Anomaly and Superconnection

Hayato Kanno
Yukawa Institute for Theoretical Physics (YITP), Kyoto University

Based on arXiv:2106.01591 [hep-th]

Work with Shigeki Sugimoto (YITP).

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

What is “anomaly”? (1)

Anomaly (Quantum Anomaly)

An classical action have some symmetries, but sometimes these symmetries disappear in quantum theory.

e.g.) $\pi^0 \to 2\gamma$

- In massless QCD, there is a chiral symmetry $U(N_f)_L \times U(N_f)_R$.

N_f: # of flavors

$$S = \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \frac{1}{2g^2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right] \right\}$$
$$= \int d^4x \left\{ \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi - \frac{1}{2g^2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right] \right\}$$

What is “anomaly”? (2)

e.g.) $\pi^0 \to 2\gamma$

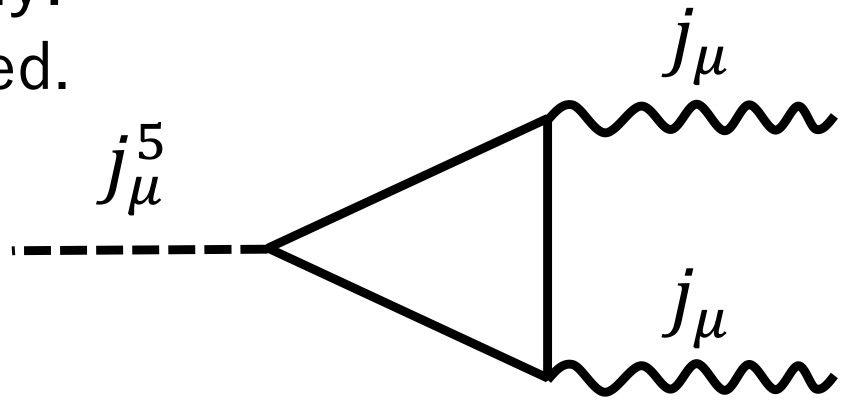
- In massless QCD, there is a chiral symmetry $U(N_f)_L \times U(N_f)_R$.
  - If you add mass term, this chiral symmetry is broken.

$$S = \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \frac{1}{2g^2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right] + m \bar{\psi} \psi \right\}$$

- In massless QCD, if there is NO anomaly, π^0 never decay.
- However, π^0 decay into 2γ, because of an anomaly!

Even if QCD is massless, $\pi^0 \to 2\gamma$ is not prohibited.

$U(N_f)_L \times U(N_f)_R \supset U(1)_A$ has an anomaly.



- Anomaly is very important!

Theories what we want to think (1)

Let us consider 4dim action contains fermions.

$$S = \int d^4x \bar{\psi} i \not{D} \psi = \int d^4x \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi$$

- This action is massless, so it has a chiral symmetry $U(N_f)_L \times U(N_f)_R$.
- There also be a $U(1)_A$ anomaly.

- Add mass term
  - Mass term breaks the chiral symmetry.

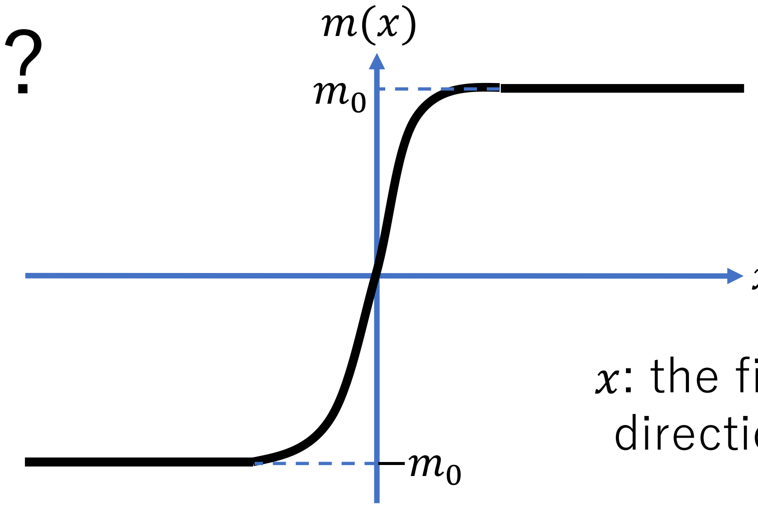
$$S = \int d^4x \bar{\psi} \left(i \not{D} + m \right) \psi$$

- Let the mass depend on the spacetime.
  - This mass is almost same as the Higgs field.
  - How change the symmetry and the anomaly?

$$S = \int d^4x \bar{\psi} \left(i \not{D} + m(x) \right) \psi$$

The spacetime dependent mass

What is “the spacetime dependent mass”?



x: the fifth direction

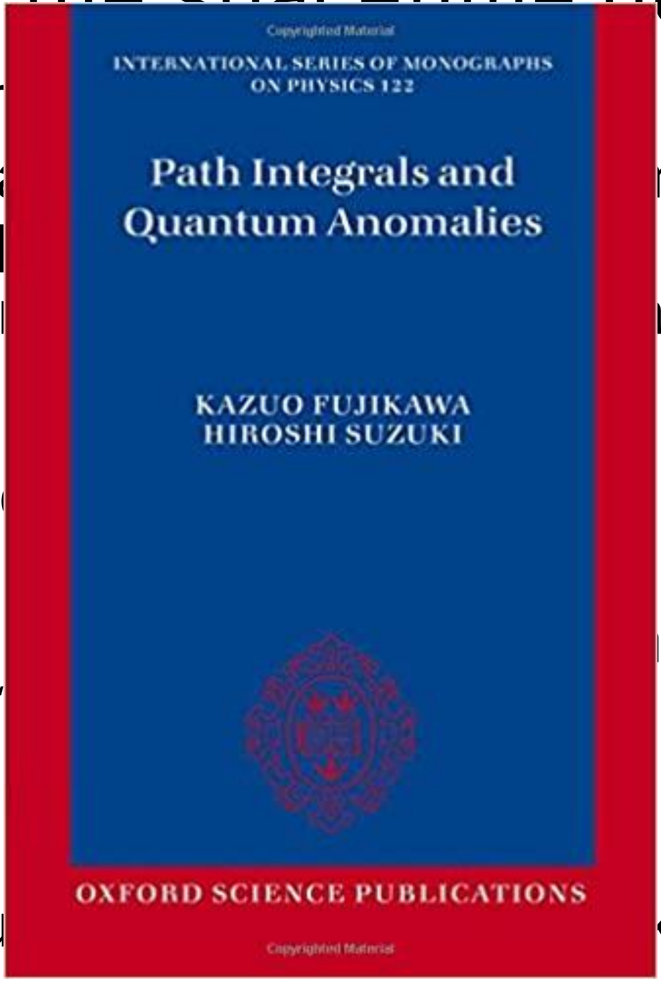
- e.g.) Domain wall fermions
  - One way to realize chiral fermions on the lattice.
  - Consider 5dim spacetime, and realize 4dim fermions on $m(x) = 0$ subspace.
- Chiral anomalies on Higgs fields
  - If Higgs fields change as bifundamental under the $U(N_f)_L \times U(N_f)_R$ chiral symmetry, the action is invariant for the symmetry.
  - It is known that chiral anomalies are not changed by adding Higgs fields.
  - See Fujikawa-san’s text book.

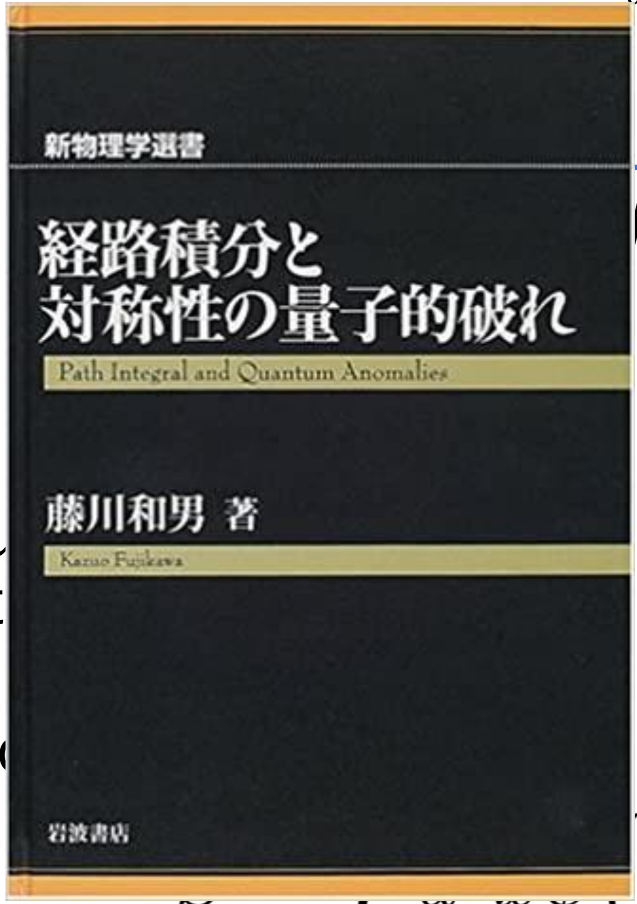
$$S = \int d^4x \bar{\psi}\left(i\not{D} + h(x)\right)\psi$$

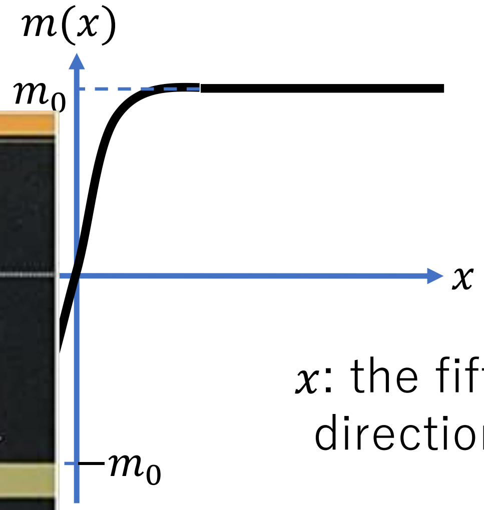
The spacetime dependent mass

What is "the spacetime dependent mass"?

- e.g.) Dom
  - One wa nions on the
  - Consid d realize 4di fermio nce.







x: the fifth direction

- Chiral an ds
  - If Higg ndamental u $U(N_f)_L$ etry, the act invaria
  - It is kn lies are not by add
  - See Fu k.

$$\int \ldots \Big(i\not{D} + h(x)\Big)\psi$$

Theories what we want to think (2)

How about the spacetime dependent mass?

- The chiral anomaly is changed by the mass!!
  - Deference between Higgs and mass
    - Higgs field : bounded (the value of the field never diverge!)
    - Spacetime dependent mass : unbounded

$$S = \int d^d x \bar{\psi}\Big(i\not{D} + m(x)\Big)\psi$$

- If the mass diverge at some points, it contribute to the anomaly.
  - This contribution might be unknown.
  - We can find the anomaly in any dimension.

- The anomaly can be written by "superconnection."

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

Plan

1. Introduction (5)
    - What is anomaly?
    - Theories what we want to think
2. Fujikawa method (4)
    - How to calculate the anomaly
    - Calculation for massive case
3. Superconnection (3)
    - Definition of superconnection
    - Application for the anomaly
4. Application (5)
    - Kink
    - Vortex
    - General codimension case
5. Conclusion (1)

How to calculate anomalies ['79 Fujikawa]

Fujikawa method

- There are some ways to calculate anomalies.
- Today, we focus on Fujikawa method.
  - Consider path integral for fermions.
  - Anomaly = Jacobian comes from path integral measure
  - We only consider perturbative anomalies.
- We calculate $\log \mathcal{J}$ for anomalies in the last part of this talk.
- We focus on 4dim case at first.

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathrm{e}^{-S}$$

e.g.) $U(1)_V$ transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{-i\alpha(x)}$$

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{J}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$
$$= \mathrm{e}^{-i\int d^4x\alpha(x)\mathcal{A}(x)}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

anomaly

$$\log \mathcal{J} = -i\int d^4x\alpha(x)\mathcal{A}(x)$$

The anomalies for massless cases

e.g.) fermions in 4dim

- Mass less case
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - $U(1)_V$ anomaly is written by the field strengths.

$$S = \int d^4x \bar{\psi} i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A^R_\mu & 0 \\ 0 & A^L_\mu \end{pmatrix} \right\} \psi$$

$$\log \mathcal{J} = \frac{i}{32\pi^2} \int d^4x \alpha(x) \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\left[F^R_{\mu\nu} F^R_{\rho\sigma} - F^L_{\mu\nu} F^L_{\rho\sigma} \right]$$

$$= \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L \right]$$

- With a Higgs field
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - The $U(1)_V$ anomaly is same for massless case.

$$S = \int d^4x \bar{\psi} \Big(i\not{D} + h(x) \Big) \psi$$

- How about the massive case?

For massive case

Let us consider spacetime dependent mass!

- The action for general even dim has $U(N_f)_L \times U(N_f)_R$ symmetry.

$$S = \int d^4x \bar{\psi} \left[i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \right] \psi$$

- For odd dim case, there is only $U(N_f)$ sym, we put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$.
- We take $m(x)$ divergent.
  - I is some directions $m(x)$ change the values.

$$|m(x^I)| \to \infty \quad (|x^I| \to \infty)$$

- We calculated $U(1)_V$ anomaly for this action by Fujikawa method.
  - It is easy to get the anomaly for any dimension.
  - It is also easy to get the anomaly for $U(N_f)_L \times U(N_f)_R$, not only for $U(1)_V$.

The anomaly for massive case

The $U(1)_V$ anomaly is,

$$\tilde{m} = m/\Lambda$$

Λ is UV cut-off comes from heat kernel regularization.

$$\begin{aligned}
\log \mathcal{J} =& \frac{i}{(2\pi)^2} \int d^4x \alpha(x) \mathrm{tr} \Bigg[\epsilon^{\mu\nu\rho\sigma} \Bigg\{ \frac{1}{8} \left(F^R_{\mu\nu} F^R_{\rho\sigma} - F^L_{\mu\nu} F^L_{\rho\sigma} \right) \\
&+ \frac{1}{12} \Big(D_\mu \tilde{m}^\dagger D_\nu \tilde{m} F^R_{\rho\sigma} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger F^L_{\rho\sigma} + F^R_{\mu\nu} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} \\
&- F^L_{\mu\nu} D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger - D_\mu \tilde{m} F^R_{\nu\rho} D_\sigma \tilde{m}^\dagger + D_\mu \tilde{m}^\dagger F^L_{\nu\rho} D_\sigma \tilde{m} \Big) \\
&+ \frac{1}{24} \left(D_\mu \tilde{m}^\dagger D_\nu \tilde{m} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger \right) \Bigg\} \Bigg] \mathrm{e}^{-\tilde{m}^\dagger \tilde{m}}
\end{aligned}$$

- This result seems very complicated…
- Can we write it more simple way?

3. Superconnection

Superconnection (1) ['85 Quillen]

- We define the superconnections for even and odd dimensions.
- This is made by Quillen, who is a mathematician, in 1985.

Even dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

$A_R : U(N_f)_R$ gauge field (1-form)

$A_L : U(N_f)_L$ gauge field (1-form)

$T : U(N_f)_L \times U(N_f)_R$ bifundamental scalar field (0-form)

- Field strength

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

$$\equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

- Supertrace

$$\mathrm{Str} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

Superconnection (2)

['85 Quillen]

Odd dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix}$$

$A : U(N_f)$ gauge field (1-form)

$T : U(N_f)$ adjoint scalar field (0-form)

- Field strength

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix}$$

- Supertrace

$$\mathrm{Str} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

We apply superconnection to write the anomaly.

Rewrite the anomaly

- We can rewrite the $U(1)_V$ anomaly by superconnection.

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x) \mathrm{Str}\left[e^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

- For odd dimension case, put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$. Then, we get $U(1)$ anomaly.
  - In odd dimension, the definition of Str is different from even dim case.

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

- It is easy to check this for 4dim massless case.

4. Application

How can we apply the anomaly?

Mass means a wall for some cases!

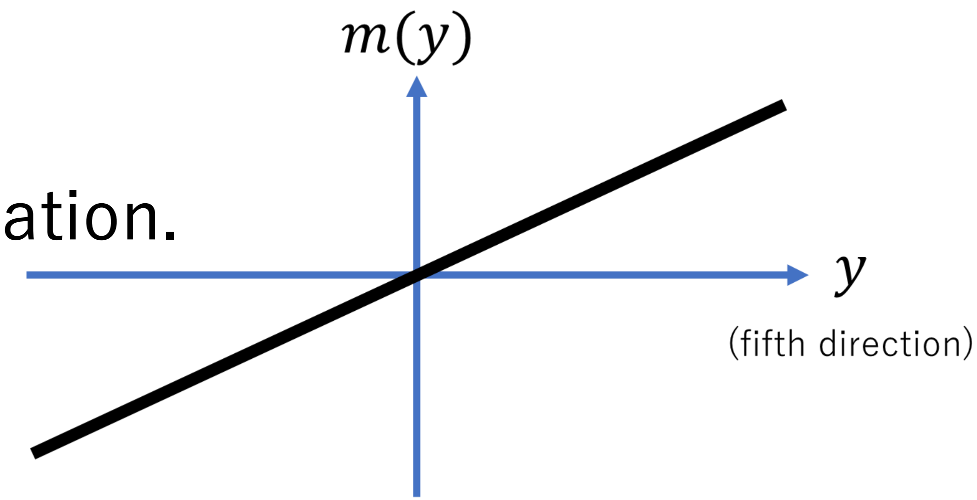
- If a fermion is massive enough, it does not have any propagating mode.
  - If the mass depends on spacetime, fermions are massless in some regions, but they can be massive in the others.
  - That means fermions localize in some areas!

  →We can make fermions localize by the mass!
- We can make some systems to decide mass configurations.
  - Kink, vortex and general codimension case
  - With boundary
- We also discuss about some index theorems.
  - APS index theorem
  - Callias type index theorem

Kink (1)

Mass kink for our set up

- For example, let's consider 5dim case.
- In this set up, "kink" means this mass configuration.



$$m(y) = uy \qquad y = x^5$$

  - This "mass" diverges at $y \to \pm\infty$.
  - 5dim fermions with $U(N_f)$ sym, and the mass depends on only y direction.

- The $U(1)$ anomaly is,

$$\log \mathcal{J} = \pm \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F \wedge F\right]$$

Corresponds to the sign of u.

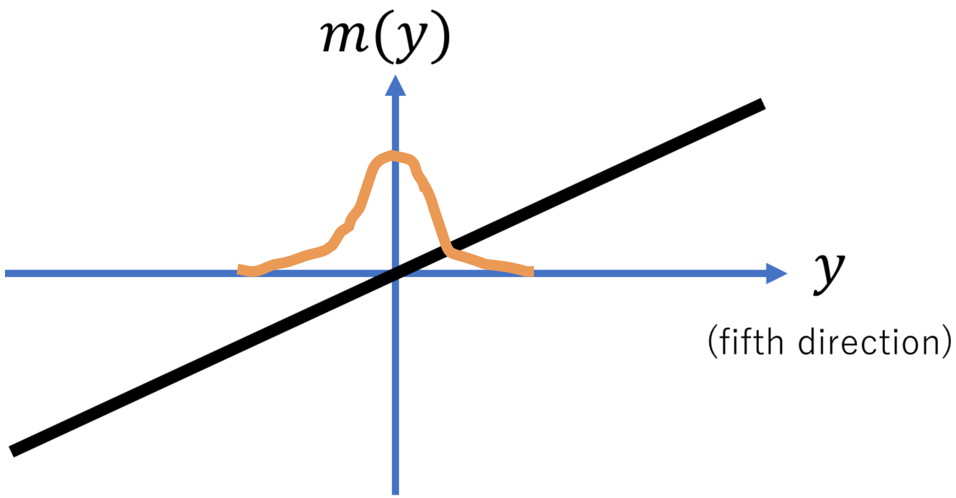
- Recall 4dim $U(1)_V$ anomaly,

$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L\right]$$

Kink (2)

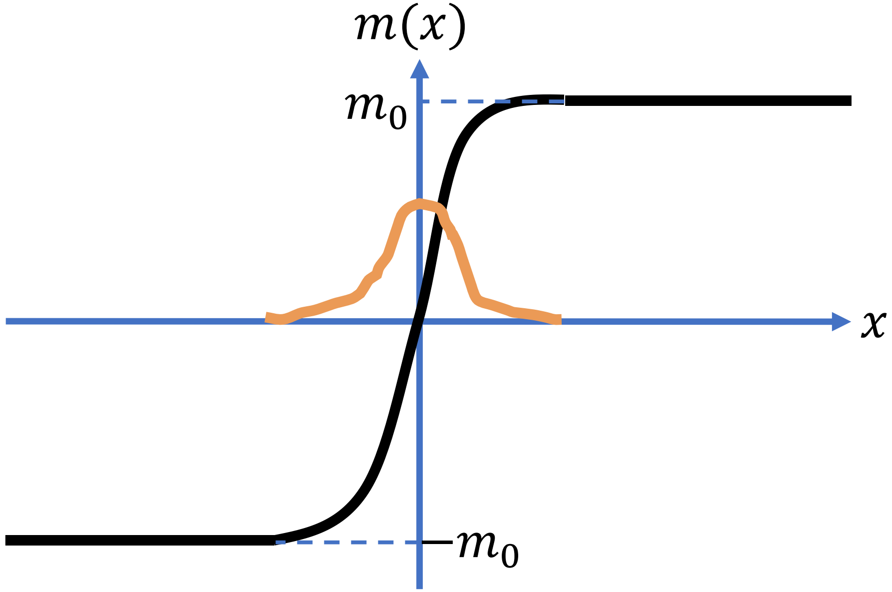
What is the meaning of the anomaly?

- 4dim Weyl fermions are localizing at $y = 0$.
    - When $u > 0$ corresponds to chirality $+$ (right-handed) fermion, and $u < 0$ corresponds to chirality – (left-handed) fermion.



Domain wall fermion

- This Weyl fermions correspond to domain wall fermions.
    - But the regularization is different, so that I don't know the correspondence in detail.



Vortex

Next, we check codim-2 case.

$$m(z) = uz\mathbf{1}_{N\times N}$$

$$z = x^{\mu=2r+1} - ix^{\mu=2r+2}$$

- Vortex is 2dim topological object.
- Let us consider $2r + 2$ dim.
  - $m(z)$ depends on 2 directions, and it is complex valued "mass".
  - This mass diverges at $|z| \to \infty$.
- For simplicity, we put $A_L = A_R$ in $2r + 2$dim.
- The $U(1)_V$ anomaly is,

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^r \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^F\right]\Big|_{2r-\mathrm{form}}$$

- This is $2r$dim $U(1)$ anomaly with $U(N_f)_R$ gauge field.
  - If you want to get chirality – (left-handed) result, use $m(\bar{z}) = u\bar{z}$, instead.

General defects

We can apply this formula to general codimension cases.

- When we think d dim system with n dim topological defects, we get $d-n$ dim $U(1)$ anomalies.
  - If $d-n$ is odd, we get nothing because odd dim mass less fermions are anomaly-free.
  - The mass configurations for general codimension is,

$$m(x) = u \sum_{I=1}^{n} \Gamma^I x^I$$

$$\gamma^I = \Gamma^I \qquad (n = odd)$$

$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix} (n = even)$$

- This results correspond to “tachyon condensation” in string theory.
  - We will discuss it in the next section.

Conclusion

- We discussed about perturbative anomaly with spacetime dependent mass.
  - $U(N_f)_L \times U(N_f)_R$ chiral symmetry for even dimension
  - $U(N_f)$ flavor symmetry for odd dimension
  - We focused on $U(1)$ anomalies for these systems.

- The anomaly can be written by superconnection.

$$\log \mathcal{J} = -i \left(\frac{i}{2\pi} \right)^{\frac{d}{2}} \int \alpha(x) \mathrm{Str} \left[\mathrm{e}^{\mathcal{F}} \right] \Big|_{d-\mathrm{form}}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- There are some applications.
  - Kink, vortex, …
  - With boundary
  - Index theorem