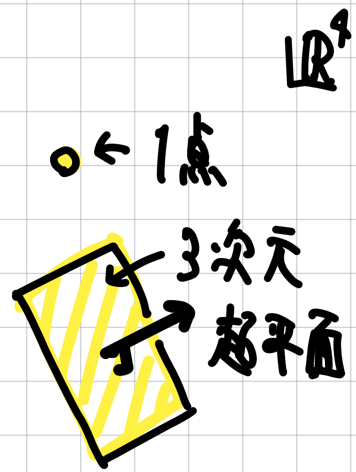
反自己双対ヤン・ミルズ方程式の新しいソリトン解

浜中真志

(名古屋大・多元数理)

Note

- NOT instanton
- BUT codim 1 soliton



参考文献

[GHHN] Gilson-H-Huang-Nimmo, JPA 53 (2020) 404002, 2004.01718

ダルブー変換によるロンスキアン解の構成

[HH1] MH & S.C. Huang, JHEP 10 (2020) 101, arXiv:2004.09248

1ソリトン解の解析

[HH2] MH & S.C. Huang, JHEP 01 (2021) 039, 2106.01353

多重ソリトン解の(漸近)解析

[HHK] MH, S.C. Huang, H. Kanno, work in progress

4次元WZW模型(N=2開弦の場の理論)への応用

0′

§1 イントロダクション

4次元反自己双対ヤン・ミルズ方程式

Anti-Self-Dual Yang-Mills (ASDYM)

- 場の理論・幾何学・可積分系において重要
- 次元還元により さまざまな可積分eqs. (KdV, NLS, Toda, Painlevé, …) に帰着 (Ward予想)

cf. [Mason-Woodhouse]

動機1：可積分系の新しい定式化

(Ward予想)

4次元 $\boxed{\begin{array}{c}\text{ASDYM} \\ F_{\mu\nu} = -*F_{\mu\nu}\end{array}}$ ↔ ツイスター理論

- Penrose-Ward対応
- Atiyah-Ward仮設解
  ⇝ NON-Wronskian解

reduction ↓

低次元 $\boxed{\begin{array}{c}\text{Toda, KdV,} \\ \text{NLS, } \cdots\end{array}}$ ↔ 佐藤理論, …

- 可積分階層, τ関数
- 多重ソリトン解 Wronskian!

split計量 (++−−)

動機1：可積分系の新しい定式化

(Ward予想)

4次元

$$\boxed{\begin{array}{c}\text{ASDYM}\\ F_{\mu\nu} = -{*}F_{\mu\nu}\end{array}}$$

↔ ツイスター理論

・Penrose-Ward対応

・Atiyah-Ward仮設解

Wronskian解？

(構成した！)

reduction ↓

低次元

$$\boxed{\begin{array}{l}\text{Toda, KdV,}\\ \text{NLS, } \cdots\end{array}}$$

↔ 佐藤理論, …

・可積分階層, τ関数

・多重ソリトン解 Wronskian！

接点！？

split計量 (++−−)

動機2：物理への応用（G=U(N)としたい）

四

(E) Euclid 計量 (++++)

- QFTへの応用
- Wilson line を入れてフーリエ変換
  ⇝ Yang-Mills-Higgs の古典解（大発見!?）

(U) Split 計量 (++−−)

- N=2 開弦の場の理論への応用
- 可積分系と弦理論の新しい接点

Plan of Talk

§1 イントロダクション (5分)

§2 ASDYM方程式とダルブー変換 (5分)

§3 1ソリトン解 (3分)

§4 多重ソリトン解 (3分)

§5 One more thing (4分)

(ご希望に応じて)

↙

付録A. ダルブー変換, B. Quasideterminant

§2. ASDYM方程式とダルブー変換

(Complex) ASD Yang-Mills eq. ($G = GL(N)$)

$$F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \quad F_{zw} = 0, \quad F_{\tilde{z}\tilde{w}} = 0$$

on 4-dim cpx space $(z, \tilde{z}, w, \tilde{w}) \in \mathbb{C}^4$

$(ds^2 = dz\,d\tilde{z} - dw\,d\tilde{w})$

実スライス ↓

$z = x^0 - x^2, \quad w = x^1 - x^3$

$\tilde{z} = x^0 + x^2, \quad \tilde{w} = x^1 + x^3$

今日

実4次元空間 with $(++--)$ 計量

(cf. $(++++)$ ⇐ $z = x^1 + ix^2, \quad w = x^3 + ix^4$
$\tilde{z} = x^1 - ix^2, \quad \tilde{w} = -(x^3 - ix^4)$)

§2. ASDYM方程式 & ダルブー変換

(Complex) ASD Yang-Mills eq. ($G = GL(N)$)

$$F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \quad F_{zw} = 0, \quad F_{\tilde{z}\tilde{w}} = 0$$

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実スライス ↓

$z = x^0 - x^2, \quad w = x^1 - x^3$

$\tilde{z} = x^0 + x^2, \quad \tilde{w} = x^1 + x^3$　　今日

⇕ 等価

ヤンの方程式　　実4次元空間 with $(++--)$ 計量

$$\partial_{\tilde{z}}(\partial_z \underbrace{J}_{N\times N} \cdot J^{-1}) - \partial_{\tilde{w}}(\partial_w J \cdot J^{-1}) = 0 \rightsquigarrow \text{ASD } A_\mu(x) \text{ を再現}$$

Lax 表示： 7

N×N 対角行列 (右作用)

$$(*)\begin{cases} L\phi = J\partial_w(J^{-1}\phi) - (\partial_{\tilde{z}}\phi)\zeta = 0 \\ M\phi = J\partial_z(J^{-1}\phi) - (\partial_{\tilde{w}}\phi)\zeta = 0 \end{cases}$$

両立条件 $L(M\phi) - M(L\phi) = 0$ ⇒ ヤンの方程式

ダルブー変換 [Nimmo-Gilson-Ohta] [GHHN]

$$(D)\begin{cases} \tilde{\phi} = \phi\zeta - \theta\Lambda\theta^{-1}\phi \\ \tilde{J} = -\theta\Lambda\theta^{-1}J \end{cases}$$

Λ: ζ の特殊値

$\theta = \phi(\zeta\to\Lambda)$ 特殊解

ダルブー変換の下, (*) は不変 (i.e. $\tilde{L}\tilde{\phi} = 0$, $\tilde{M}\tilde{\phi} = 0$

ダルブー変換の反復で種子解から新しい解が生成

$$J_{[0]} \xrightarrow{(D)} J_{[1]} \xrightarrow{(D)} J_{[2]} \xrightarrow{(D)} \cdots \xrightarrow{(D)} J_{[n]} \longrightarrow \cdots$$

種子解 (e.g. 自明解) $\cdots\to$ **Wronskian-type!**

$J_{[0]} = 1_{N\times N}$

結果

$$J_{[n]} = \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}$$

↖ quasideterminant

$\Theta = (\theta_1, \cdots, \theta_n)$

$\Theta^{(k)} = \Theta\Lambda^k$

$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

(θ_i, λ_i) : $\partial_w \theta_i = \partial_{\tilde{z}} \theta_i \lambda_i$

$\partial_z \theta_i = \partial_{\tilde{w}} \theta_i \lambda_i$

§3 1ソリトン解 (G=SU(2))

[HH1]

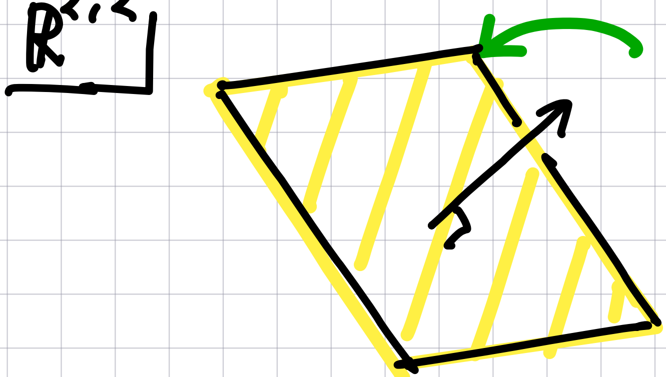
$$J=-\theta\Lambda\theta^{-1},\quad \theta=\begin{pmatrix} ae^{L} & be^{-\bar{L}} \\ -\bar{b}e^{-L} & \bar{a}e^{\bar{L}} \end{pmatrix},\quad L=\lambda\beta z+\alpha\tilde{z}+\lambda\alpha w+\beta\tilde{w}$$

$$\Lambda=\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

$\lambda=\bar{\lambda}$ で自明解

$$\mathrm{Tr}F^2=8(\alpha\bar{\beta}-\bar{\alpha}\beta)^2(\lambda-\bar{\lambda})^2\left(2\,\mathrm{sech}^2X-3\,\mathrm{sech}^4X\right)\in\mathbb{R}$$

作用密度



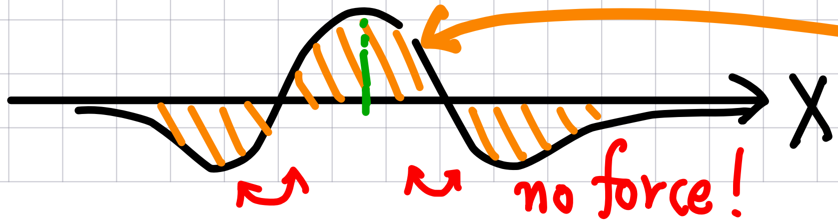
$$X=L+\bar{L}+\log\frac{|a|}{|b|}=0 \quad \text{3次元超曲面}$$

"Soliton Wall" (codim 1)

積分するとゼロ! (not ∞)

$$\int d^4x\,\mathrm{Tr}F^2=0$$

主ピーク: $X=0$



注：“ソリトン解”は簡単に作れる（ところが自明！）

(i) $G = SU(2)$, 't Hooft ansatz

$$A_\mu = i\,\eta^{(+)}_{\mu\nu}\,\partial^\nu \log\phi\,,$$

w/ $\partial^2\phi = 0$

4-dim Laplace eq.

- $\phi = 1 + \dfrac{\lambda}{r^2}$ ⇝ 't Hooft instanton
- $\phi = \dfrac{e^K + e^{-K}}{2}$ 　 $K = k_\mu x^\mu$ 　 $(k^2 = 0)$

↓ 1-ソリトン？

$$\mathrm{Tr}F^2 \propto (k^2)^2 (4\,\mathrm{sech}^2 K - 5\,\mathrm{sech}^4 + 2) \equiv 0$$ ← 自明！

(ii) $G = GL(2)$, Atiyah-Ward ansats ⇝ $\mathrm{Tr}F^2 \equiv 0$

⇒ 2スペクトル・パラメータの記述が本質的？？

§4 多重ソリトン解

$G=SU(2)$ n ソリトン解 (Wronskian type!)

$$
J_{[n]} = \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}
$$

← quasideterminant (Apx. B)

$$\Theta = (\theta_1, \cdots, \theta_n)$$

$$\Theta^{(k)} = \Theta \Lambda^k$$

$$\Lambda = \mathrm{diag}(\Lambda_1, \cdots, \Lambda_n)$$

$$
\theta_i = \begin{pmatrix} a_i e^{L_i} & b_i e^{-\bar{L}_i} \\ -b_i e^{-L_i} & \bar{a}_i e^{\bar{L}_i} \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \bar{\lambda}_i \end{pmatrix}
$$

split 計量

$$L_i = \lambda_i \beta_i z + \alpha_i \tilde{z} + \lambda_i \alpha_i w + \beta_i \tilde{w}$$

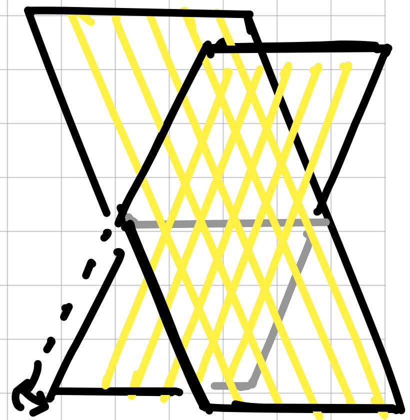
nソリトン解の漸近形 [HH2]

$\quad\;\;\,\|$

n個の soliton-wall の「非線形重ね合わせ」

$$J \xrightarrow[\text{comoving with the Ith soliton}]{r\to\infty} -\theta'_I \Lambda_I \theta_I'^{-1} \cdot C_I$$ ←定数

$$\theta'_I = \begin{pmatrix} a'_I e^{L_I} & b'_I e^{-\bar{L}_I} \\ -\bar{b}'_I e^{-L_I} & \bar{a}'_I e^{\bar{L}_I} \end{pmatrix}$$



主ピーク: $X_I = L_I + \bar{L}_I + \log\frac{|a_I|}{|b_I|} + \sum_{k=1 (\neq I)}^{n} (\pm) \log\left|\frac{\lambda_I - \lambda_k}{\lambda_I - \bar{\lambda}_k}\right|$ ↓ズレ

KdVソリトン等で典型的な現象 ⟶「位相のずれ」

13

注：ヤンの方程式 ＝ 4次元WZW模型の運動方程式
＝ $N=2$ 開弦の場の理論の運動方程式

⇓

ソリトン・ウォール解はこの弦理論において
"交差ブレーン"として実現(?)
(作用密度は $\mathrm{Tr}F^2$ ではない) → WZW4を解析せよ！

§5 4次元WZW模型 [HHK]

この作用密度を計算した

(詳細は 8月19日13時5分公開)

$$S_0 = \int_{M^4} \omega \wedge \mathrm{Tr}\left((\partial J)J^{-1} \wedge (\tilde{\partial} J)J^{-1}\right) + \int_{M^5} \overset{dA}{\tilde{\omega}} \wedge \mathrm{Tr}\left((d\tilde{J})\tilde{J}^{-1}\right)^3$$ [Donaldson]

$M^5 = M^4 \times [0,1] \ni t \qquad \tilde{J}(t=1) = J \quad \tilde{J}(t=0) = 1$

$$= \underbrace{\int_{M^4} \mathrm{Tr}\left((\partial_m J)J^{-1}(\partial^m J)J^{-1}\right) d^4x}_{\text{EOM: Yang's eq}} + \underbrace{\int_{M^4} A \wedge \mathrm{Tr}\left\{(dJ)J^{-1}\right\}^3}_{\text{topological term}}$$

ω : Kähler form, $\quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \tilde{\partial} = d\tilde{w}\,\partial_w + d\tilde{z}\,\partial_{\tilde{z}}$

付録A. ダルブー変換 (for 非可換 KdV eq.)

$u = u(t,x)$: $\mathbb{H}$-valued fcn.

(NC KdV eq.)

$$u_t + u_{xxx} + 3(u_x u + u u_x) = 0 \quad \cdots ①$$

① is derived from compatibility condition of the linear system:

$$[L, M] = 0$$

$$\begin{cases} L\phi = 0 \\ M\phi = 0 \end{cases} \quad \cdots ②$$

ϕ : $\mathbb{H}$-valued

$$\begin{cases} L_\xi := \partial_x^2 + u - \xi \\ M := \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x \end{cases}$$

ξ : spectral parameter

$$\phi = \phi(t, x; \xi) \qquad u = u(t, x)$$

(Darboux trf.)

θ : "special" sol. of ② i.e. $\begin{cases} L_\lambda \theta = 0 \\ M \theta = 0 \end{cases}$ … ③

$\swarrow \zeta \to \lambda$: special value (fix)

$\theta(\lambda) := \phi(\zeta \to \lambda)$

$G_\theta := \theta \, \partial_x \, \theta^{-1} = \partial_x - \theta_x \theta^{-1}$ … ④ $\quad \swarrow \partial_x \theta^{-1} = -\theta^{-1}\theta_x\theta^{-1}$

$\left(\Leftrightarrow \; G_\theta \overset{\forall}{f} = \theta \, \partial_x (\theta^{-1} f) = \partial_x f - \theta_x \theta^{-1} f \right)$

The following trf. is called the Darboux trf.

$$(D) \begin{cases} L \mapsto \tilde{L} := G_\theta L G_\theta^{-1} \quad (= \partial_x^2 + \tilde{u} - \zeta : \text{form invariant}) \\ M \mapsto \tilde{M} := G_\theta M G_\theta^{-1} \\ \phi \mapsto \tilde{\phi} := G_\theta \phi \overset{④}{=} \phi_x - \theta_x \theta^{-1} \phi \end{cases}$$

(D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$ … ⑤

$\because$ (D) $\Leftrightarrow$ $\tilde{L} G_\theta = G_\theta L$, $\tilde{M} G_\theta = G_\theta M$ $\underset{③}{\Rightarrow}$ ⑤ $\boxtimes$

Strategy [Gilson-Nimmo, 2007]

$$\begin{cases} L_\zeta \phi = (\partial_x^2 - \zeta)\phi = 0 \\ M\phi = (\partial_t + 4\partial_x^3)\phi = 0 \end{cases}$$

$$\boxed{\theta_k := \phi(\zeta \to \lambda_k)}$$

[0] $\Big(\overset{0}{\overset{\shortparallel}{u}}, \overset{\text{一般解}}{\phi}, \overset{\text{特殊解}}{\theta_1}\Big)$: initial seed sol.

$\downarrow$ (D) $\phi_{[1]} = \phi' - \theta_1' \theta_1^{-1} \phi$ ←ビフン→

[1] $\Big(\underset{\underset{2(\theta_1' \theta_1^{-1})'}{\shortparallel}}{u_{[1]}}, \phi_{[1]}, \quad \Big)$

$\downarrow$

[2]

$\downarrow$

[3]

Strategy [Gilson–Nimmo, 2007]

$$\begin{cases} L_\zeta \phi = (\partial_x^2 - \zeta)\phi = 0 \\ M\phi = (\partial_t + 4\partial_x^3)\phi = 0 \end{cases}$$

一般解　特殊解

[0] $\left(\overset{0}{\underset{}{u}}, \phi, \theta_1\right)$: initial seed sol.

←ビブン→

$$\downarrow \text{(D)} \quad \phi_{[1]} = \phi' - \theta_1' \theta_1^{-1} \phi$$

$$\boxed{\theta_k := \phi(\zeta \to \lambda_k)}$$

[1] $(u_{[1]}, \phi_{[1]}, \theta_{[1]})$

$u_{[1]} = 2(\theta_1' \theta_1^{-1})'$

$\zeta \to \lambda_2$

$$\theta_{[1]} := \phi_{[1]}(\zeta \to \lambda_2) = \underbrace{\phi'(\zeta \to \lambda_2)}_{=\theta_2'} - \theta_1' \theta_1^{-1} \underbrace{\phi(\zeta \to \lambda_2)}_{=\theta_2}$$

$$\downarrow \text{(D)} \quad \phi_{[2]} = \phi_{[1]}' - \theta_{[1]}' \theta_{[1]}^{-1} \phi_{[1]}$$

代入していけば
initial data だけで
解が逐次求まる.

[2] $(u_{[2]}, \phi_{[2]}, \quad)$

$\downarrow$

(結果) $\Theta := (\theta_1, \cdots, \theta_n)$: set of sols of ③

$$\phi_{[n]} = \begin{vmatrix} \Theta & \phi \\ \Theta^{(1)} & \phi^{(1)} \\ \vdots & \vdots \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}$$

[Etingof-Gelfand-Retakh, 1997]

(Quasi-Wronskian)

$$u_{[n]} = 2\left(\sum_{k=1}^{n} \theta_{[k]}^{(1)}\, \theta_{[k]}^{-1}\right)_x = -2 \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}_x$$

Quasideterminant で compact に書ける!

付録B. Quasideterminants

$A = (a_{ij})_{1\leq i,j \leq n}$ $\quad a_{ij} \in$ Division Ring (斜体)

$\oslash^{-1}$ is assumed to exist. (e.g. $\mathbb{H}$) noncommutative

<u>Def</u>

Let $A = (a_{ij})$ be an $n \times n$ square matrix, and $B = (b_{ij})$ be A^{-1}.

b_{ji}^{-1} is (i,j)-quasideterminant of A and represented:

$$b_{ji}^{-1} =: |A|_{ij} \overset{\text{or}}{\equiv} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \boxed{a_{ij}} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \qquad \left(\xrightarrow[\text{limit}]{\text{com.}} (-1)^{i+j} \frac{|A|}{|A^{ij}|} \right)$$

suffix or box

逆行列の公式

Rmk Block decomposition

square → A, square → D

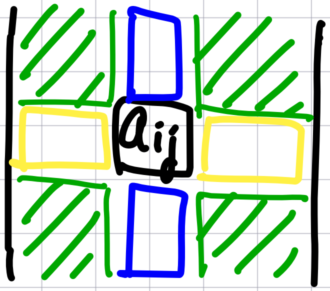
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \quad \cdots ⑥$$

これが逆行列の1要素

w/ $S := D - CA^{-1}B$ (Schur compliment)

$$D \equiv a_{ij} \Rightarrow S \equiv |A|_{ij} = a_{ij} - \left(i\text{-th row}\right)\left(A'_{ij}\right)^{-1}\left(j\text{-th column}\right)$$

deleting i-th row & j-th column



Ex

$n=1)$ $\quad |A| = a$

$n=2)$ $\quad \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21}, \quad \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}\, a_{21}^{-1}\, a_{22}, \quad \cdots$

$$\therefore \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

all squared

blocked

$n=3)$

$$\left|\begin{array}{c|cc} \boxed{a_{11}} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right| = a_{11} - (a_{12}\ a_{13}) \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$$

$$= a_{11} - a_{12} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{31}$$

$$- a_{13} \begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} a_{31}, \quad \cdots$$

Useful Identity

(NC Jacobi identity)
NonCommutative

$$
\begin{array}{c|c}
 & \begin{array}{ccc} N & 1 & 1 \end{array} \\
\begin{array}{c} N \\ 1 \\ 1 \end{array} & \left|\begin{array}{ccc} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{array}\right|
\end{array}
= \left|\begin{array}{cc} A & C \\ E & \boxed{i} \end{array}\right| - \left|\begin{array}{cc} A & B \\ E & \boxed{h} \end{array}\right| \left|\begin{array}{cc} A & B \\ D & \boxed{f} \end{array}\right|^{-1} \left|\begin{array}{cc} A & C \\ D & \boxed{g} \end{array}\right| \cdots ⑦
$$

$(N+2)\times(N+2)$

気分 "$i - hf^{-1}g$" (Schur comp.)

$(N+1)\times(N+1)$

∵ (S.C. Huang, PhD thesis, 2112.10702)

$$
(\mathrm{LHS}) = i - \begin{pmatrix} E & h \end{pmatrix} \underbrace{\begin{pmatrix} A & B \\ D & f \end{pmatrix}^{-1}}_{⑥\ (\text{ブロック分解})^{-1}} \begin{pmatrix} C \\ g \end{pmatrix} = (\mathrm{RHS}) \quad \blacksquare
$$

(Derivative formula)

derivative

k-th row of A

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & e_k \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & \boxed{(B^k)'} \end{vmatrix}$$

$n \quad 1$

k-th column of A

$$= \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix} \begin{vmatrix} A & B \\ {}^t e_k & \boxed{0} \end{vmatrix}$$

$${}^t e_k = (0 \cdots \overset{k\text{-th}}{1} \cdots 0)$$

$$\left(\because (A \cdot A^{-1})' = 0 \rightarrow \right) \quad (A^{-1})' = -A^{-1}A'A^{-1} \rightarrow$$

$$\because \begin{vmatrix} A & B \\ C & d \end{vmatrix}' = (d - CA^{-1}B)' = d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'$$

$\sum_{k=1}^{n} e_k {}^t e_k$: unit matrix

$$= d' - C'A^{-1}B + \sum_{k=1}^{n} (CA^{-1}e_k)(\overbrace{{}^t e_k A'}^{(A^k)'} A^{-1}B) - \sum_{k=1}^{n} CA^{-1}e_k (B^k)' \quad \blacksquare$$

§3 Proofs: Page 6 の結果の $\phi_{[n]}, \theta_{[n]}$ が

$$\phi_{[n+1]} = \phi'_{[n]} - \theta'_{[n]}\,\theta_{[n]}^{-1}\,\phi_{[n]}$$ をみたすか?

$$\phi_{[n]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}$$

$$\phi'_{[n]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} + \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \vdots & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix} \phi_{[n]}$$

only $k=n$ survive

$$\theta_{[n]} = \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n)} & \boxed{\theta_{n+1}^{(n)}} \end{vmatrix}$$

$$\theta'_{[n]} = \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} + \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \vdots & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix} \theta_{[n]}$$

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \theta_{n+1} & \phi \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \theta_{n+1}^{(n)} & \phi^{(n)} \\ \Theta^{(n+1)} & \theta_{n+1}^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix}$$

$$\therefore\ \phi'_{[n]} - \theta'_{[n]}\,\theta_{[n]}^{-1}\,\phi_{[n]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} - \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} \theta_{[n]}^{-1}\,\phi_{[n]} \overset{\text{Jacobi}}{=} \phi_{[n+1]}$$

OK!

$$
u_{[n+1]} = -2 \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}_x
$$

$$
\overset{\text{Jacobi}}{=} +2\partial_x \left\{ \begin{vmatrix} \ddots & & & \theta_n \\ & & & \vdots \\ & & & \theta_n^{(n-2)} \\ \theta_1^{(n)} & \cdots & \theta_{n-1}^{(n)} & \boxed{\theta_n^{(n)}} \end{vmatrix} \; \overbrace{\begin{vmatrix} \ddots & & & \theta_n \\ & & & \vdots \\ & & & \theta_n^{(n-2)} \\ \theta_1^{(n-1)} & \cdots & \theta_{n-1}^{(n-1)} & \boxed{\theta_n^{(n-1)}} \end{vmatrix}}^{\text{Quasi Wronskian}}{}^{-1} \right\}
$$

"gapped"

$\downarrow$ commutative limit

$$
= 2\partial_x \left\{ Wr(\theta_1, \cdots, \theta_n)' \, Wr(\theta_1, \cdots, \theta_n)^{-1} \right\}
$$

$$
= 2\partial_x^2 \log Wr(\theta_1, \cdots, \theta_n)
$$

(実は一般解)

Hirota trf. & Wronskian sol.