5D AGT correspondence of supergroup gauge theories from Ding-Iohara-Miki algebra

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based on an ongoing work

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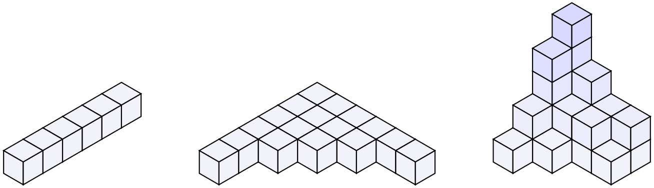
Introduction

- AGT correspondence [Alday-Gaiotto-Tachikawa 2009]
  Nekrasov partition functions ↔ conformal blocks of Liouville/Toda CFT
  supersymmetric gauge theory ↔ algebra (Virasoro, W_N algebra,...)
- There is a 5D lift-up of the correspondence (5D AGT).
- On the algebra side, quantum algebras appear (q-Virasoro, q-W_N, q-$Y_{L,M,N}$...).
- They are understood as truncations of **Ding-Iohara-Miki algebra** (quantum toroidal $\mathfrak{gl}_1$).

- One application of quantum toroidal algebras is the intertwiner formalism [Awata-Feigin-Shiraishi 2011].
- Using two basic representations "crystal" and "vertex operator" representations, we can construct algebraic objects called "intertwiners". Composition of them gives Nekrasov partition functions.



Crystal representations

- Changing the algebra itself or the representations, we can get different intertwiners leading to different correspondences with gauge theories.

Algebra	Geometry	Gauge theory	Representation
QT $\mathfrak{gl}_1$	$\mathbb{C}_{q_1}\times\mathbb{C}_{q_2}\times\mathbb{C}_{q_3}$	5D $\mathcal{N}=1$ on $\mathbb{C}^2_{q_1,q_2}\times S^1$	$\mathcal{F}_3^{\mathrm{cryst}}$, $\mathcal{F}_3^{\mathrm{op}}$
QT $\mathfrak{gl}_1$	$\mathbb{C}_{q_1}\times\mathbb{C}_{q_2}\times\mathbb{C}_{q_3}$	3D $\mathcal{N}=2$ on $\mathbb{C}_{q_1}\times S^1$	$\mathcal{V}_1^{\mathrm{cryst}}$, $\mathcal{F}_3^{\mathrm{op}}$

- How about **supergroup** gauge theories?? [Vafa 2001, Okuda-Takayanagi 2006, Dijkgraaf-Heidenreich-Jefferson-Vafa 2016]
- Nekrasov partition function [Kimura-Pestun 2019], anti-refined topological vertex [Kimura-Sugimoto 2020],....
  $\Rightarrow$ Unified descripition using quantum algebra

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Supergroup gauge theory [Vafa 2001, Okuda-Takayanagi 2006, Dijkgraaf-Heidenreich-Jefferson-Vafa 2016]

- Yang-Mills theory
  - Defining data: Space-time $\mathbb{R}^4$, gauge group G, $\mathfrak{g} = \mathrm{Lie}\, G$, gauge fields $A_\mu(x)$
  - Action:

$$S_{\mathrm{YM}} = \frac{1}{g_{\mathrm{YM}}^2} \int d^4x \, |F|^2 = -\frac{1}{g_{\mathrm{YM}}^2} \int \mathrm{Tr}\,(F \wedge *F), \quad F = dA + A \wedge A$$

  - Usually, the gauge group is chosen to be a compact Lie group so that we have a gauge invariant **positive Killing form** (e.g. Tr). Physically, this is necessary for the energy to be bounded from below.
- What if G is a supergroup?
  - For this talk, unitary group $G = U(N_+ \,|\, N_-)$ is enough.
  - The $U(N_+ \,|\, N_-)$ invariant trace is the **supertrace**

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathrm{Str}\, U := \mathrm{Tr}\, A - \mathrm{Tr}\, D.$$

- Yang-Mills theory with $G = U(N_+ \mid N_-)$:

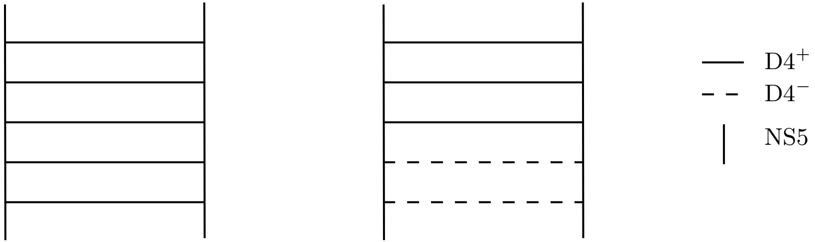
$$S_{\mathrm{YM}} = -\frac{1}{g_{\mathrm{YM}}^2} \int d^4x \, \mathrm{Str}\,(F \wedge *F),$$
$$A_\mu : \mathfrak{g}\,\text{valued}, \quad A_\mu \to g A_\mu g^{-1} + g \partial_\mu g^{-1} \ (g \in G)$$

- Is this physical?
  - The gauge fields A_μ take values in the superalgebra, which means the components of them include Grassmann numbers. → **breaks spin-statistics theorem**
  - The action is not positive semidefinite, so the spectrum is unbounded and thus the theory is **non-unitary**

$$S_{\mathrm{YM}} = -\frac{1}{g_{\mathrm{YM}}^2} \int d^4x \, \mathrm{Tr}_{\mathbb{C}^{N_+}} \, (F \wedge *F)^+ - \left(-\frac{1}{g_{\mathrm{YM}}^2} \right) \int d^4x \, \mathrm{Tr}_{\mathbb{C}^{N_-}} \, (F \wedge *F)^-$$

  The kinetic term of the negative part has a wrong sign.
  - Seems NOT physical (?) Not much studies have been done from the physical view-point.

- Although the physical meaning of this theory is still not clear, it seems we can study non-perturbative effects for supersymmetric gauge theories. We also have brane realizations of it.
- In particular, we are interested in 4D $\mathcal{N}=2$ or 5D $\mathcal{N}=1$ supersymmetric gauge theories.
- Brane construction: including "ghost"-branes (not anti-branes) give supergroup gauge theories



- Taking T-duality gives the 5D theory.

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Instantons [Kimura-Pestun 2019, Kimura review]

- We also have instantons in supergroup gauge theories.
- They are characterized by the topological number

$$\frac{1}{g_{\mathrm{YM}}^2}\int \mathrm{Str}\, F\wedge F = k_+ - k_- \equiv k \in \mathbb{Z}$$

- $k_+ > 0 \cdots$ positive instanton number
  $k_- > 0 \cdots$ negative instanton number
- We can add the so-called θ-term

$$S_\theta[A] = -\frac{i\theta}{8\pi^2}\int \mathrm{Str}\, F\wedge F$$

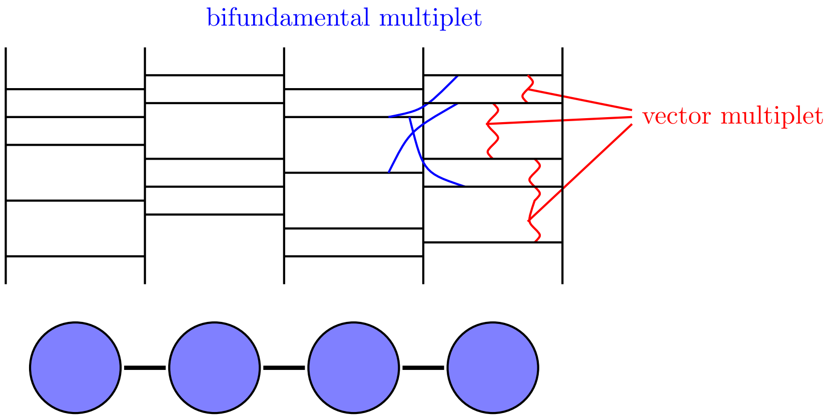
- Evaluating the total action around the k-instanton configuration, the path integral is schematically derived as

$$\mathcal{Z} = \int [\mathcal{D}\mathcal{A}]\, e^{-S_{\mathrm{tot}}} = \sum_{k} \mathfrak{q}^k \int [\mathcal{D}A^{(k)}_{\mathrm{inst}}] \int [\mathcal{D}\delta A]\, e^{-S_{\mathrm{fluc}}[\delta A]} = \sum_{k\in\mathbb{Z}} \mathfrak{q}^k \mathcal{Z}_k$$

- We can introduce matters and the partition function similarly decomposes into topological sectors.
- Computing the paritition function is difficult but under suitable number of supersymmetries and Ω background, we can compute it explicitly.
  $\rightarrow$ instanton partition function, Nekrasov partition function [Kimura-Pestun 2019]

Instanton partition function [Kimura-Pestun 2019]

- Non-supergroup linear quiver gauge theory:



- Assign a Young diagram λ and a parameter v to each D-brane.
- The Nekrasov factor is defined as

$$N_{\lambda_1\lambda_2}(Q) = \prod_{x\in\lambda_1}\left(1 - Qq_1^{l_{\lambda_1}(x)+1}q_2^{-a_{\lambda_2}(x)}\right)\prod_{x\in\lambda_2}\left(1 - Qq_1^{-l_{\lambda_2}(x)}q_2^{a_{\lambda_1}(x)+1}\right), \quad Q = v_1/v_2.$$

- Vector multiplet contribution from D-brane (v_1, λ_1) to (v_2, λ_2):

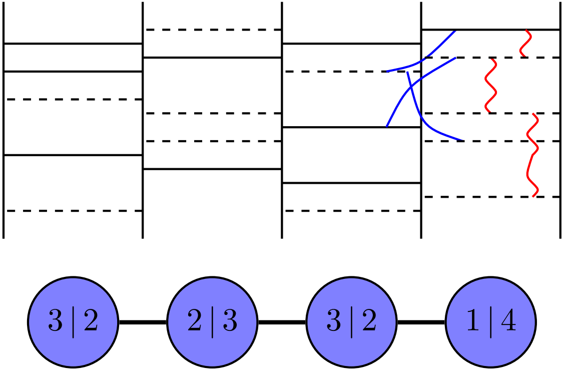
$$N_{\lambda_1\lambda_2}(v_1/v_2)^{-1}$$

- Bifundamental contribution from D-brane (v_1, λ_1) to (v_2, λ_2) with bifundamental mass μ:

$$N_{\lambda_1\lambda_2}(\mu^{-1}v_1/v_2)^{+1}$$

- After taking the composition of each contributions, take the sum for all possible Young diagrams.

- Supergroup case $\Rightarrow$ need to include **negative branes**



vector multiplet
$D4^+ \leftrightarrow D4^+$
$D4^- \leftrightarrow D4^-$
$D4^+ \leftrightarrow D4^-$

bifundamental multiplet
$D4^+ \leftrightarrow D4^+$
$D4^- \leftrightarrow D4^-$
$D4^+ \leftrightarrow D4^-$

- Assign a Young diagram λ and a parameter v to each D-brane. Additionally, we need to distinguish the parities of the D-branes. We assign $\sigma = \pm 1$ to each D-brane depending on its parity. Namely, each D-brane is labeled by (v, λ, σ).
- Define four Nekrasov factors:

$$N^{\sigma_1\sigma_2}_{\lambda_1\lambda_2}(v_1/v_2) = \prod_{x\in\lambda_1}\left(1 - \frac{\chi_x^{(\sigma_1)}}{q_3 v_2}\right)\prod_{x\in\lambda_2}\left(1 - \frac{v_1}{\chi_x^{(\sigma_2)}}\right)\prod_{\substack{x\in\lambda_1\\ y\in\lambda_2}} S\left(\frac{\chi_x^{(\sigma_1)}}{\chi_y^{(\sigma_2)}}\right),$$

$$S(z) = \frac{(1-q_1 z)(1-q_2 z)}{(1-z)(1-q_3^{-1}z)}, \quad \chi_x^{(\sigma)} = v q_3^{\frac{1-\sigma}{2}} q_1^{\sigma(i-1)} q_2^{\sigma(j-1)}, \quad \sigma, \sigma_{1,2} = \pm 1$$

- $N^{++}_{\lambda_1,\lambda_2}(Q)$ is just the Nekrasov factor defined a while ago.

- Vector multiplet from D-brane $(v_1, \lambda_1, \sigma_1)$ to $(v_2, \lambda_2, \sigma_2)$:

$$N^{\sigma_1\sigma_2}_{\lambda_1\lambda_2}(v_1/v_2)^{-\sigma_1\sigma_2}$$

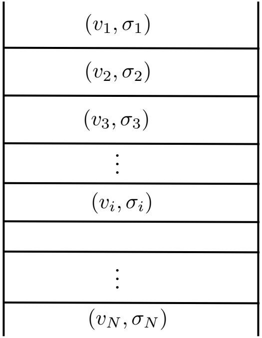
- Bifundamental multiplet from D-brane $(v_1, \lambda_1, \sigma_1)$ to $(v_2, \lambda_2, \sigma_2)$ with bifundamental mass μ:

$$N^{\sigma_1\sigma_2}_{\lambda_1\lambda_2}(\mu^{-1}v_1/v_2)^{\sigma_1\sigma_2}$$

- Also have Chern-Simons term, topological term, (anti)fundamental part,....

- Nekrasov partition function of pure $U(N\,|\,\sigma_1,\ldots,\sigma_N)$ gauge theory:

$$\mathcal{Z}^{\text{inst}}_{N,(\sigma_1,\cdots,\sigma_N)}=\sum_{\lambda_1,\cdots,\lambda_N}\mathfrak{q}^{\sum_{i=1}^{N}\sigma_i|\lambda_i|}\left(\prod_{i=1}^{N}\prod_{x\in\lambda_i}\left(\chi_x^{(\sigma_i)}\right)^{\sigma_i\kappa}\right)\prod_{i,j=1}^{N}N^{\sigma_i\sigma_j}_{\lambda_i\lambda_j}(v_i/v_j)^{-\sigma_i\sigma_j},$$



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Quantum toroidal $\mathfrak{gl}_1$ [Miki, Ding-Iohara, Feigin-Feigin-Jimbo-Miwa-Mukhin,...]

- Currents:

$$x^{\pm}(z) = \sum_{m\in\mathbb{Z}} x^{\pm}_m z^{-m}, \quad \psi^{\pm}(z) = \sum_{r\geq 0} \psi^{\pm}_{\pm r} z^{\mp r}, \quad \hat{\gamma}, \quad \psi^{+}_0/\psi^{-}_0,$$

- Defining relations:

$$[\psi^{\pm}(z), \psi^{\pm}(w)] = 0, \quad \psi^{+}(z)\psi^{-}(w) = \frac{g(\hat{\gamma}z/w)}{g(\hat{\gamma}^{-1}z/w)}\psi^{-}(w)\psi^{+}(z),$$

$$\psi^{\pm}(z)x^{+}(w) = g\left(\hat{\gamma}^{\pm\frac{1}{2}}z/w\right)x^{+}(w)\psi^{\pm}(z), \quad \psi^{\pm}(z)x^{-}(w) = g\left(\hat{\gamma}^{\mp\frac{1}{2}}z/w\right)^{-1}x^{-}(w)\psi^{\pm}(z),$$

$$x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z),$$

$$[x^{+}(z), x^{-}(w)] = \frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})}\left(\delta(\hat{\gamma}w/z)\psi^{+}\left(\hat{\gamma}^{\frac{1}{2}}w\right) - \delta(\hat{\gamma}^{-1}w/z)\psi^{-}\left(\hat{\gamma}^{-\frac{1}{2}}w\right)\right),$$

where

$$g(z) = \frac{(1-q_1z)(1-q_2z)(1-q_3z)}{(1-q_1^{-1}z)(1-q_2^{-1}z)(1-q_3^{-1}z)}.$$

Representations

- Representations are specified by central charges $(\hat{\gamma}, \psi_0^-) = (\gamma^{l_1}, \gamma^{l_2})$, $\gamma = q_3^{1/2}$.
  $\rightarrow$ Level (l_1, l_2) representation.
- Crystal representations:
  - Level $(0,1)$: Well known representation

$$\mathcal{Y}_\lambda^{(+)}(z) \coloneqq (1 - v/z) \prod_{x\in\lambda} S(\chi_x^{(+)}/z), \quad \chi_x^{(+)} = v q_1^{i-1} q_2^{j-1},$$

$$x^+(z)\left|v,\lambda\right\rangle = \sum_{x\in A(\lambda)} \delta\big(z/\chi_x^{(+)}\big) \operatorname*{Res}_{z=\chi_x} \frac{1}{z\mathcal{Y}_\lambda^{(+)}(z)} \left|v,\lambda+x\right\rangle,$$

$$x^-(z)\left|v,\lambda\right\rangle = \gamma^{-1} \sum_{x\in R(\lambda)} \delta\big(z/\chi_x^{(+)}\big) \operatorname*{Res}_{z=\chi_x} z^{-1}\mathcal{Y}_\lambda^{(+)}(zq_3^{-1}) \left|v,\lambda-x\right\rangle,$$

$$\psi^\pm(z)\left|v,\lambda\right\rangle = [\Psi_\lambda^{(+)}(z)]_\pm \left|v,\lambda\right\rangle$$

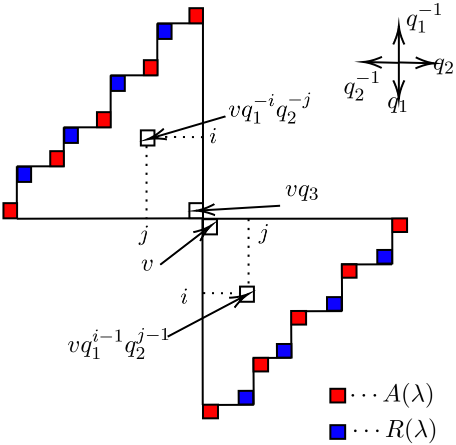
- Level $(0,-1)$: Known to exist, but only one paper used it to study [Bourgine 2018].

$$\mathcal{Y}_{\lambda}^{(-)}(z) = (1 - v/z) \prod_{x\in\lambda} S\left(\chi_x^{(-)}/z\right), \quad \chi_x^{(-)} = v q_1^{-i} q_2^{-j},$$

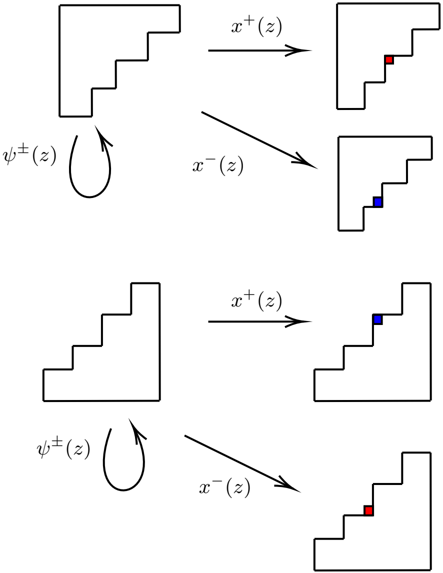
$$x^{+}(z)\,|v,\lambda\rangle = \gamma \sum_{x\in R(\lambda)} \delta\left(z/\chi_x^{(-)}\right) \operatorname*{Res}_{z=\chi_x^{(-)}} z^{-1}\mathcal{Y}_{\lambda}^{(-)}(z)\,|v,\lambda - x\rangle\,,$$

$$x^{-}(z)\,|v,\lambda\rangle = \sum_{x\in A(\lambda)} \delta\left(z/\chi_x^{(-)}\right) \operatorname*{Res}_{z=\chi_x^{(-)}} z^{-1}\mathcal{Y}_{\lambda}^{(-)}(q_3^{-1}z)^{-1}\,|v,\lambda + x\rangle\,,$$

$$\psi^{\pm}(z)\,|v,\lambda\rangle = \left[\Psi_{\lambda}^{(-)}(z)\right]_{\pm}|v,\lambda\rangle\,,$$



q_1^{-1}
q_2
q_2^{-1}
q_1
$vq_1^{-i}q_2^{-j}$
i
vq_3
j
j
v
i
$vq_1^{i-1}q_2^{j-1}$
$\cdots A(\lambda)$
$\cdots R(\lambda)$



$x^+(z)$
$\psi^{\pm}(z)$
$x^-(z)$
$x^+(z)$
$\psi^{\pm}(z)$
$x^-(z)$

- Vertex operator representations with level $(1, n)$

$$x^{+}(z) \mapsto u\gamma^{n} z^{-n}\eta(z), \quad x^{-}(z) \mapsto u^{-1}\gamma^{-n} z^{n}\xi(z), \quad \psi^{\pm}(z) \mapsto \gamma^{\mp n}\varphi^{\pm}(z), \quad \hat{\gamma} \mapsto \gamma,$$

$$\eta(z) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^{n}}{n}q_{1}^{n}(1-q_{2}^{n})a_{-n}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{z^{-n}}{n}(1-q_{1}^{-n})a_{n}\right),$$

$$\xi(z) = \exp\left(\sum_{n=1}^{\infty}\frac{z^{n}}{n}q_{1}^{n}(1-q_{2}^{n})\gamma^{n}a_{-n}\right)\exp\left(\sum_{n=1}^{\infty}\frac{z^{-n}}{n}(1-q_{1}^{-n})\gamma^{n}a_{n}\right),$$

$$\varphi^{+}(z) = \exp\left(\sum_{n>0}\frac{z^{-n}}{n}\gamma^{-\frac{n}{2}}(1-q_{1}^{-n})(q_{3}^{n}-1)a_{n}\right),$$

$$\varphi^{-}(z) = \exp\left(\sum_{n>0}\frac{z^{n}}{n}q_{1}^{n}\gamma^{-\frac{n}{2}}(1-q_{2}^{n})(q_{3}^{n}-1)a_{-n}\right),$$

$$[a_{n}, a_{m}] = n\delta_{n+m,0}.$$

- How about level $(-1, n)$?? $\rightarrow$ Later discuss (future work).

Intertwiners

- Intertwiner relations:

$$\rho_{u'}^{(1,n\pm1)}(g(z))\Phi_{\pm}^{(n)}[u,v]=\Phi_{\pm}^{(n)}[u,v](\rho_{v}^{(0,\pm1)}\otimes\rho_{u}^{(1,n)})\Delta(g(z)),$$

$$(\rho_{u}^{(1,n)}\otimes\rho_{v}^{(0,\pm1)})\Delta(g(z))\Phi_{\pm}^{(n)*}[u,v]=\Phi_{\pm}^{(n)*}[u,v]\rho_{u'}^{(1,n\pm1)}(g(z)),$$

- Positive and negative intertwiners:

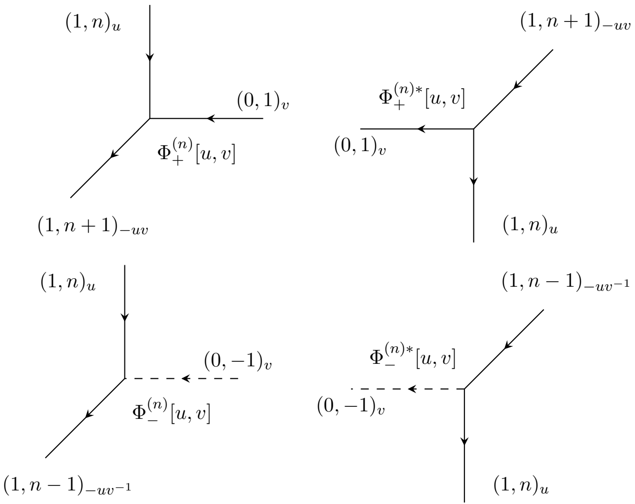
$$\Phi_{\pm}^{(n)}[u,v]:(0,\pm1)_{v}\otimes(1,n)_{u}\to(1,n\pm1)_{u'},\quad u'=-uv^{\pm1}$$

$$\Phi_{\pm}^{(n)}[u,v]=\sum_{\lambda}a_{\lambda}^{(\pm)}\langle v,\lambda|\otimes\Phi_{\pm,\lambda}^{(n)}[u,v],$$

- Positive and negative dual intertwiners:

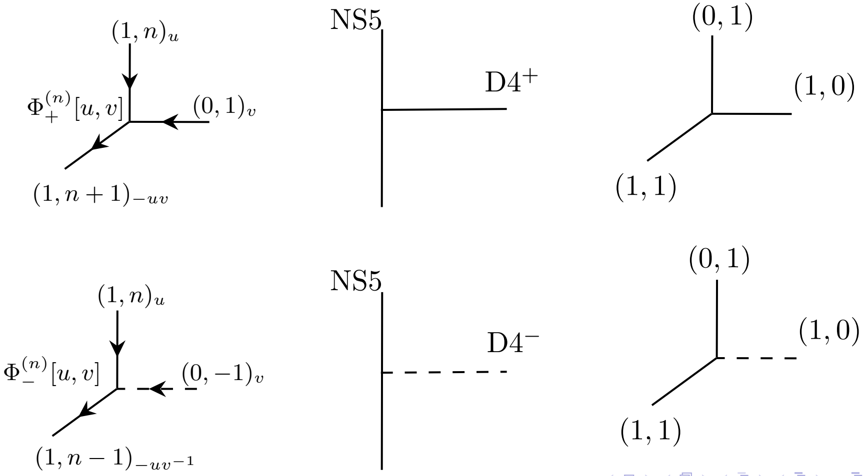
$$\Phi_{\pm}^{(n)*}[u,v]:(1,n\pm1)_{u'}\to(1,n)_{u}\otimes(0,\pm1)_{v},$$

$$\Phi_{\pm}^{(n)*}[u,v]=\sum_{\lambda}a_{\lambda}^{(\pm)}\Phi_{\pm,\lambda}^{(n)*}[u,v]\otimes|v,\lambda\rangle$$

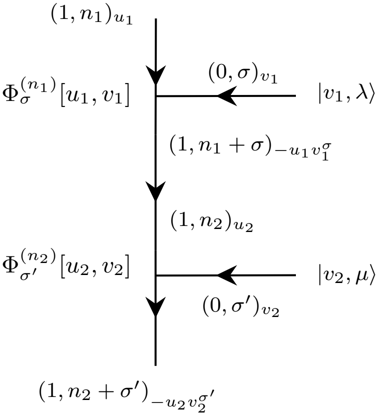

$(1,n)_u$
$(0,1)_v$
$\Phi_+^{(n)}[u,v]$
$(1,n+1)_{-uv}$
$(1,n+1)_{-uv}$
$\Phi_+^{(n)*}[u,v]$
$(0,1)_v$
$(1,n)_u$
$(1,n)_u$
$(0,-1)_v$
$\Phi_-^{(n)}[u,v]$
$(1,n-1)_{-uv^{-1}}$
$(1,n-1)_{-uv^{-1}}$
$\Phi_-^{(n)*}[u,v]$
$(0,-1)_v$
$(1,n)_u$

- Correspondences with brane realization

Representation | Type IIA | Type IIB



- Contractions of the intertwiners give the Nekrasov factors and the perturbative contribution



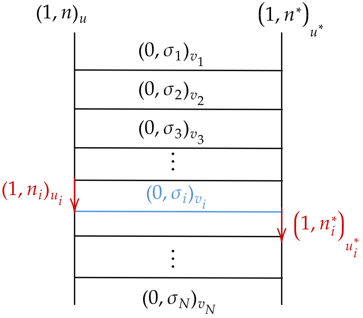
$$
\begin{aligned}
&= \Phi^{(n_2)}_{\sigma',\mu}[u_2,v_2]\Phi^{(n_1)}_{\sigma,\lambda}[u_1,v_1] \\
&= \left(\mathcal{G}\left(q_3^{-1}v_1/v_2\right)\right)^{\sigma\sigma'} N^{\sigma\sigma'}_{\lambda\mu}(v_1/v_2)^{-\sigma\sigma'} \\
&\qquad \times :\Phi^{(n_2)}_{\sigma',\mu}[u_2,v_2]\Phi^{(n_1)}_{\sigma,\lambda}[u_1,v_1]:
\end{aligned}
$$

- Similarly for the dual intertwiners.

Example: Pure supergroup gauge theory

- In realizing pure supergroup gauge theory using D-branes, we need to specify the order of them.
- We denote a supergroup gauge theory with N-D4(D5) branes with signatures $\sigma_1, \ldots, \sigma_N$ from the top to the bottom as $U(N \,|\, \sigma_1, \sigma_2 \cdots, \sigma_N)$.

$\mathcal{Z}_{N,(\sigma_1,\ldots,\sigma_N)} =$



$$= \begin{array}{ccc} \widehat{0} & & \widehat{0} \\ \Phi^{(n_1)}_{\sigma_1}[u_1, v_1] & \cdot & \Phi^{(n_1^*)*}_{\sigma_1}[u_1^*, v_1] \\ \Phi^{(n_2)}_{\sigma_2}[u_2, v_2] & \cdot & \Phi^{(n_2^*)*}_{\sigma_2}[u_2^*, v_2] \\ \vdots & & \vdots \\ \Phi^{(n_N)}_{\sigma_N}[u_N, v_N] & \cdot & \Phi^{(n_N^*)*}_{\sigma_N}[u_N^*, v_N] \\ \underbrace{0} & & \underbrace{0} \end{array}$$

- The spectral parameters obey the conservation law:

$$n_i = n + \sum_{l=1}^{i-1} \sigma_l, \quad u_i = u \prod_{l=1}^{i-1} (-v_l^{\sigma_l}), \quad n_i^* = n^* - \sum_{l=1}^{i} \sigma_l, \quad u_i^* = u^* \prod_{l=1}^{i} (-v_l^{-\sigma_l})$$

- Result:

$$\mathcal{Z}_{N,(\sigma_1,\cdots,\sigma_N)} = \prod_{i<j} \mathcal{G}\left(q_3^{-1}\frac{v_i}{v_j}\right)^{\sigma_i\sigma_j} \mathcal{G}\left(\frac{v_i}{v_j}\right)^{\sigma_i\sigma_j}$$
$$\times \sum_{\lambda_1,\cdots,\lambda_N} \mathfrak{q}^{\sum_{i=1}^{N}\sigma_i|\lambda_i|} \left(\prod_{i=1}^{N}\prod_{x\in\lambda_i}\left(\chi_x^{(\sigma_i)}\right)^{\sigma_i\kappa}\right) \prod_{i,j=1}^{N} N_{\lambda_i\lambda_j}^{\sigma_i\sigma_j}(v_i/v_j)^{-\sigma_i\sigma_j},$$

where

$$\mathfrak{q} = -\frac{u}{u^*}\gamma^{n-n^*}\prod_{i=1}^{N}(-v_i)^{\sigma_i}, \quad \kappa = n^* - n - \sum_{i=1}^{N}\sigma_i.$$

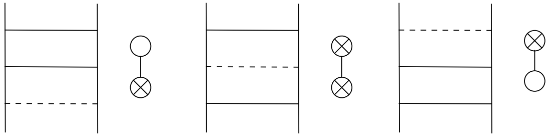
- The result matches with the localization formula.
- The partition function does not depend on the order of the signatures

$$\mathcal{Z}_{N,(\sigma_1,\ldots,\sigma_N)} = \mathcal{Z}_{N,(\omega\cdot\sigma_1,\ldots,\omega\cdot\sigma_N)}, \quad \omega\cdot\sigma_i = \sigma_{\omega(i)}, \quad \omega \in \mathfrak{S}_N$$

under redefinition of the Coulomb vev parameters

$$v_i \to \omega\cdot v_i = v_{\omega(i)}.$$

- Actually, the independence of the order of the signatures is related to the underlying superalgebra structure.



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Refined topological vertex and anti-vertex [Iqbal-Kozcaz-Vafa 2007, Awata-Kanno 2005, 2008, Kimura-Sugimoto 2020]

- The positive intertwiner is known to be related with the refined topological vertex.
- How about the negative intertwiner? Actually, recently, Kimura-Sugimoto defined the anti refined topological vertex.

$$C^{(\sigma)}_{\lambda\mu\nu}(t,q) = t^{-\frac{1}{2}||\mu^{\mathrm{T}}||^2} q^{\frac{1}{2}(||\mu||^2+\sigma||\nu||^2)} \widetilde{Z}_\nu(t^\sigma, q^\sigma)$$

$$\times \sum_\eta \left(\frac{q}{t}\right)^{\frac{1}{2}(|\eta|+|\lambda|-|\mu|)} s_{\lambda^{\mathrm{T}}/\eta}(t^{-\sigma\rho} q^{-\sigma\nu}) s_{\mu/\eta}(q^{-\sigma\rho} t^{-\sigma\nu^{\mathrm{T}}}), \quad \sigma = \pm$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} \left(1 - t^{l_\nu(i,j)+1} q^{a_\nu(i,j)}\right)^{-1}$$

- Taking nontrivial matrix elements, we obtain the topological vertices

$$\langle\mu|\,\Phi^{(n)}_{\sigma,\lambda}[u,v]\,|\nu\rangle \propto C^{(\sigma)}_{\mu^{\mathrm{T}}\nu\lambda}(q,t), \quad \langle\nu|\,\Phi^{(n)*}_{\sigma,\lambda}[u,v]\,|\mu\rangle \propto C^{(\sigma)}_{\mu\nu^{\mathrm{T}}\lambda^{\mathrm{T}}}(t,q), \quad q_1 = q, \quad q_2 = t^{-1}$$

- The gluing rules also match with the gluing of the topological vertices (the framing factors match exactly).

$$\begin{matrix} & \widehat{\underline{\nu_1}} & \\ \Phi^{(N)}_{\sigma}[u,v_1] & & |\lambda_1\rangle \\ \Phi^{(N+\sigma)}_{\sigma'}[-uv_1^{\sigma},v_2] & & |\lambda_2\rangle \\ & \underbrace{\overline{\nu_2}} & \end{matrix} \propto \sum_{\mu}\left(-\frac{v_1}{v_2}\right)^{|\mu|}\left(\frac{q}{t}\right)^{\frac{|\mu|}{2}} f_{\mu}(q,t)C^{(\sigma)}_{\mu^{\mathrm{T}}\nu_1\lambda_1}(q,t)C^{(\sigma')}_{\nu_2^{\mathrm{T}}\mu\lambda_2}(q,t)$$

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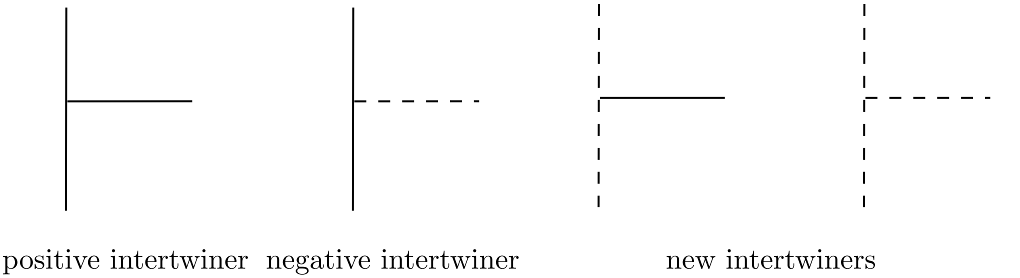
Conclusion

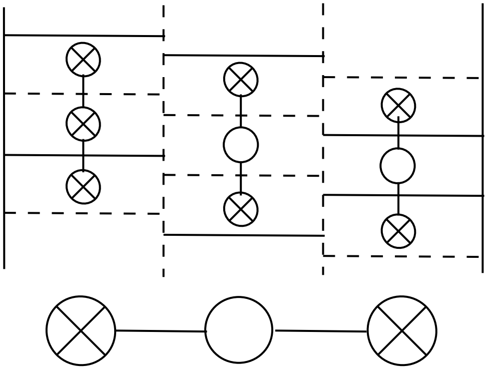
- Supergroup gauge theory is a non-unitary theory whose physical usefulness is not studied well.
- Instanton partition functions can be derived by localization and the basic components are four Nekrasov factors.
- From the AGT correspondence perspective, there should be a quantum algebraic structure lying behind it. Using the representation theory of quantum toroidal $\mathfrak{gl}_1$, we construct a new intertwiner called the negative intertwiner.
- Composition of the positive and negative intertwiners give the Nekrasov factors and reproduce the partition function of the supergroup gauge theory.
- Non-trivial matrix elements of the intwerners indeed reproduce the (anti-) refined topological vertex.
- Many things to do: generalizations to 2d/3d/4d/5d/6d, integrable structure from algebra side (R-matrix, q-KZ eq,..), relation with BPS crystals,....

Future directions 1: Superquiver gauge theory

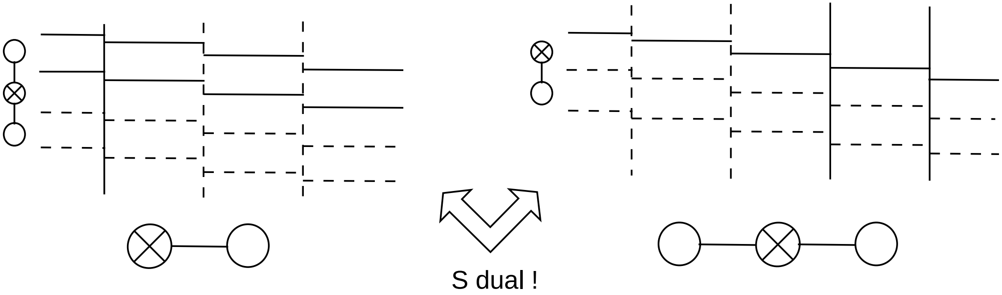
- In deriving the negative intertwiner, we newly introduced the level $(0,-1)$ representation. We should also introduce the level $(-1,n)$ representation. This should give new intertwiners.



- Taking the contractions of the new intertwiners, we should obtain partition functions of *supersymmetric supergroup superquiver gauge theories.* ⟵conjecture

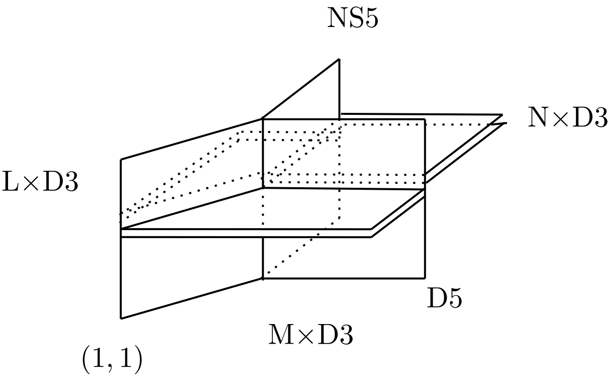


- We should have a duality between supergroup gauge theories and superquiver gauge theories, which comes from S-duality. This should be a strong consistency check.



Future directions 2: Corner (q-)VOA [Gaiotto-Rapcak 2017 etc.]

- Corner VOA $Y_{L,M,N}$ is obtained by stacking D3-branes in junctions of (p,q) branes.



- The D3-branes do not have to be positive branes. We should insert negative D3-branes too.

- The corner VOA should be generalize to an algebra

$$Y_{L,M,N}, \quad L = (L_+, L_-), \quad M = (M_+, M_-) \quad N = (N_+, N_-)$$

- $Y_{0,0,(N_+,N_-)}$ should be the AGT dual algebra of $U(N_+ \,|\, N_-)$ supergroup gauge theory.
- The q-deformed version of this generalized CVOA should be obtained by taking tensor products of the well known representations $\mathcal{F}_i(u)$ $(i = 1, 2, 3)$ with positive central charges and $\bar{\mathcal{F}}_i(u)$ $(i = 1, 2, 3)$ with negative central charges.
- I have studied only the algebra $\bar{\mathcal{F}}_3(u_1) \otimes \bar{\mathcal{F}}_3(u_2)$ and actually it is just the q-Virasoro algebra. This is not special since the AGT dual gauge theories $U(2\,|\,0)$ and $U(0\,|\,2)$ give the same partition functions and are equivalent.
- Other algebras are yet to be studied. I still do not know how to remove the extra Heisenberg algebra.

Example: q-Virasoro(?)

- $\bar{\mathcal{F}}_3(u_1) \otimes \bar{\mathcal{F}}_3(u_2)$
  - Current

$$T(z) = u_1 \Lambda_1(z) + u_2 \Lambda_2(z)$$
$$\Lambda_1(z) = \xi(z^{-1}) \otimes 1, \quad \Lambda_2(z) = \varphi^+(\gamma^{1/2} z^{-1}) \otimes \xi(\gamma z^{-1})$$

  - Quadratic relation:

$$S(w/z)T(z)T(w) - S(z/w)T(w)T(z)$$
$$= \frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})} \left\{ \delta\left(\frac{q_3 w}{z}\right) : \Lambda_1(q_3^{-1} z)\Lambda_2(z) : - \delta\left(\frac{w}{q_3 z}\right) : \Lambda_1(z)\Lambda_2(q_3 z) : \right\}$$

  - Usually, we can decouple the blue terms by dressing the currents

$$T(z) \to \alpha(z) T(z) \beta(z).$$

After mixing with reps with other colors and positive central charges, I do not know what will happen.

Appendix: Equivariant index formula

- Quiver gauge theory: $\Gamma = (\Gamma_0, \Gamma_1)$, $\Gamma_0 = \{i\}$, $\Gamma_1 = \{e : i \to j\}$
- Framing bundle $\mathbf{N} = (\mathbf{N}_i)_{i\in\Gamma_0}$, instanton bundle $\mathbf{K} = (\mathbf{K}_i)_{i\in\Gamma_0}$
- Supercharacters:

$$\operatorname{sch}\mathbf{N}_i = \operatorname{ch}\mathbf{N}_i^+ - \operatorname{ch}\mathbf{N}_i^-, \quad \operatorname{sch}\mathbf{K}_i = \operatorname{ch}\mathbf{K}_i^+ - \operatorname{ch}\mathbf{K}_i^-$$

where each is defined as

$$\operatorname{ch}\mathbf{N}_i^\sigma = \sum_{\alpha=1}^{N_{i,\sigma}} v_{i,\alpha}^{(\sigma)}, \quad \operatorname{ch}\mathbf{K}_i^\sigma = \sum_{x\in\lambda_i^{(\sigma)}} \chi_x^{(\sigma)}, \quad \sigma = \pm$$

where

$$\chi_x^{(+)} = v_{i,\alpha}^{(+)} q_1^{i-1} q_2^{j-1}, \quad (\alpha, i, j) \in \lambda_{i,\alpha}^{(+)}, \quad \alpha = 1, \ldots, N_{i,+},$$
$$\chi_x^{(-)} = v_{i,\alpha}^{(-)} q_1^{-i} q_2^{-j}, \quad (\alpha, i, j) \in \lambda_{i,\alpha}^{(-)}, \quad \alpha = 1, \ldots, N_{i,-},$$

- Vector multiplet:

$$\operatorname{sch}\mathbf{V}_i^{\text{inst}} = -\operatorname{sch}\mathbf{N}_i^\vee \operatorname{sch}\mathbf{K}_i - q_3 \operatorname{sch}\mathbf{K}_i^\vee \operatorname{sch}\mathbf{N}_i + \operatorname{ch}\wedge\mathbf{Q}^\vee \operatorname{sch}\mathbf{K}_i^\vee \operatorname{sch}\mathbf{K}_i,$$
$$\operatorname{sch}\mathbf{V}_i^{\text{inst}} = \sum_{\sigma,\sigma'=\pm} \sigma\sigma' \operatorname{ch}\mathbf{V}_{i,\sigma\sigma'}^{\text{inst}},$$
$$\operatorname{ch}\mathbf{V}_{i,\sigma\sigma'}^{\text{inst}} = -\operatorname{ch}\mathbf{N}_i^{\sigma\vee} \operatorname{ch}\mathbf{K}_i^{\sigma'} - q_3 \operatorname{ch}\mathbf{K}_i^{\sigma\vee} \operatorname{ch}\mathbf{N}_i^{\sigma'} + \operatorname{ch}\wedge\mathbf{Q}^\vee \operatorname{ch}\mathbf{K}_i^{\sigma\vee} \operatorname{ch}\mathbf{K}_i^{\sigma'},$$

where

$$\operatorname{ch}\wedge\mathbf{Q} = (1-q_1)(1-q_2),$$
$$\operatorname{ch}\mathbf{X} = \sum_X x \longrightarrow \operatorname{ch}\mathbf{X}^\vee = \sum_X x^{-1}.$$

- The partition functions are obtained by taking the index

$$\mathbb{I}[\mathbf{X}] = \prod_{x\in\mathbf{X}} (1 - x^{-1})$$

Appendix: Quantum toroidal $\mathfrak{gl}_1$

- Coproduct:

$$\Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(\hat{\gamma}_{(1)}^{1/2} z) \otimes x^+(\hat{\gamma}_{(1)} z),$$

$$\Delta(x^-(z)) = x^-(\hat{\gamma}_{(2)} z) \otimes \psi^+(\hat{\gamma}_{(2)}^{1/2} z) + 1 \otimes x^-(z),$$

$$\Delta(\psi^+(z)) = \psi^+(\hat{\gamma}_{(2)}^{1/2} z) \otimes \psi^+(\hat{\gamma}_{(1)}^{-1/2} z),$$

$$\Delta(\psi^-(z)) = \psi^-(\hat{\gamma}_{(2)}^{-1/2} z) \otimes \psi^-(\hat{\gamma}_{(1)}^{1/2} z),$$

$$\Delta(\hat{\gamma}) = \hat{\gamma} \otimes \hat{\gamma},$$

where $\hat{\gamma}_{(1)} = \hat{\gamma} \otimes 1$ and $\hat{\gamma}_{(2)} = 1 \otimes \hat{\gamma}$.

- Miki automorphism $\mathcal{S}$:

$$\mathcal{S}^2 \cdot x^\pm(z) = -x^\mp(z^{-1}), \quad \mathcal{S}^2 \cdot \psi^\pm(z) = \psi^\mp(z^{-1}), \quad \mathcal{S}^2 \cdot (c, \bar{c}) = (-c, -\bar{c})$$