Conformal three-point functions in momentum space

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and in progress
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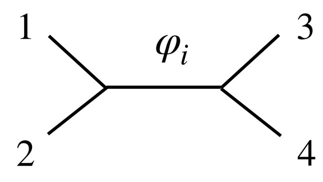
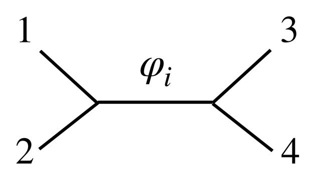
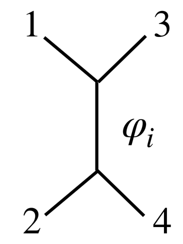
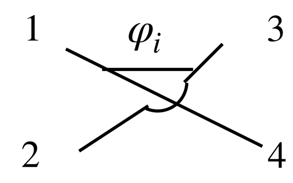
Introduction

- Conformal field theory appears in various fields in theoretical physics ; Critical phenomena AdS/CFT Inflation etc,,,
- Conformal bootstrap: constrain the spectrum and the interactions from theoretical consistency.

Bootstrap 1, Operator product expansion $O_1(x_1)O_2(x_2) \sim \sum_i C_{12i}\,\varphi_i(x_2) \longrightarrow \langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle \sim \sum_i C_{12i}C_{34i}\langle\varphi_i(x_2)\,\varphi_i(x_3)\rangle$



2, Consistency with Crossing symmetry $\sum_i$  $= \sum_i$  $= \sum_i$ 

⟶ 3 point function is a building block of 4 point function : $\langle O_1(x_1)O_2(x_2)\varphi_i(x_3)\rangle = C_{12i}\,\langle\varphi_i(x_2)\varphi_i(x_3)\rangle$

- In position space, 3 point functions of general operators have been constructed.

#scalar and conserved currents — Erdmenger, Osborn '97
#general spinning operators — Costa et.al [1107.3554]

- In momentum space, 3 point function of three scalars was constructed by Ferrara et.al '74 .

$$\langle\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)\rangle = J_{0\{0,0,0\}}(p_1,p_2,p_3)$$

triple-K integral : $J_{N\{k_1,k_2,k_3\}}(p_1,p_2,p_3) = \int_0^\infty \frac{dz}{z} z^{2d-\Delta_t-k_t+N}(p_1 z)^{\nu_1+k_1}K_{\nu_1+k_1}(p_1 z)(p_2 z)^{\nu_2+k_2}K_{\nu_2+k_2}(p_2 z)(p_3 z)^{\nu_3+k_3}K_{\nu_3+k_3}(p_3 z)$,

where $\nu_i = \Delta_i - \frac{d}{2}$

40 years later, 3 point functions of scalars and conserved currents were constructed in terms of triple-K integral by Bzowski et.al [1304.7760] .

⟶ Our work: 3 point functions of scalar, conserved currents and a spinning operator

Another motivation is Cosmological Bootstrap.

- Space-time during inflation is approximated by de Sitter.

⟶ Inflationary correlator is fixed by D=3 conformal symmetry (+ analyticity of its momentum).

N. Arkani-Hamed et.al [1811.00024] (talk by G.L.Pimentel)

Our result

D ≧ 4

Ansatz

$$\langle O_1(p_1)\,\epsilon_2^2\cdot T(p_2)\,\epsilon_3^s\cdot\varphi^s(p_3)\rangle' = (\epsilon_2^2\cdot\Pi_2\cdot p_1^2)\sum_{n=0}^{s}(\epsilon_3\cdot p_2)^n(\epsilon_3\cdot(p_1+p_2))^{s-n}\boldsymbol{A_n(p_1,p_2,p_3)}$$
$$+(\epsilon_2^2\cdot\Pi_2\cdot p_1\epsilon_3)\sum_{n=0}^{s-1}(\epsilon_3\cdot p_2)^n(\epsilon_3\cdot(p_1+p_2))^{s-n-1}\boldsymbol{B_n(p_1,p_2,p_3)}$$
$$+(\epsilon_2^2\cdot\Pi_2\cdot\epsilon_3^2)\sum_{n=0}^{s-2}(\epsilon_3\cdot p_1)^n(\epsilon_3\cdot(p_1+p_2))^{s-n-2}\boldsymbol{C_n(p_1,p_2,p_3)}$$

$(\Pi_2)^{ij}_{kl}$: traceless, transverse and symmetric projector

- Initial condition: in terms of triple-K integrals

$$A_s = C_A J_{s+2\{0,0,0\}} \qquad B_{s-1} = 2C_A J_{s+1\{1,0,0\}} + C_B J_{s-1\{0,0,0\}}$$
$$C_{s-1} = C_A J_{s+1\{2,0,0\}} + C_B J_{s-1\{1,0,0\}} + C_C J_{s-2\{0,0,0\}}$$

- Lower functions: given by differential operator

$$A_n = \frac{(\Xi+n-1)_{s-n}}{(-s+n)!(\Delta_3-1+n)_{s-n}}A_s + \sum_{t=0}^{s-1-n}\frac{(\Xi+n-1)_t}{(-s+n)_{t+1}(\Delta_3-1+n)_{t+1}}\frac{1}{p_3}\frac{\partial}{\partial p_3}B_{n+t} \qquad C_n = \frac{(\Xi+n+3)_{s-2-n}}{(-s+n+2)!(\Delta_3+1+n)_{s-2-n}}C_{s-2}$$

$$B_n = \frac{(\Xi+n+1)_{s-1-n}}{(-s+1+n)!(\Delta_3+n)_{s-1-n}}B_{s-1} + 2\sum_{t=0}^{s-2-n}\frac{(\Xi+n+1)_t}{(-s+1+n)_{t+1}(\Delta_3+n)_{t+1}}\frac{1}{p_3}\frac{\partial}{\partial p_3}C_{n+t}$$

where $\Xi = \frac{1}{2}\left(-s+\Delta_1-\Delta_2+\Delta_3-p_1\frac{\partial}{\partial p_1}+p_2\frac{\partial}{\partial p_2}-\frac{p_1^2-p_2^2}{p_3}\frac{\partial}{\partial p_3}\right)$

D = 3

Ansatz

$$\langle\hat{T}^-\hat{T}^-\varphi_s^{(-h)}\rangle = (\epsilon_1^-\cdot\epsilon_2^-)\,(\epsilon_2^-\cdot\epsilon_3^-)\,(\epsilon_3^-\cdot\epsilon_1^-)\,(\epsilon_3^-\cdot p_1)^{h-2}\,p_1^2p_2^2p_3^h\,\boldsymbol{F^{(-h)}}(p_1,p_2,p_3)$$

- Maximal helcity:

$$s \leqq 4 \qquad F^{(-s)}(p_1,p_2,p_3) = p_3^{-2s-2}\left(\frac{p_1+p_2-p_3}{p_3}\right)^{-s}{}_2F_1\left(6-\Delta, 3+\Delta, 5-s, \frac{p_3-p_1-p_2}{2p_3}\right)$$

$$s \geqq 4 \qquad F^{(-s)}(p_1,p_2,p_3) = p_3^{-2s-2}\left(\frac{p_1+p_2-p_3}{p_3}\right)^{-4}{}_2F_1\left(2+s-\Delta, -1+s+\Delta, -3+s, \frac{p_3-p_1-p_2}{2p_3}\right)$$

- Lower helicity: given by recursion relation

$$\left[-2(h-2)+2(p_2+p_3)\frac{\partial}{\partial p_1}+2(p_1-p_3)\frac{\partial}{\partial p_2}+2(p_1+p_2)\frac{\partial}{\partial p_3}-2\frac{h\,p_1\left(\frac{\partial}{\partial p_1}+\frac{\partial}{\partial p_2}\right)-h+2}{p_1-p_2+p_3}p_3\right]F^{(-h)}$$

$$+\frac{(s-h)(2-\Delta+h)}{4}(p_1^2-(p_2+p_3)^2)F^{(-h-1)} = -2\frac{(s+h)(-2+h+\Delta)}{p_3(p_1-p_2+p_3)}F^{(-h+1)}$$

Outlook

- Construction of crossing symmetric 4 point function with external spinning field (talk by H.Isono)

- Inflationary correlators with external spinning field

Appendix

Conformal three point functions in D ≧ 4 dimensions

Ansatz

- Constraints from momentum conservation, rotation invariance and current's conservation law are

$$\langle O_1(p_1)\,\epsilon_2^2\cdot T(p_2)\,\epsilon_3^s\cdot\varphi^s(p_3)\rangle' = (\epsilon_2^2\cdot\Pi_2\cdot p_1^2)\sum_{n=0}^{s}(\epsilon_3\cdot p_2)^n(\epsilon_3\cdot(p_1+p_2))^{s-n}\frac{1}{n!}A_n(p_1,p_2,p_3)$$

$$+(\epsilon_2^2\cdot\Pi_2\cdot p_1\epsilon_3)\sum_{n=0}^{s-1}(\epsilon_3\cdot p_2)^n(\epsilon_3\cdot(p_1+p_2))^{s-n-1}\frac{1}{n!}B_n(p_1,p_2,p_3)$$

$$+(\epsilon_2^2\cdot\Pi_2\cdot\epsilon_3^2)\sum_{n=0}^{s-2}\frac{1}{n!}(\epsilon_3\cdot p_1)^n(\epsilon_3\cdot(p_1+p_2))^{s-n-2}C_n(p_1,p_2,p_3)$$

Arbitrary functions determined by conformal symmetry

#momentum conservation

$$\langle O(p_1)T^{\mu\nu}(p_2)\varphi^s(p_3)\rangle = (2\pi)^d\delta(p_1+p_2+p_3)\langle O(p_1)T^{\mu\nu}(p_2)\varphi^s(p_3)\rangle'$$

#conservation law and traceless condition

$$p_{2\mu}\cdot\langle O(p_1)\,T^{\mu\nu}(p_2)\,\varphi^s(p_3)\rangle = 0$$
$$g_{\mu\nu}\cdot\langle O(p_1)\,T^{\mu\nu}(p_2)\,\varphi^s(p_3)\rangle = 0$$
$$\longrightarrow\quad \langle O(p_1)\,T^{\mu\nu}(p_2)\,\varphi^s(p_3)\rangle' = \Pi^{\mu\nu}_{2\rho\sigma}\cdot\bigcirc^{\rho\sigma}$$

$\Pi^{\mu\nu}_{2\rho\sigma}$:transverse, traceless and symmetric projector

#We introduced polarization vectors ϵ_2 and ϵ_3 to contract with tensor indices

ϵ_2 : arbitrary vector $\quad\epsilon_3$: null vector due to contraction with traceless-symmetric tensor

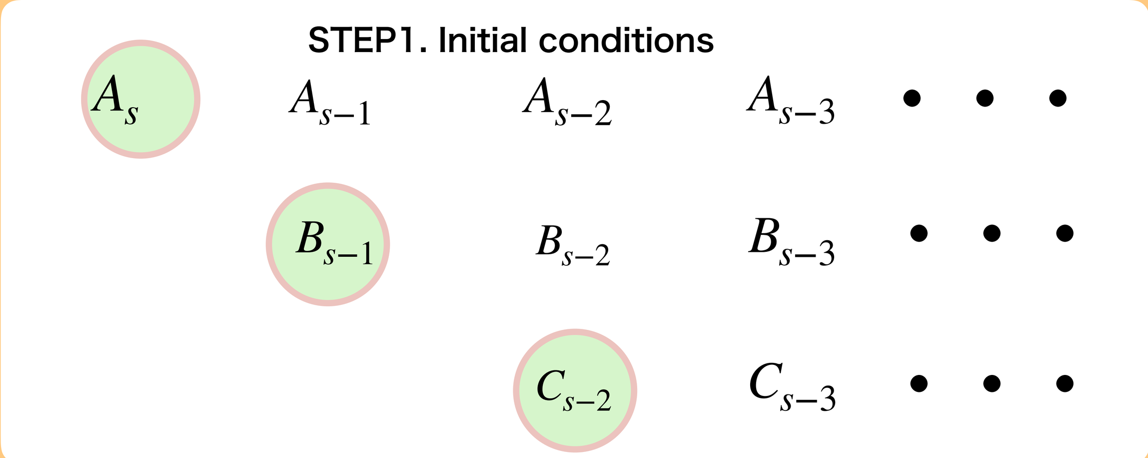
Constraints from special conformal symmetry

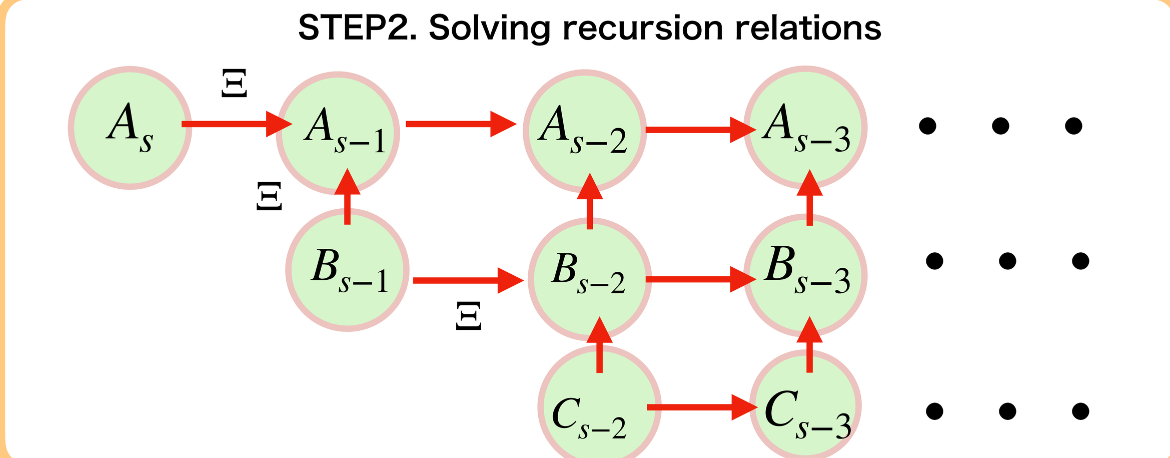
- Special conformal Ward-Takahashi identity are

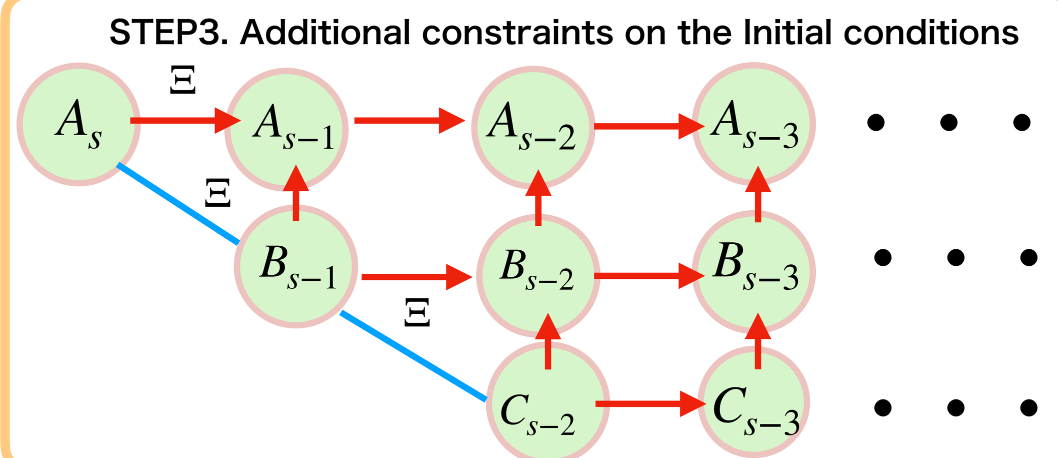
$$\left[\sum_{i=1}^{2}\left[b\cdot\partial_i\left(-2\left(\Delta_i-d+1\right)+2p_i\cdot\partial_i\right)-(b\cdot p_i)\,\partial_i^2\right]+2(\epsilon_2\cdot\partial_2)(b\cdot\partial_{\epsilon_2})-2(b\cdot\epsilon_2)(\partial_2\cdot\partial_{\epsilon_2})\right]\times\ \langle O_1(p_1)\epsilon_2^2\,.\,T(p_2)\epsilon_3^s\,.\,\varphi_3(-p_1-p_2)\rangle' = 0$$

b is an arbitrary parameter ⟶ Four identities: $b\cdot p_1\times\cdots=0,\ b\cdot p_2\times\cdots=0,\ b\cdot\epsilon_3\times\cdots=0,$ and $b\cdot\epsilon_2\times\cdots=0$

Strategy: We solve the four identities in an appropriate order.







Conformal three point functions in D = 3 dimensions

The above methodology fails in d=3

- We took the ansatz with 3s free parameters.

However, tensor structure is not independent in D = 3; $(\epsilon_2^2\cdot\Pi_2\cdot\epsilon_3^2) = \alpha\ (\epsilon_2^2\cdot\Pi_2\cdot p_1^2) + \beta\ (\epsilon_2^2\cdot\Pi_2\cdot p_1\epsilon_3)$.

Essentially, the number of free parameters reduces to 2s+1 .

$$\alpha = \frac{16}{J^4}\Big[\{(p_1\cdot p_2)^2+2p_2^2(p_1\cdot p_2)^2-p_1^2p_2^2\}(\epsilon_3\cdot p_1)(\epsilon_3\cdot p_2) - p_2^4(\epsilon_3\cdot p_1)^2-(p_1\cdot p_2)^2(\epsilon_3\cdot p_2)^2\Big]$$

$$\beta = \frac{4}{J^2}\{p_2^2-(p_1\cdot p_2)\}(\epsilon_3\cdot p_1)(\epsilon_3\cdot p_2)$$

Ansatz based on **Helicity** is good ・・・ 2s+1 parameter! (parity imposed)

Ansatz based on Helicity

- helicity mode: $\hat{T}^-(p) = \frac{T^-(p)}{p} = \frac{\epsilon_-^i\epsilon_-^j}{p}T_{ij}(p) \qquad \hat{\varphi}_s^{(-h)}(p) = \frac{\varphi_s^{(-h)}(p)}{p^{\Delta-2}} = \frac{\epsilon_-^{i_1}\epsilon_-^{i_2}\cdots\epsilon_-^{i_h}\epsilon_0^{i_{h+1}}\cdots\epsilon_0^{i_s}}{p^{\Delta-2}}\varphi^{i_1\cdots i_s}(p)$

- We can take an ansatz:

$$\langle\hat{T}^-\hat{T}^-\varphi_s^{(+h)}\rangle = (\epsilon_1^-\cdot\epsilon_2^-)\,(\epsilon_2^-\cdot\epsilon_3^+)(\epsilon_3^+\cdot\epsilon_1^-)\,(\epsilon_3^+\cdot p_1)^{h-2}p_1^2p_2^2p_3^hF^{+h}(p_1,p_2,p_3)$$

$$\langle\hat{T}^-\hat{T}^-\varphi_s^{(-h)}\rangle = (\epsilon_1^-\cdot\epsilon_2^-)\,(\epsilon_2^-\cdot\epsilon_3^-)\,(\epsilon_3^-\cdot\epsilon_1^-)\,(\epsilon_3^-\cdot p_1)^{h-2}p_1^2p_2^2p_3^hF^{-h}(p_1,p_2,p_3)$$

Arbitrary functions

#Spinor helicity variables

$m^2 = det\left(p_\mu(\sigma^\mu)_{ab}\right) = 0$ (rank 1)

decomposed into two bosonic spinors $\lambda_a\ \bar{\lambda}_b$

we added

- To write helicity vectors, we introduce spinor helicity variables ; $\lambda_a\bar{\lambda}_b = -\frac{1}{2}p_\mu(\sigma^\mu)_{ab}$, where $p^\mu = (|p|, p_1, p_2, p_3)$.

Helicity vectors: $\quad(\epsilon^+)^i = \frac{(\sigma^i)_{ab}\lambda^a\lambda^b}{p} \qquad (\epsilon^0)^i = \frac{p^i}{p} \qquad (\epsilon^-)^i = \frac{(\sigma^i)_{ab}\bar{\lambda}^a\bar{\lambda}^b}{p}$

#we neglected dot and dotted indices because they have the same transformation property under 3D rotation.

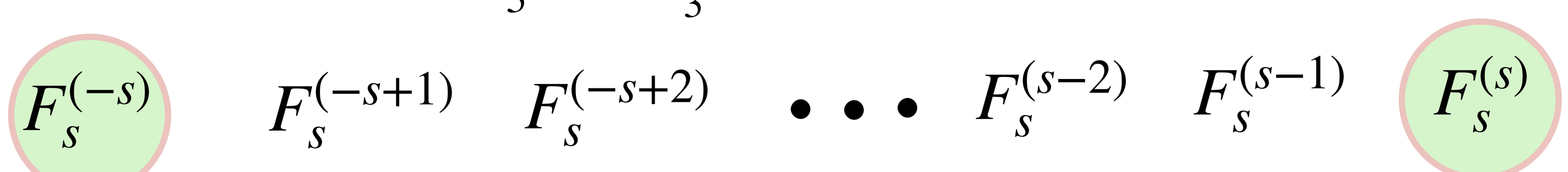
Constraints from special conformal symmetry

- Special Conformal Ward-Takahashi identity with spinor helicity variables is

$$\sum_{l=1}^{3}b_l(\sigma^l)^{ab}\frac{\partial}{\partial\lambda_l^a}\frac{\partial}{\partial\bar{\lambda}_l^b}\langle\hat{T}^-\hat{T}^-\varphi_s^{(-h)}\rangle = \frac{(s+\Delta-2h-1)(2-s-\Delta)}{p_3}\frac{b\cdot p_3}{p_3}\langle\hat{T}^-\hat{T}^-\varphi_s^{(-h)}\rangle + \frac{2h(2-s-\Delta)}{p_3}b\cdot\epsilon_3\langle\hat{T}^-\hat{T}^-\varphi_s^{(-h)}\rangle - \frac{2(s-h)(2-\Delta+h)}{p_3}b_i\langle\hat{T}^-\hat{T}^-\varphi_s^{i(-h)}\rangle$$

We can take three choices of b ; $b=\epsilon_3^-$, $b=\epsilon_3^0$, and $b=\epsilon_3^+$. ⟶ Three identities

STEP1 : Maximal helicity state (Initial condition for $d\geq 4$) from $b=\epsilon_3^-$ $b=\epsilon_3^0$



STEP2 : Recursion relation from $b=\epsilon_3^+$