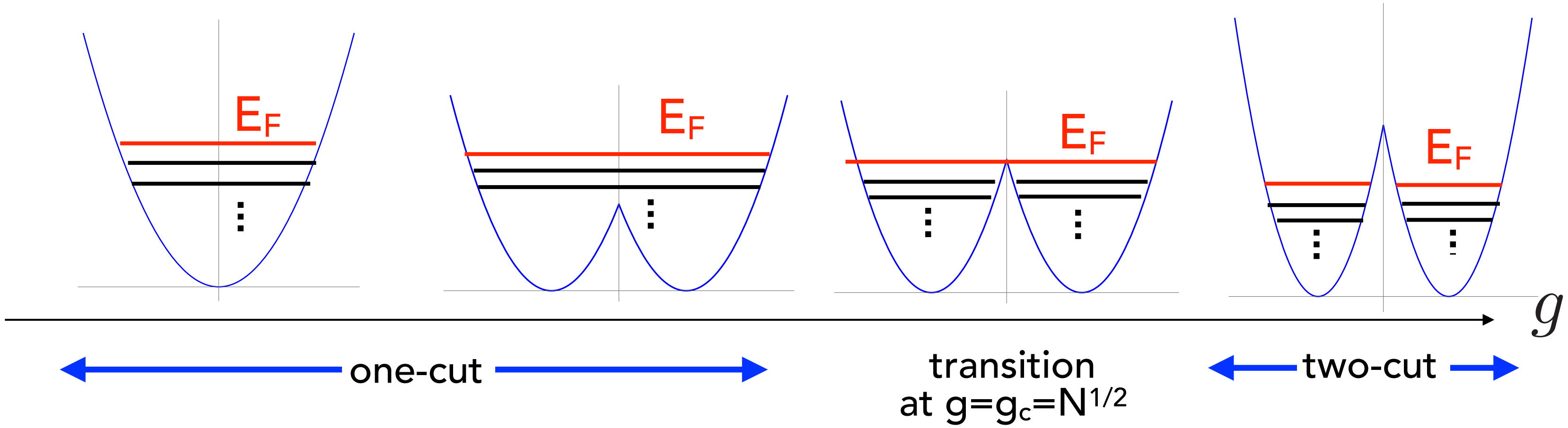
$$= (\delta E_F)^{5/3} + \cdots$$

⇒ The 2nd derivative diverges.
So it is a 2nd order phase transition.

∗ 1dim. HMM with cusp potential

$$V(x) = \frac{1}{2}\begin{cases} (x-g)^2 & (x>0) \\ (x+g)^2 & (x<0) \end{cases}$$

∗ Phase structure



∗ This phase transition is NOT a GWW-type transition.

$$\delta F = F_{\rm one}|_{g\to g_c} - F_{\rm two}|_{g\to g_c}$$
$$= (\delta E_F)^{5/3} + \cdots$$

∵ The 2nd derivative diverges.
So it is a 2nd order phase transition.

※ In the case of phase transitions of smooth potentials

$$\delta F = (\delta E_F)^2 \log(\delta E_F) + \cdots$$

⇒ The 3rd derivative diverges.

∗ In addition, we show that not only this cusp potential case, but orders of phase transitions of general cusp potentials are universally 2nd!

⇒ Our Claim : 1dim.HMMs with cusp potentials might have a new universality class near critical points.

* Conclusions

＊ Conclusions

＊ In this study, we investigate 0 and 1 dimensional HMMs with cusp potentials at large-N.

＊ In the case of the 0dim.HMMs, we show that there is no phase transition at finite coupling.

＊ On the other hand, in the case of the 1dim.HMMs, we show that the orders of the large-N phase transitions of these models are universally 2nd (not 3rd).

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</table>

Thank you for your attention!

App. Motivations for HMM with Cusp

∗ We expect that the critical phenomena of HMMs with cusp potentials have physical roles just like the cases of the ordinary smooth ones.

∗ Recently similar problems of HMMs with cusp potentials at large-N appear in several models.

① N≧2 SUSY Chern-Simons matter theories on S^3

(called CS matrix models including the ABJM matrix model)

☆ By considering special solutions, Cusp potentials appear in these models. [Morita,KS'17],[Morita,KS'18]

☆ Cusp potentials appear in the cases coupled massive matters.

[Barranco,Russo'14],[Santilli,Tierz'18]

② Higher rank Wilson loops in N=4 SYM

☆Generating functions of the Wilson loops at some special limit are analyzed by using a HMM with a cusp potential. [Okuyama'17]