Progress in the matrix model formulation of N = 2 SUSY gauge theories & Painlevé

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1. Introduction

The β-deformed matrix models are the integral representation of two dimensional conformal/irregular conformal block. They correspond the instanton partition function of $\mathcal{N}=2$ supersymmetric gauge theories. For $N_f=2$ $SU(2)$ case, the partition function can be evaluated by the matrix model which has the rational potential as well as the logarithmic potential. This model depends on two integration contours C_L and C_R .

$$Z(N_L,N_R)=\left(\prod_{i=1}^{N_L}\int_{C_L}dz_i\right)\left(\prod_{j=1}^{N_R}\int_{C_R}dz_{N_L+j}\right)\Delta(z)^{2\beta}\exp\Big(\sqrt{\beta}\sum_{I=1}^{N_L+N_R}W(z_I)\Big)$$

The partition function which is represented as a Painlevé system is not of the above partition function, but instead given by a finite N analog of Fourier transformation.

$$\underline{Z}(N,\mu_L,\mu_R)=\sum_{N_L+N_R=N}\frac{\mu_L^{N_L}}{N_L!}\frac{\mu_R^{N_R}}{N_R!}Z(N_L,N_R)$$

$\underline{Z}(N,\mu_L,\mu_R)$ is defined by eigenvalue integral along single integration path. Moreover, in the case that β = 1 and the potential $W_U(z)$ has no branch cut, we can regard $\underline{Z}(N,\mu_L,\mu_R)$ as the unitary matrix model. We evaluate the partition function in this case and show that the partition function is the tau function of Painlevé III equation. Furthermore, we derive the various equations which characterize the partition function and study the relation between Painlevé equations. In double scaling limit, we derive Painlevé II equation and show that the spectral curve in the matrix model turns into the Seiberg-Witten curve for the Argyres-Douglas theory.

2. Set up & Method

Partition function

$$\underline{Z}_{U(N)}=\frac{1}{N!}\int\prod_{i=1}^{N}d\mu(z_i)\Delta(z)\Delta(1/z),\qquad \Delta(z)=\prod_{i<j}(z_i-z_j)$$

where the measure $d\mu(z)$ and potential $W_U(z)$ are

$$d\mu(z)=\frac{dz}{2\pi iz}\exp\Big(W_U(z)\Big),\;\; W_U(z)=-\frac{1}{2\underline{g}_s}\left(z+\frac{1}{z}\right)+M\log z$$

The size of matrix N corresponds the sum of masses of two hypermultiplets, the coefficient of logarithmic term M and

$$\underline{g}_s=\frac{g_s}{\Lambda_2},\;\; g_s^2=-\epsilon_1\epsilon_2$$

Λ_2 : dynamical scale of $N_f=2$ theory

$\epsilon_{1,2}$: Omega background parameters

Orthogonal Polynomials

$$\int d\mu(z)p_n(z)\tilde{p}_n(1/z)=h_n\delta_{n,m},$$

$$p_n(z)=z^n+\sum_{k=0}^{n-1}A_k^{(n)}z^k,\quad \tilde{p}_n(1/z)=z^{-n}+\sum_{k=0}^{n-1}B_k^{(n)}z^{-k}.$$

The partition function can be written in terms of h_k, A_j and B_j :

$$\underline{Z}_{U(N)}=\prod_{k=0}^{N-1}h_k=h_0^N\prod_{j=0}^{N-1}(1-A_jB_j)\quad A_j\equiv A_0^{(j)},\quad B_j\equiv B_0^{(j)}.$$

String equations

Using the special case $(k,\,\ell,\,m)=(-1,n,n-1),\,(0,n,n),\,(1,n-1,n)$ of constraints

$$0=\int dz\frac{\partial}{\partial z}\left[\frac{z^k}{2\pi i}\exp\Big(W_U(z)\Big)p_\ell(z)\tilde{p}_m(1/z)\right],$$

we can obtain various polynomial equations A_j and B_j.

3. Results

The partition function can be written in terms of the determinant of the modified Bessel function of the first kind $I_\nu(z)$:

$$\underline{Z}_{U(N)}=(-1)^{MN}K_M^{(N)}\qquad K_\nu^{(n)}\equiv\det\left(I_{j-i+\nu}(1/\underline{g}_s)\right)_{1\le i,j\le n}$$

If we define

$$\tau(s)\equiv s^{(1/2)MN}K_M^{(N)}(2\sqrt{s}),\quad \sigma(s)\equiv s\frac{d}{ds}\log\tau(z)\quad 2\sqrt{s}\equiv\frac{1}{\underline{g}_s}$$

$\sigma(s)$ satisfies the σ-form of Painlevé III' equation:

$$(s\sigma'')-4(\sigma-s\sigma')\sigma'(\sigma'-1)-\left((N-M)-\frac{N}{2}\right)^2=0$$

Hence, the partition function $\underline{Z}_{U(N)}$ is the tau function of Painlevé III' equation up to an overall factor $s^{(1/2)MN}$.

The string equations

$$A_{n+1}=-A_{n-1}+\frac{2ng_sA_n}{1-A_nB_n},\qquad B_{n+1}=-B_{n-1}+\frac{2ng_sB_n}{1-A_nB_n},$$

$$A_nB_{n+1}-A_{n+1}B_n=2M\underline{g}_s$$

are equivalent to the alternate discrete Painlevé II equation.

Let $A_n=R_nD_n$ and $B_n=R_n/D_n$, the string equations turns into the equation for R_n :

$$(1-R_n^2)\Big(\sqrt{R_n^2\,R_{n+1}^2+M^2\,\underline{g}_s^2}+\sqrt{R_n^2\,R_{n-1}^2+M^2\,\underline{g}_s^2}\Big)=2\,n\,\underline{g}_s\,R_n^2$$

or

$$0=\eta_n^2\Big[\xi_n^2(1-\xi_n)^2-\eta_n^2\,\xi_n^2+\zeta^2(1-\xi_n)^2\Big]$$
$$+\frac{1}{2}\eta_n^2\,\xi_n\,(1-\xi_n)^2(\xi_{n+1}-2\,\xi_n+\xi_{n-1})-\frac{1}{16}(1-\xi_n)^4(\xi_{n+1}-\xi_{n-1})^2$$

where $\xi_n\equiv R_n^2,\;\eta_n\equiv n\,\underline{g}_s,\;\zeta\equiv M\,\underline{g}_s.$

In the planar limit $(\xi_n,\eta_n,\zeta)\to(\xi,\eta,\zeta)$, above equation turns into

$$\xi^3\,(\xi-2)=0$$

at $\eta=\pm1,\;\zeta=0.$

Therefore, let us define the double scaling limit:

$$x\equiv n/N,\;a^3\equiv1/N,\;\;\eta_n=\widetilde{S}\,x=1-(1/2)a^2\,t,\quad \zeta=a^3\,\widetilde{S}\,M$$

$$\xi(x)=\xi(n/N)=\xi_n=a^2\,u(t)$$

equation for ξ_n becomes Painlevé II equation

$$u''=\frac{(u')^2}{2\,u}+u^2-\frac{1}{2}\,t\,u-\frac{M^2}{2\,u}$$

We can see that the spectral curve turns into Seiberg-Witten curve of Argyres-Douglas theory in the double scaling limit.

4. Summary & Perspective

The partition function of $N_f=2$ $\mathcal{N}=2$ $SU(2)$ supersymmetric gauge theory is the tau function of Painlevé III. It is also characterized by the alternate discrete Painlevé II which turns into the Painlevé II in the double scaling limit.

We can also evaluate the unitary matrix models which contain the higher multicritical point. They are matrix model representation of the $\hat{A}_{m,n}$ theory.