The Geometry of Double Field Theory

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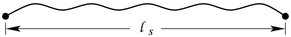
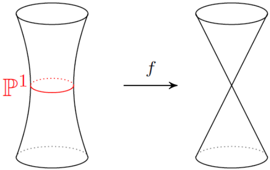
String Geometry



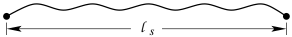
- Strings see geometry in different ways than particles do
- Sometimes strings see **deformations of geometry**
- **E.g. Resolutions of singularities:**



- Probes of Planck scale quantum geometry:

  Spacetime uncertainty $\Delta x \geq \ell_s$ related to noncommutative spacetime structure?

String Geometry



- Spacetime geometry in string theory is an approximate notion:
  Valid at sizes $R \gg \ell_s$, but breaks down at $R \sim \ell_s$ due to non-locality

- Isolate geometry from non-locality: Geometry makes sense in decoupling limit $\alpha' = \ell_s^2 \longrightarrow 0$ with R finite

- Not all spacetime geometries are ordinary geometric spaces,
  e.g. noncommutative spaces can arise as decoupling limits

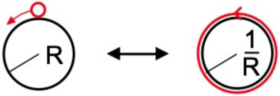
- One can use effective field theories as probes of geometry:
  Introduce D-branes and take decoupling limit $\Longrightarrow$
  Noncommutative worldvolume gauge theories in an NS–NS B-field background (Douglas & Hull '97; Seiberg & Witten '99; Cornalba & Schiappa '01; Herbst, Kling & Kreuzer '01; . . .)

Outline

- T-duality and doubled geometry
- Non-geometric backgrounds
- Supergravity and generalized geometry
- Basic double field theory
- Global aspects of double field theory
- Double field theory and para-Hermitian geometry
- Further developments

T-Duality

- T-duality is a string symmetry relating distinct spacetimes, some of which are "non-geometric"
- Simplest example: $\mathsf{T} : R \longrightarrow R' = \ell_s^2/R$
- String theory on S^1 of radius R is physically equivalent to string theory on S^1 of radius ℓ_s^2/R (automorphism of CFT)
- Exchanges discrete momentum p and winding w



- Exchanges S^1 coordinate x with dual S^1 coordinate $\tilde{x}$
- Acts on a "doubled circle" with coordinates $(x, \tilde{x})$:

Strings "see" a doubled geometry

T-Duality

- For a d-torus T^d with background fields (g, B), worldsheet theory is

$$S = \int \mathrm{d}^2\sigma \; E_{ij}(x)\,\partial_+ x^i\,\partial_- x^j \qquad , \qquad E = g + B$$

- T-duality symmetry $O(d, d; \mathbb{Z})$:

$$E' = (a\,E + b)\,\frac{1}{c\,E + d} \qquad , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d, d; \mathbb{Z})$$

- Acts on d discrete momenta and d winding numbers, preserves $\eta = 2\,\mathrm{d}x^i\,\mathrm{d}\tilde{x}_i$: String theory "sees" a doubled torus T^{2d}
- More generally, if M is a T^d-bundle, then string theory "sees" a torus bundle with doubled torus fibres T^{2d}:

T-duality $O(d, d; \mathbb{Z}) \subset GL(2d, \mathbb{Z})$ acts geometrically

Non-Geometric Backgrounds

(Hellerman, McGreevy & Williams '02; Dabholkar & Hull '02; Kachru, Schulz, Tripathy & Trivedi '02; Hull '04)

- New features of T-duality when $H = \mathrm{d}B \neq 0$
- Prototypical examples come from torus bundles $M \xrightarrow{T^d} W$ (with H-flux $[H] \in H^3(M, \mathbb{Z})$)

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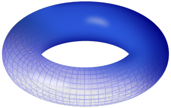
- E.g. $W = S^1$, M = twisted torus, $H = 0$:

Twisted torus

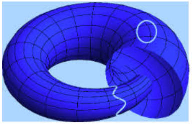


$\downarrow$

S^1_x

$\xrightarrow{\mathrm{T}_z}$

T-fold



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S^1_x

Patching: Diffeos

Patching: T-duality

Non-Geometric Backgrounds

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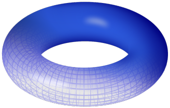

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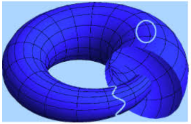


$\downarrow$

S^1_x

$\downarrow \mathrm{T}_x$

Essentially doubled space

Generalized Flux Backgrounds

$M = T^3$ with H-flux $H = m\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, $B = m\,x\,\mathrm{d}y \wedge \mathrm{d}z$ gives geometric and non-geometric fluxes (Hull '05; Shelton, Taylor & Wecht '05; Dabholkar & Hull '06; ...)

$$H_{ijk} \xrightarrow{\mathsf{T}_i} f^i{}_{jk} \xrightarrow{\mathsf{T}_j} Q^{ij}{}_k \xrightarrow{\mathsf{T}_k} R^{ijk}$$

(T^3,H-flux): $[H] = m$

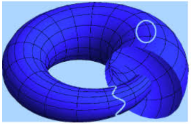
$\uparrow \mathsf{T}_y$

Nilfold (f) $\xrightarrow{\mathsf{T}_z}$ T-fold (Q)

$m \; S^1 \downarrow$ (Nilfold $\to T^2$)

$\downarrow$ (T-fold $\to S^1_x$)

T^2 $\qquad$ S^1_x

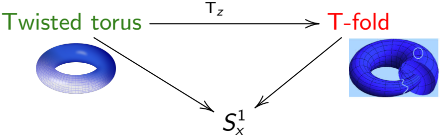


$\downarrow \mathsf{T}_x$

Essentially doubled (R)

Doubled Geometry

- Doubled torus $\xrightarrow{T^4} S^1_x$: (Hull '05)



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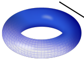
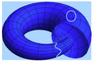
Twisted torus $\xrightarrow{\mathrm{T}_z}$ T-fold





Twisted torus $\to S^1_x$, T-fold $\to S^1_x$

- Doubled twisted torus $\xrightarrow{T^4} S^1_x \times S^1_{\tilde{x}}$: (Hull & Reid-Edwards '07)

Doubled torus $\xrightarrow{\mathrm{T}_x}$ Essentially doubled torus

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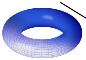

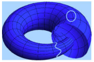

Twisted torus $\longrightarrow S^1_x$, T-fold $\longrightarrow S^1_x$

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Essentially doubled torus $\xrightarrow{T^4} S^1_{\tilde{x}}$

- Geometrization of non-geometry: $GL(4,\mathbb{Z}) \supset O(2,2;\mathbb{Z}) \subset O(3,3)$

Target Space Perspective: Supergravity

- (g, B) satisfy field equations that determine a CFT

- Reproduced from target space theory ($d = 10$):

$$S_{\rm SUGRA}[g, B] = \int \mathrm{d}^d x \, \sqrt{g} \, \Big(R(g) - \frac{1}{12} H^2 \Big) \quad , \quad H = \mathrm{d}B$$

  Low energy effective theory $\equiv$ supergravity

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- (g, B) and (g', B') give same CFT if related by:

  S1. Diffeomorphisms and B-field gauge transformations

  S2. (Factorized) T-dualities

- S1. captured as transition functions in Generalized Geometry

(Hitchin '02; Gualtieri '04)

Target Space Perspective: Supergravity

- String Hamiltonian $h = \frac{1}{2}\,\mathcal{H}_{IJ}\,P^I\,P^J$ with:

$$\mathcal{H}(g,B) = \begin{pmatrix} g - B\,g^{-1}\,B & B\,g^{-1} \\ -g^{-1}\,B & g^{-1} \end{pmatrix} \quad , \quad P = \begin{pmatrix} w^i \\ p_i \end{pmatrix}$$

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- Generalized Geometry doubles tangent bundle

$$TM \longrightarrow \mathbb{T}M = TM \oplus T^*M$$

with structure of a Courant algebroid, twisted by a B-field

- $\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ $O(d,d)$-structure (fibre metric of $\mathbb{T}M$),
$\mathcal{H}^{-1} \ = \ \eta^{-1}\,\mathcal{H}\,\eta^{-1}$, bracket of sections is the Courant bracket

- $\mathcal{H}(g,B) \in O(d,d)/O(d)\times O(d)$ Generalized metric on $\mathbb{T}M$,
P is a section of $\mathbb{T}M$

Courant Algebroids

(Courant '90; Liu, Weinstein & Xu '97; Uchino '02)

Quadruple $\left(E \xrightarrow{\mathbb{R}^{2d}} M\ ,\ [-,-]\ ,\ \langle -,-\rangle\ ,\ \rho : E \longrightarrow TM\right)$ satisfying:

1. Jacobi: $[[A,B],C] + \text{cyclic} = \frac{1}{3}\,\mathcal{D}\langle [A,B],C\rangle + \text{cyclic}$
2. Leibniz: $[A, f\,B] = f\,[A,B] + \big(\rho(A)f\big)B - \langle A,B\rangle\,\mathcal{D}f$
3. Compatibility:

$$\rho(C)\langle A,B\rangle = \langle [C,A] + \mathcal{D}\langle C,A\rangle, B\rangle + \langle [C,B] + \mathcal{D}\langle C,B\rangle, A\rangle$$

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Additional properties:

4. Homomorphism: $\rho([A,B]) = [\rho(A),\rho(B)]$ $\quad (A,B,C \in \Gamma(E))$
5. "Strong constraint": $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$ $\quad (f,g \in C^\infty(M))$

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Courant bracket:

$$[A,B]_K = \left({\rho_J}^i\,A^J\,\partial_i B_K - \tfrac{1}{2}\,{\rho_K}^i\,A^J\,\partial_i B_J\right) - (A \leftrightarrow B) + T(A,B,e_K)$$

Supergravity on Courant Algebroids

- When $E = \mathbb{T}M = TM \oplus T^*M$ with natural frame $(e_I) = (\partial_i, \mathrm{d}x^i)$ and $O(d,d)$-invariant metric $\langle \partial_i, \mathrm{d}x^j \rangle = \delta_i{}^j$, axioms give fluxes (H, f, Q, R) and Bianchi identities of supergravity

- Type II supergravity can be entirely formulated in terms of Generalized Geometry (Graña, Minasian, Petrini & Waldram '08; Coimbra, Strickland-Constable & Waldram '11)

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- S2. **not** a manifest symmetry: T-duality is an isomorphism between (twisted) Courant algebroids of T^d-bundles (Cavalcanti & Gualtieri '10)

Double Field Theory

(Duff '90; Tseytlin '90; Siegel '93;
Hull & Zwiebach '09; Hohm, Hull & Zwiebach '10; Hohm & Kwak '11; ...)

- Duality-covariantization of supergravity:

  $O(d,d)$ symmetry is manifest

- Consequence of string field theory on torus T^d:

$$\psi(p,w) \xrightarrow{\text{Fourier}} \psi(x,\tilde{x})$$

- Strings see doubled spacetime $M \longrightarrow \mathcal{M} = M \times \widetilde{M}$:

$$\mathbb{X}^I = (x^i, \tilde{x}_i) \quad , \quad \partial_I = (\partial_i, \tilde{\partial}^i)$$

- Needed to describe non-geometric backgrounds and generalized T-duality; doubled geometry is physical and dynamical

- $O(d,d)$-structure η / generalized metric $\mathcal{H}(g,B)$

Double Field Theory

- Einstein-Hilbert type action from generalized Ricci scalar $\mathcal{R}(\mathcal{H})$:

$$S_{\rm DFT}[\mathcal{H}] = \int \mathrm{d}^{2d}\mathbb{X}\ \mathcal{R}(\mathcal{H})$$

- Invariance under generalized Lie derivative:

$$\delta_\epsilon \mathcal{H}^{IJ} \ = \ (\partial^I \epsilon_K - \partial_K \epsilon^I)\, \mathcal{H}^{KJ} + (I \leftrightarrow J) + \epsilon^K\, \partial_K \mathcal{H}^{IJ} \ =: \ \mathsf{L}_\epsilon \mathcal{H}^{IJ}$$

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- Strong constraint: $\partial^I \partial_I f = 0$ (level matching) , $\partial^I f\, \partial_I g = 0$
  Solutions select polarisations defining d-dimensional 'physical' null submanifolds of doubled space, DFT reduces to supergravity in different duality frames related by $O(d,d)$-transformations

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- Supergravity frame: $\tilde\partial^i f = 0$ $(w^i = 0)$, $S_{\rm DFT}[\mathcal{H}] \longrightarrow S_{\rm SUGRA}[g, B]$

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- Supergravity frame: $\tilde{\partial}^i f = 0$ $(w^i = 0)$, $S_{\mathrm{DFT}}[\mathcal{H}] \longrightarrow S_{\mathrm{SUGRA}}[g,B]$

- C-bracket: Closure $[\mathsf{L}_{\epsilon_1}, \mathsf{L}_{\epsilon_2}] = \mathsf{L}_{[\![\epsilon_1,\epsilon_2]\!]}$ after strong constraint:

$$[\![\epsilon_1,\epsilon_2]\!]^J \ = \ \epsilon_1^K\,\partial_K \epsilon_2^J - \tfrac{1}{2}\,\epsilon_1^K\,\partial^J \epsilon_{2\,K} - (\epsilon_1 \leftrightarrow \epsilon_2)$$

  Reduces to (standard) Courant bracket after polarisation

Global Aspects of Double Field Theory

- What is full DFT with dynamical doubled geometry beyond strong constraint?
- When $\mathcal{M}$ is T^{2d} or a T^{2d}-bundle, DFT on $\mathcal{M}$ can be reduced to string theory on T^d or a T-fold
- Background independent formulation suggests writing DFT on more general doubled manifolds $\mathcal{M}$ (Hohm, Hull & Zwiebach '10)
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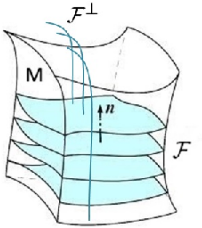
— meaning of $\tilde{x}$ for general spacetimes M?
- Strong constraint picks out rank d maximally η-isotropic $L_+ \subset T\mathcal{M}$, with $\mathsf{L}_V f = V^I \, \partial_I f = 0$ for $V \in \Gamma(L_+)$, which is integrable:

$$\eta(V, W) = 0 \quad , \quad [V, W] \in \Gamma(L_+)$$

E.g. $V = \tilde{v}_i \, \tilde{\partial}^i$

Global Aspects of Double Field Theory

- Polarisation selects physical spacetime as a **quotient** $M = \mathcal{M}/\mathcal{F}$ by action on leaves of foliation $L_+ = T\mathcal{F}$ (Hull & Reid-Edwards '09; Vaisman '12; Park '13; Lee, Strickland-Constable & Waldram '15)



$$T\mathcal{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$$

T-fold: Singular quotient (e.g. orbifolds)

Essentially doubled space: No foliation

(Marotta & Sz '19)

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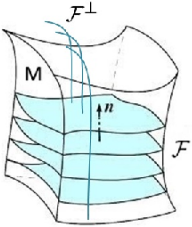

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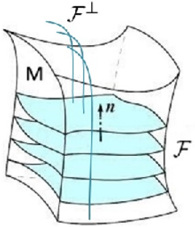

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- Flat metric η too restrictive: $\mathcal{M} = \mathbb{R}^{2d}/\Gamma$ locally
  $\Longrightarrow$ allow more general (Cederwall '14; Marotta & Sz '19)

$$\eta = 2\,\mathrm{d}x^i\,\mathrm{d}\tilde{x}_i + h_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

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- Top-down approach: Para-Hermitian geometry (understand reduction to flat space DFT and Generalized Geometry, and emergence of non-geometric backgrounds)

Doubling, Splitting and Projecting

(Chatzistavrakidis, Jonke, Khoo & Sz '18; Svoboda '18)

- Local model for doubled space: Quotient means $\mathcal{M} = T^*M$ (or quotients) with fibres $\mathcal{F}$; then $T\mathcal{M}\big|_M \simeq TM \oplus T^*M = \mathbb{T}M$
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Pre-Courant and Metric Algebroids

- $(L_+ \xrightarrow{\mathbb{R}^{2d}} \mathcal{M} \ , \ [\![-,-]\!] \ , \ \langle\!\langle -,- \rangle\!\rangle \ , \ \rho_+ : L_+ \longrightarrow T\mathcal{M})$ is a "DFT algebroid"; When strong constraint is imposed, ρ_+ is a homomorphism and L_+ is a Courant algebroid over $\mathcal{M} = T^*M$

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Double Field Theory and Para-Hermitian Geometry

- Para-Hermitian Geometry: A "real version" of complex Hermitian geometry
- Addresses global issues of doubled geometry, provides simple elegant framework for generalized flux compactifications and non-geometric backgrounds (Hull '04; Vaisman '12; Freidel, Rudolph & Svoboda '17; Chatzistavrakidis, Jonke, Khoo & Sz '18; Svoboda '18; Marotta & Sz '18; Mori, Sasaki & Shiozawa '19; Hassler, Lüst & Rudolph '19; ...)

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- Examples: Fibre bundles (T^*M, TM, ...), Doubled Lie groups, Drinfel'd doubles, and quotients (T^{2d}, doubled twisted torus, ...)

Para-Hermitian Manifolds

- Para-complex structure $K : T\mathcal{M} \longrightarrow T\mathcal{M}$ on $2d$-dim manifold $\mathcal{M}$ with $K^2 = +\mathbb{1}$, whose ± 1-eigenbundles $L_\pm$ have same rank d
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- (Generalized) Fluxes (Lie algebroid 3-forms) measure (lack of) weak integrability of para-Hermitian structures with respect to C-bracket

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- Generalized metric on a para-Hermitian manifold $(\mathcal{M}, K, \eta)$: Riemannian metric $\mathcal{H}$ on $\mathcal{M}$ satisfying compatibility conditions

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  Gives notion of generalized T-duality
- Double Field Theory is a limit of Born geometry:
  - Flat space limit: $\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\mathcal{H}(g) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$
  - $O(d,d)(\mathcal{M})$ B-transformation gives DFT generalized metric $\mathcal{H}(g, B)$
  - Canonical C-bracket reduces to C-bracket of DFT

Recovering the Physical Spacetime

- Polarization: Choice of para-Hermitian structure (K, η) on $\mathcal{M}$ (splitting $T\mathcal{M} = L_+ \oplus L_-$ into maximally isotropic sub-bundles)

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- Change of polarization (generalized T-duality):
$$(K, \eta) \longmapsto (K_\vartheta, \eta) \ , \quad K_\vartheta = \vartheta^{-1}\, K\, \vartheta \ , \quad \vartheta \in O(d,d)(\mathcal{M})$$

Further Developments

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- D-branes and non-geometry/doubled geometry (Hull '04; Lowe, Natase & Ramgoolam '03; Ellwood & Hashimoto '06; Grange & Schäfer-Nameki '06; Lawrence, Schulz & Wecht '06; Albertsson, Kimura & Reid-Edwards '08; Hull & Sz '19)

Background	Dp-brane	x	y	z	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
T^3 with H-flux	D0-brane	−	−	−	×	×	×
Nilfold	D1-brane	−	×	−	×	−	×
T-fold	D2-brane	−	×	×	×	−	−
Essentially doubled space	D3-brane	×	×	×	−	−	−

Further Developments

- Doubled sigma-models (Hull '05; Berman, Copland & Thompson '07; Hull & Reid-Edwards '09; Copland '11; Lee & Park '13; Marotta & Sz '19; ...)
- D-branes and non-geometry/doubled geometry (Hull '04; Lowe, Natase & Ramgoolam '03; Ellwood & Hashimoto '06; Grange & Schäfer-Nameki '06; Lawrence, Schulz & Wecht '06; Albertsson, Kimura & Reid-Edwards '08; Hull & Sz '19)

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T^3 with H-flux	D0-brane	$-$	$-$	$-$	$\times$	$\times$	$\times$
Nilfold	D1-brane	$-$	$\times$	$-$	$\times$	$-$	$\times$
T-fold	D2-brane	$-$	$\times$	$\times$	$\times$	$-$	$-$
Essentially doubled space	D3-brane	$\times$	$\times$	$\times$	$-$	$-$	$-$

  - D-branes on T-folds: T-duality monodromies of non-geometric background $\Longrightarrow$ Morita duality monodromies of noncommutative Yang-Mills theory on D-brane

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- Doubled sigma-models (Hull '05; Berman, Copland & Thompson '07; Hull & Reid-Edwards '09; Copland '11; Lee & Park '13; Marotta & Sz '19; ...)
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Background	Dp-brane	x	y	z	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
T^3 with H-flux	D0-brane	$-$	$-$	$-$	$\times$	$\times$	$\times$
Nilfold	D1-brane	$-$	$\times$	$-$	$\times$	$-$	$\times$
T-fold	D2-brane	$-$	$\times$	$\times$	$\times$	$-$	$-$
Essentially doubled space	D3-brane	$\times$	$\times$	$\times$	$-$	$-$	$-$

  - D-branes on T-folds: T-duality monodromies of non-geometric background $\Longrightarrow$ Morita duality monodromies of noncommutative Yang-Mills theory on D-brane
  - D-branes on essentially doubled spaces: D-brane gauge theory depends on $\widetilde{x}$, no conventional formulation in spacetime

Further Developments

- Heterotic DFT (Hohm & Kwak '11)
- Type II DFT (Hull '07; Hohm, Kwak & Zwiebach '11; Thompson '11; Jeon, Lee, Park & Suh '12)
- Exceptional Generalized Geometry/Field Theory for M-theory
  (Hull '07; Pacheco & Waldram '08; Berman & Perry '10; Hohm and Samtleben '13; ...)
- Membrane/threebrane sigma-models (Mylonas, Schupp & Sz '12; Chatzistavrakidis, Jonke & Lechtenfeld '15; Besso, Heller, Ikeda & Watamura '15; Kökényesi, Sinkovics & Sz '18; Chatzistavrakidis, Jonke, Khoo & Sz '18; Chatzistavrakidis, Jonke, Lüst & Sz '19)
- DFT and supergeometry (Deser & Stasheff '14; Deser & Sämann '16; Heller, Ikeda & Watamura '16)
- DFT classification of non-Riemannian geometries
  (e.g. Newton-Cartan geometry for non-relativistic strings)
  (Morand & Park '17; Berman, Blair & Otsuki '19)
- Non-geometry as closed string/M2-brane noncommutative and nonassociative geometries (Blumenhagen & Plauschinn '10; Lüst '10; Mylonas, Schupp & Sz '12; Günaydin, Lüst & Malek '16; Kupriyanov & Sz '17; ...)