# Symmetries in QFTs and applications

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# Solving Quantum Field Theories

UV ← microscopic description
- Quantum Chromodynamics
- (Lattice) quantum many-body systems
- Conformal field theories ($CFT_{UV}$)

Renormalization Group (RG) flow

IR ← macroscopic description
- Nonlinear $\sigma$-model (chiral Lagrangian)
- Fermi liquid, spin liquid (TQFT), ...
- $CFT_{IR}$.

This is a very tough problem!

# Power of Symmetry

Sometimes, we can know about low-energy dynamics using Symmetry without solving microscopic Hamiltonian.

e.g. In '60s, people didn't know about Quantum Chromodynamics (QCD).
fundamental theory of strong interaction.

current algebra (chiral effective Lagrangian)

$\Longrightarrow$ successful description of low-energy properties of strong interaction

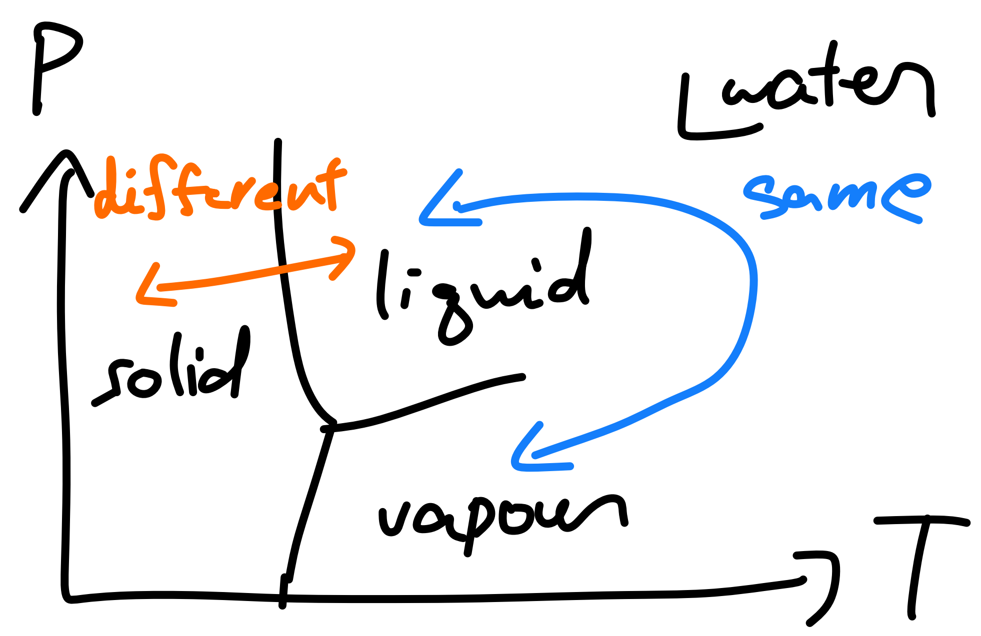
Why this was possible?

$\boxed{\text{Universality due to SSB of chiral symmetry.}}$

# (Some of) Important Theorems related to Symmetry

- **Landau's criterion of phases of matter**

  If symmetry breaking patterns are different,
  there have to be a phase transition separating those states.




- **Nambu-Goldstone theorem**

  If continuous symmetry is spontaneously broken,
  there are massless NG bosons.
  They have derivative couplings, so they interact weakly at low-energies.

- **'t Hooft anomaly matching**

  Quantum anomaly associated with gauging of global symmetry is RG invariant.

# Continuous Symmetry in QFT

Noether : If classical action $S[\phi]$ is invariant under continuous transformation, then there is a conserved current $J^\mu$ :

$$\partial_\mu J^\mu = 0 .$$

$$\Downarrow$$

In QFT, this becomes Ward-Takahashi identity :

$$\langle \partial_\mu J^\mu(x) \, O_1(x_1) \cdots O_n(x_n) \rangle = \sum_i \delta(x - x_i) \langle O_1(x_1) \cdots \delta O_i(x_i) \cdots O_n(x_n) \rangle .$$

$$\Downarrow$$

Various theorems related to symmetry.

Generalization of Symmetry

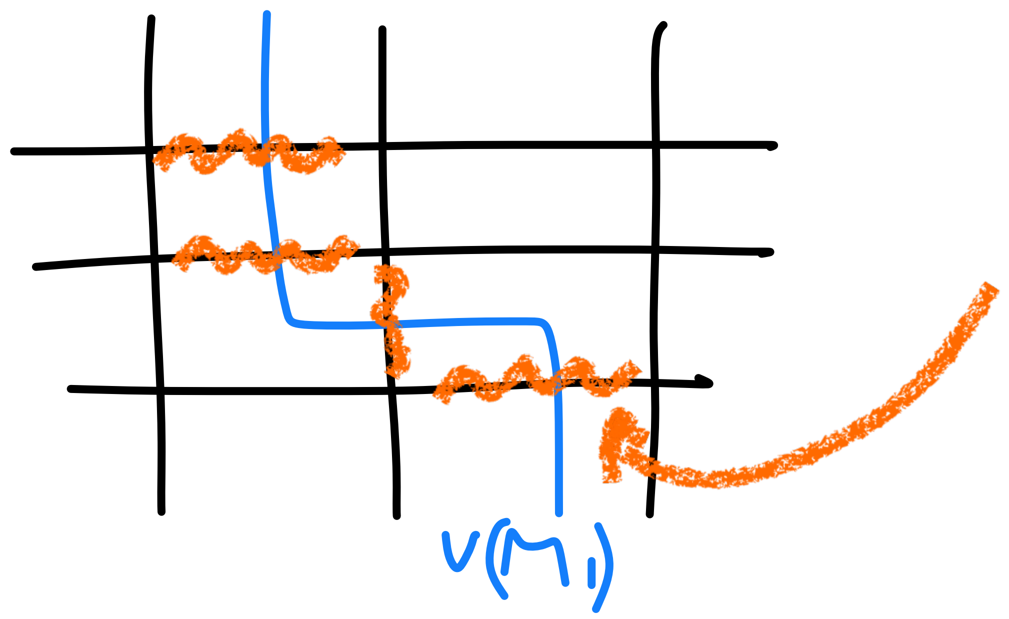
$\parallel$

Generalization of Ward-Takahashi identity

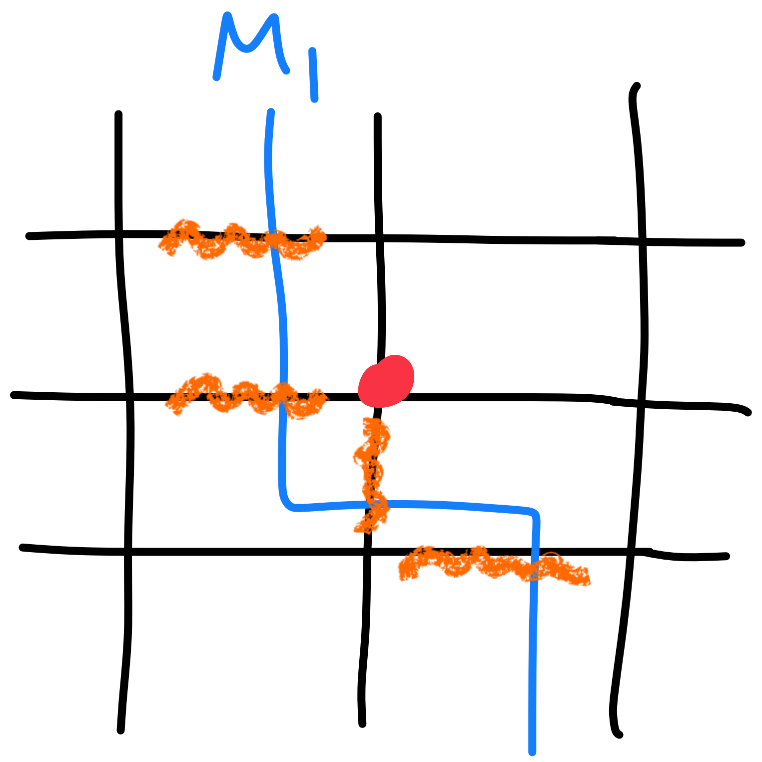
## Example WT-type identity for $\mathbb{Z}_2$ symmetry of Ising model

$$H_{\text{Ising}} = J \sum_{\langle x, x' \rangle : \text{nearest neighbors}} s(x)\, s(x') \qquad (s(x) = \pm 1)$$

$V(M_1)$ : defect operator



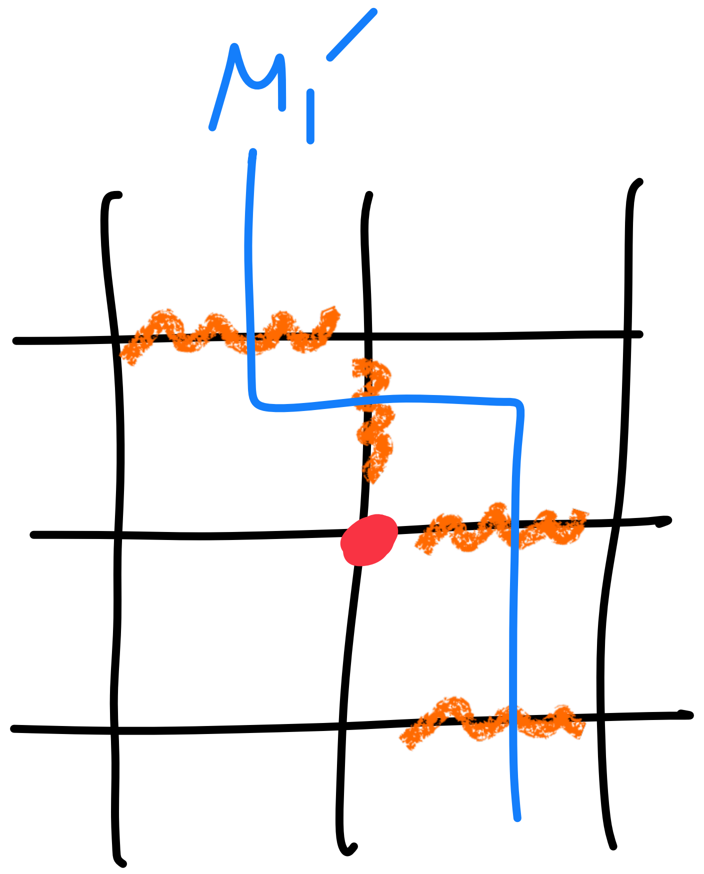

For these $\langle x, x' \rangle$, change $J \to -J$ in the Hamiltonian.




$\mathbb{Z}_2$ operation for $S(x)$ at ● :

$s \to -s$

$\Longrightarrow$




WT-type identity

$$\langle V(M_1) \cdots \rangle$$

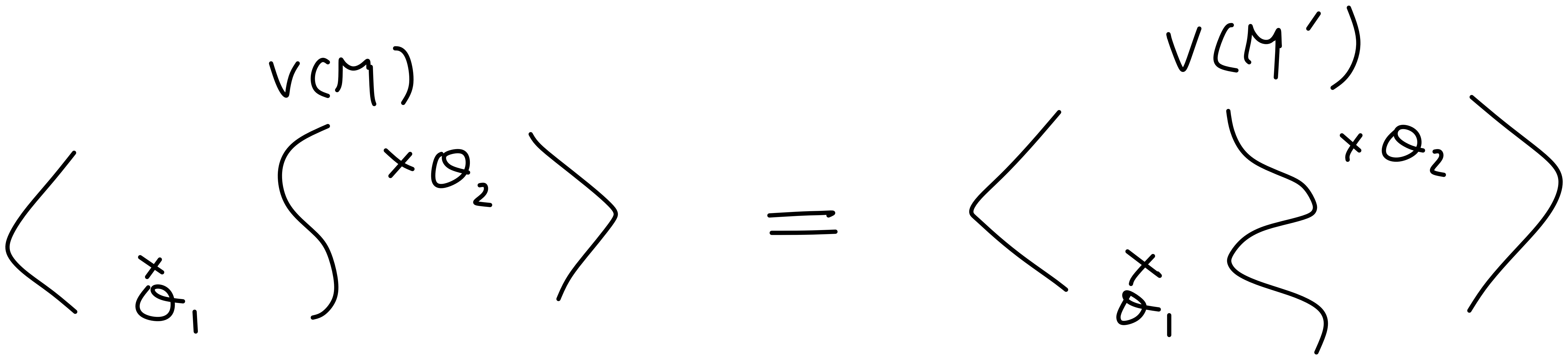
$$= \langle V(M_1') \cdots \rangle .$$

# Modern Definition of (Generalized) Symmetry

Take-home message

$$\boxed{\text{Symmetry} = \text{Topological defect operators}}$$

Topological




$\Longleftrightarrow$ Conservation Law

(Continuous sym : $V(M) = e^{i\alpha Q(M)} = e^{i\alpha \int_M *(J_\mu dx^\mu)}$.

$\mathbb{Z}_2$ of Ising : $V(M) = J$ in $H$ is replaced by $-J$ when $\langle x, x' \rangle$ crosses $M$.)

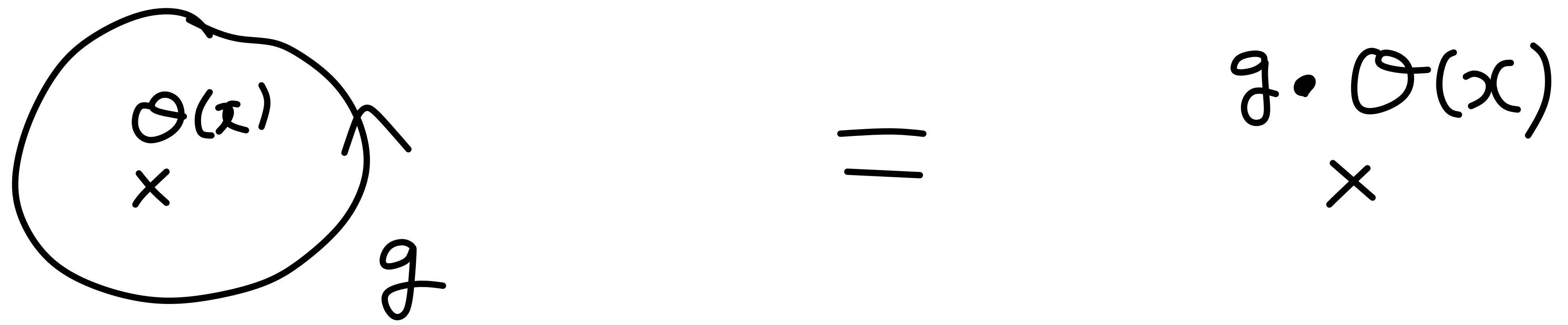
# Ordinary Symmetry in Modern Viewpoints

$d$-dim. QFT has a global symmetry $G$.

$\overset{\text{def}}{\Longleftrightarrow}$

- $\exists\, V_g(M_{d-1})$ : topological codim-1 defect operator for each $g \in G$

- 


- 


(Valid for both continuous and discrete symmetries)

# Various generalizations

- **p-form symmetry** (Gaiotto, Kapustin, Seiberg, Willet '14)

  Topological defects have codim-$(p+1)$: $V_g(M_{d-p-1})$.

  (* Esp., 1-form symmetry generalizes the center sym. in gauge theories.)

- **n-group symmetry** (Sharpe '15, Cordova, Dumitrescu, Intriligator '17, YT, Ünsal '19 ...)

  ≈ Mixture of 0-, 1-, ..., (n-1)-form symmetries.

- **non-invertible symmetry** (Bhardwaj, Tachikawa '17, .... in 2d QFTs.
  '21 ~ in ≥3d Nguyen, YT, Ünsal, Koide, Nagoya, Yamaguchi, Choi, Cordova, Hsin, Lam, Shao, Kaidi, Ohmori, Zheng, ....)

  Transformation rule does not form a group

## Application : Phase diagram of Fradkin-Shenker's model

# Fradkin - Shenker's (non-) complementarity.

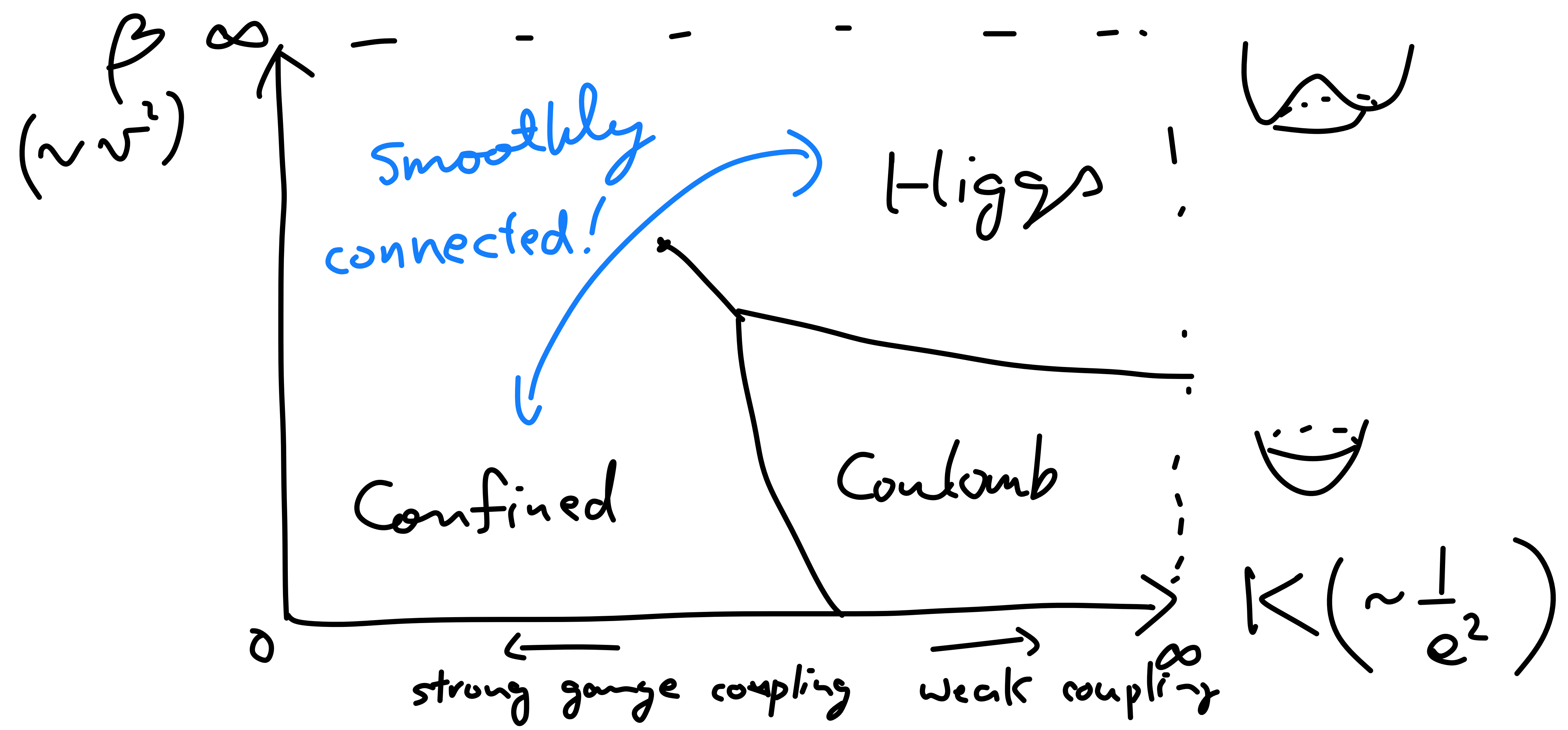
Consider the compact $U(1)$ gauge theory coupled to charge-$q$ scalar ($q=1,2,3,\cdots$)

$$S = \frac{1}{2e^2}\int da \wedge * da + \int \left\{ |(\partial_\mu + i q a_\mu)\phi|^2 + g(|\phi|^2 - v^2)^2 \right\}.$$

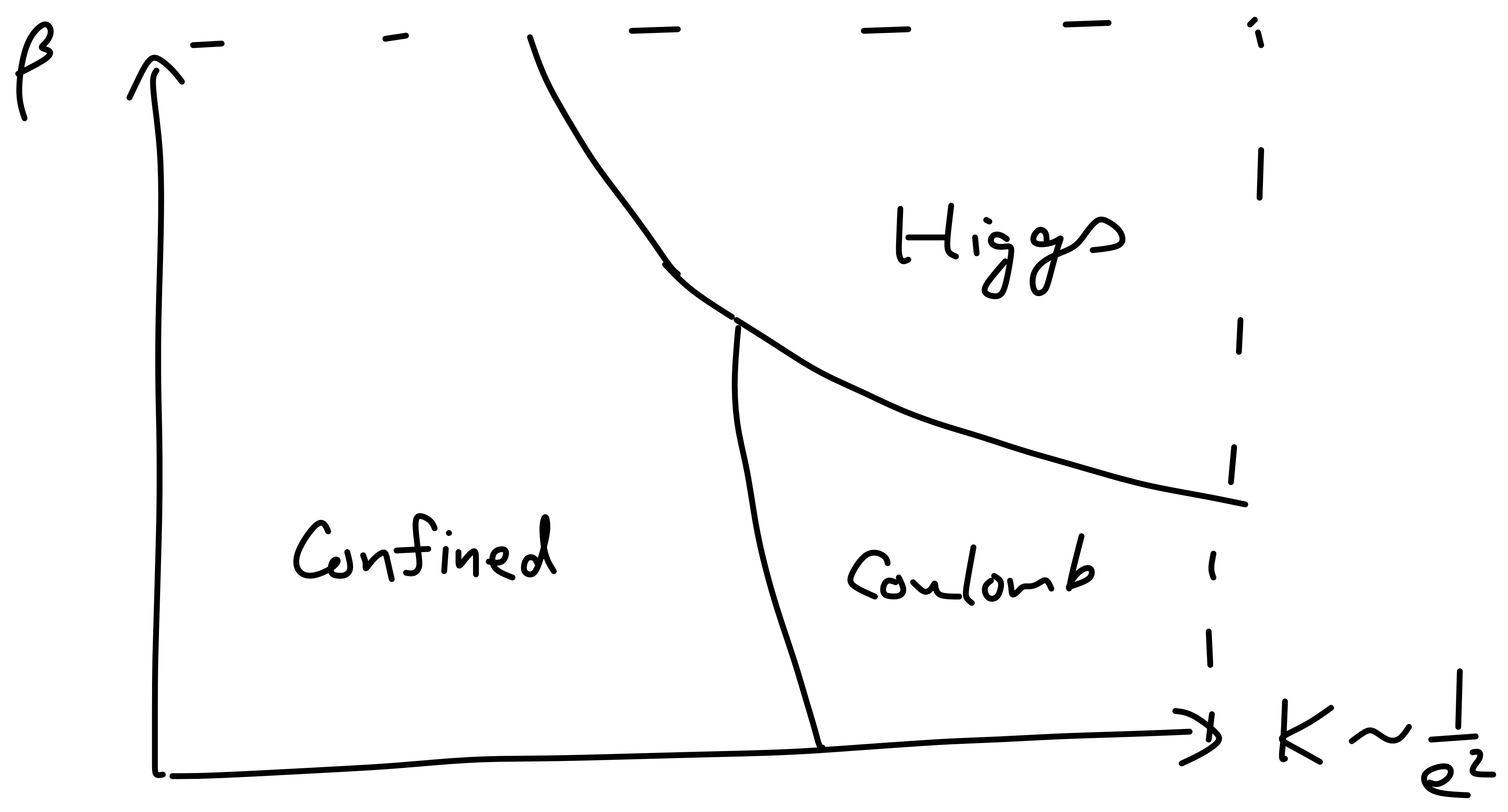
(In Fradkin, Shenker's paper ('79), the lattice version is considered, ($\phi \leftrightarrow e^{i\theta}$)

$$S = \beta \sum_i \cos(\partial_\mu \theta + q a_\mu) + K \sum_\square \cos(f_{\mu\nu}).$$
)

$q=1$ (Confined & Higgs phases are the same)




$q \geq 2$ (They are different)



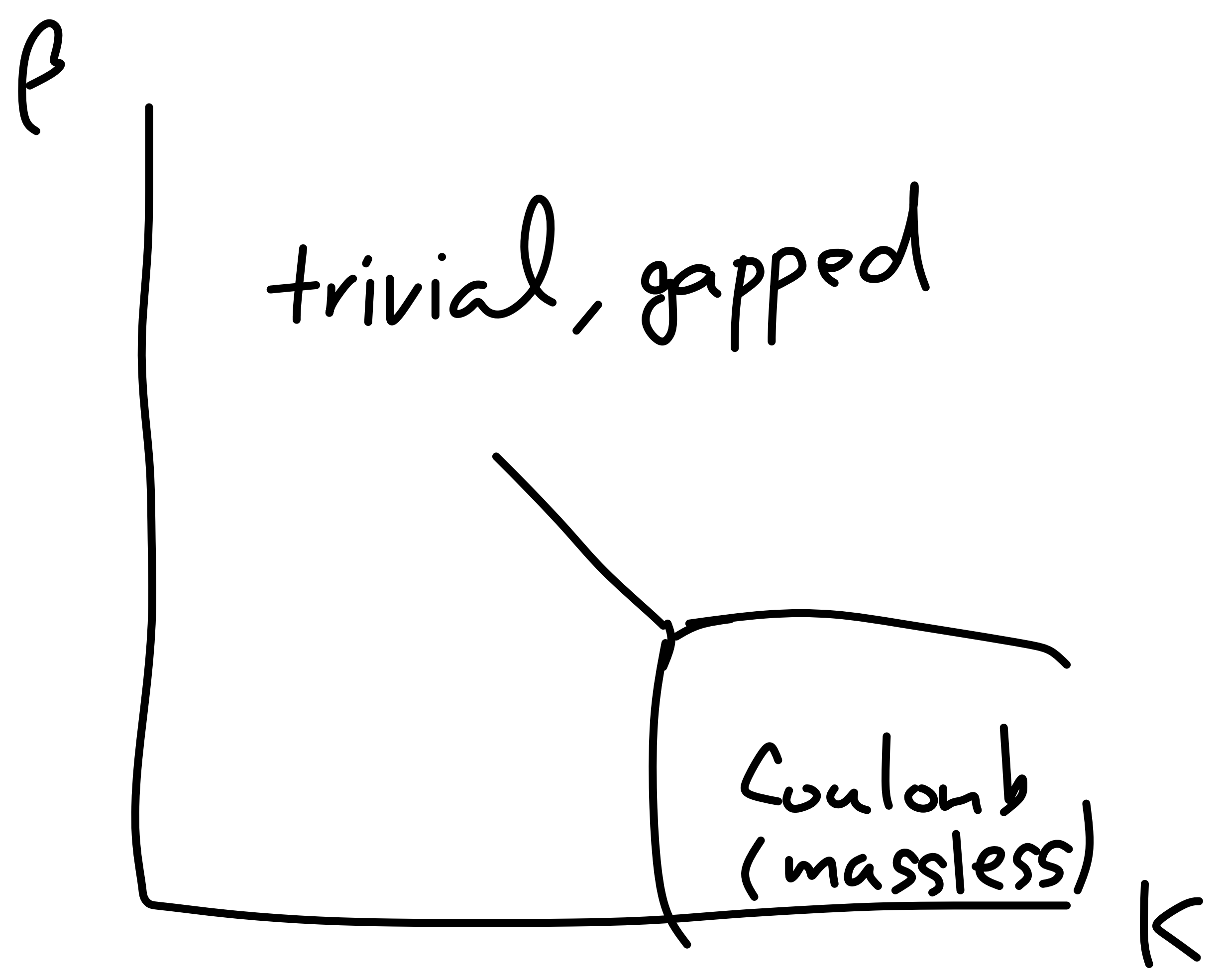
# Fradkin - Shenker **revisited** (Application of 1-form symmetry)

They consider charge-$q$ $U(1)$-Higgs model on a lattice

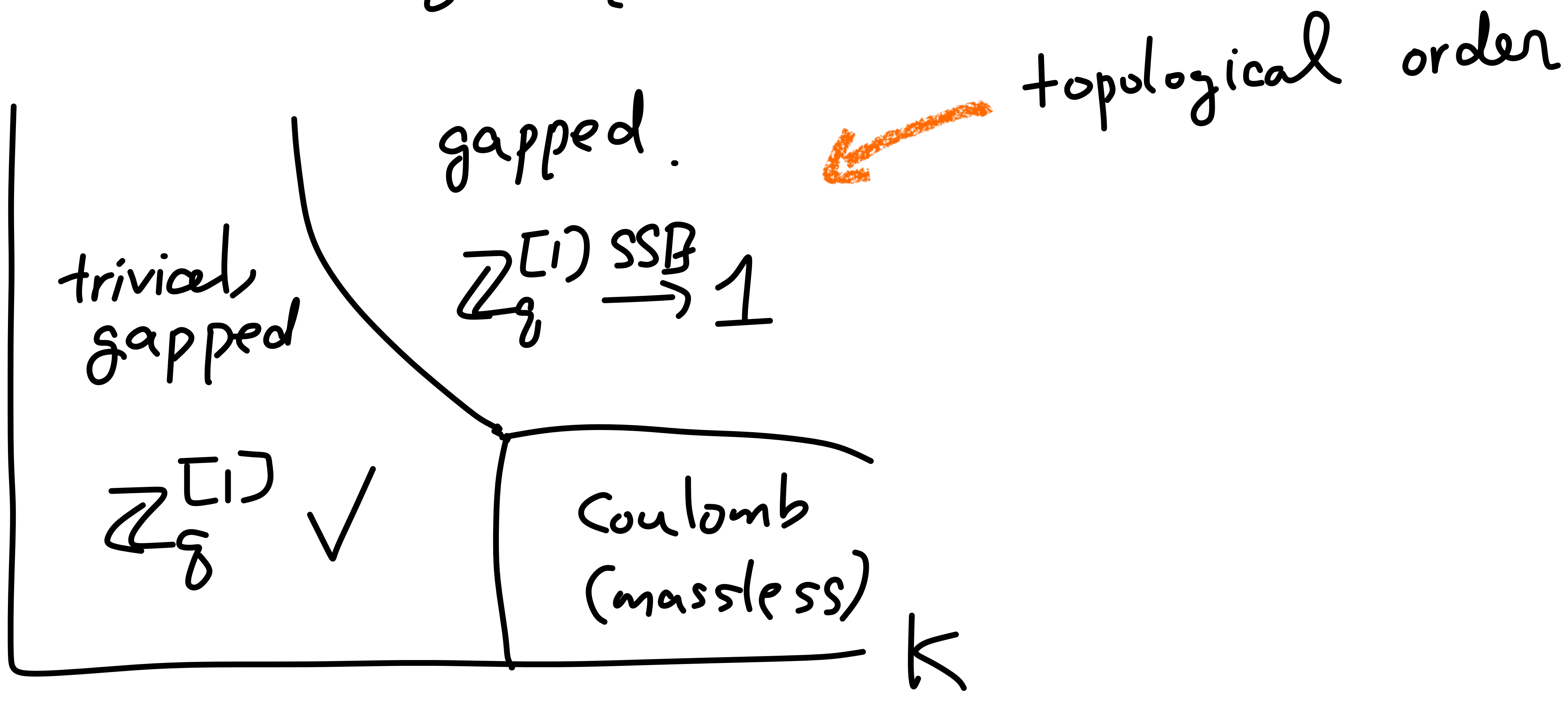
$$S = \beta \sum_{i,\mu} \cos(\partial_\mu \theta + q a_\mu) + K \sum_{\square} \cos(f_{\mu\nu})$$

$$\left( \longleftrightarrow \quad S = \frac{1}{2e^2} \int |da|^2 + \underbrace{\int \left\{ |(\partial_\mu + i q a_\mu)\phi|^2 + \quad \right\}}_{U(1)^{[1]}_E \xrightarrow{\text{explicit}} \mathbb{Z}_q^{[1]}} + \underbrace{\text{monopoles}}_{U(1)^{[1]}_M \xrightarrow{\text{explicit}} \times} \right).$$

$q = 1$ (No symmetry)




$q \geq 2$ ( $\mathbb{Z}_q^{[1]}$ symmetry)



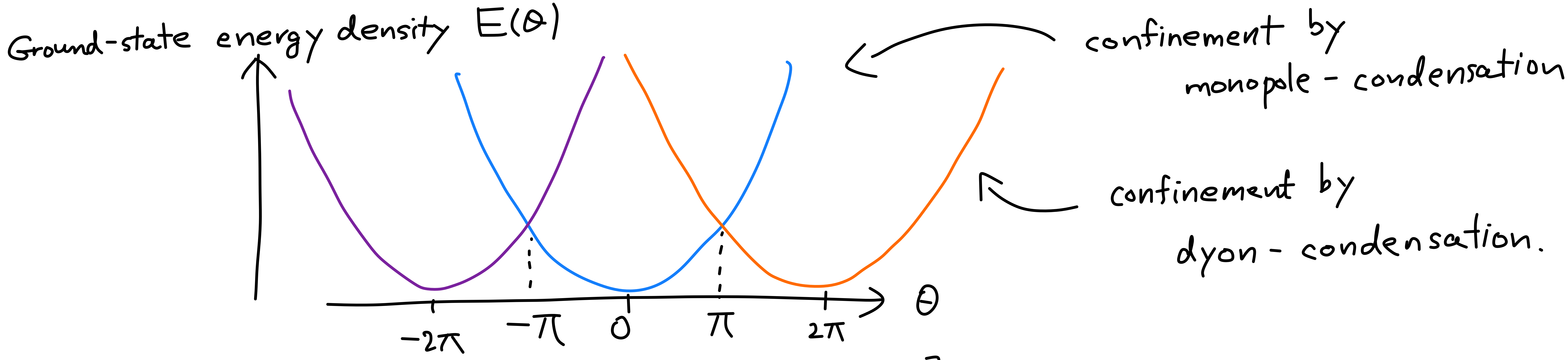
Application : $\theta$-dependence of 4d gauge theories

# YM theory at finite $\theta$.

4d gauge theory has two renormalizable terms:

$$\underbrace{\frac{1}{g^2}\int F_{\mu\nu}F_{\mu\nu}}_{\text{kinetic term}} + i\theta\underbrace{\int \varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}}_{\text{topological term (= instanton number)}}.$$

When confinement occurs at any $\theta$, the conjectured phase diagram is




**Q.** All confinement phases have unbroken $\mathbb{Z}_N^{[1]}$. Is the phase transition at $\theta=\pi$ accidental?

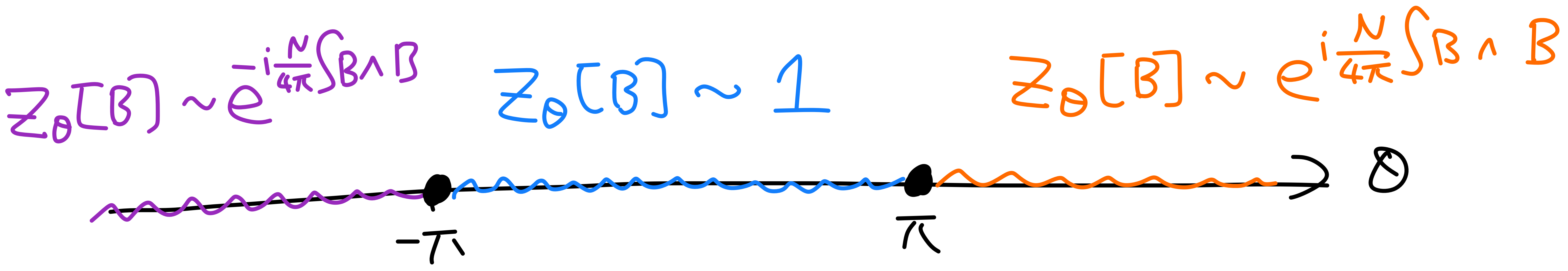
**A.** These confinement states are different as Symmetry-Protected Topological (SPT) states,

$\Rightarrow$ Phase transition is mandatory. (Gaiotto, Kapustin, Komargodski, Seiberg '17)

$B$ : $\mathbb{Z}_N$ 2-form gauge field (= Background gauge field for $\mathbb{Z}_N^{[1]}$)

$$Z_{\theta+2\pi}[B] = \underbrace{e^{i\frac{N}{4\pi}\int B\wedge B}} \times Z_\theta[B].$$

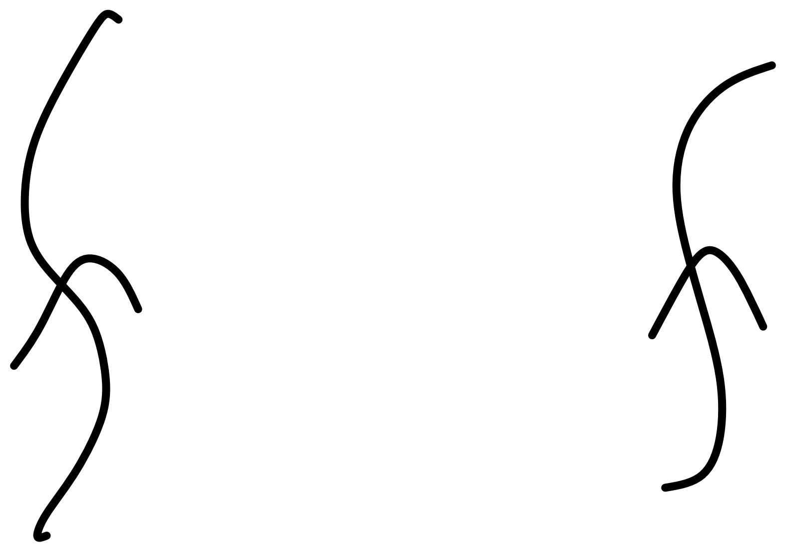
$\uparrow$ $2\pi$-periodicity of $\theta$ is violated by a local counterterm of $B$.



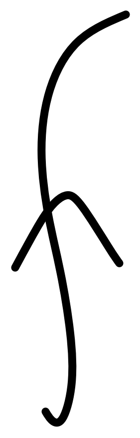
Application : Duality Symmetry as Noninvertible Symmetry

# Noninvertible Symmetry

Usually, symmetry forms a group :



$g_1 \quad g_2 \in G \quad \Longrightarrow \quad$

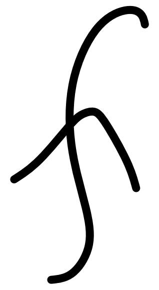


$g_1 \cdot g_2 \in G$

Is this necessary?

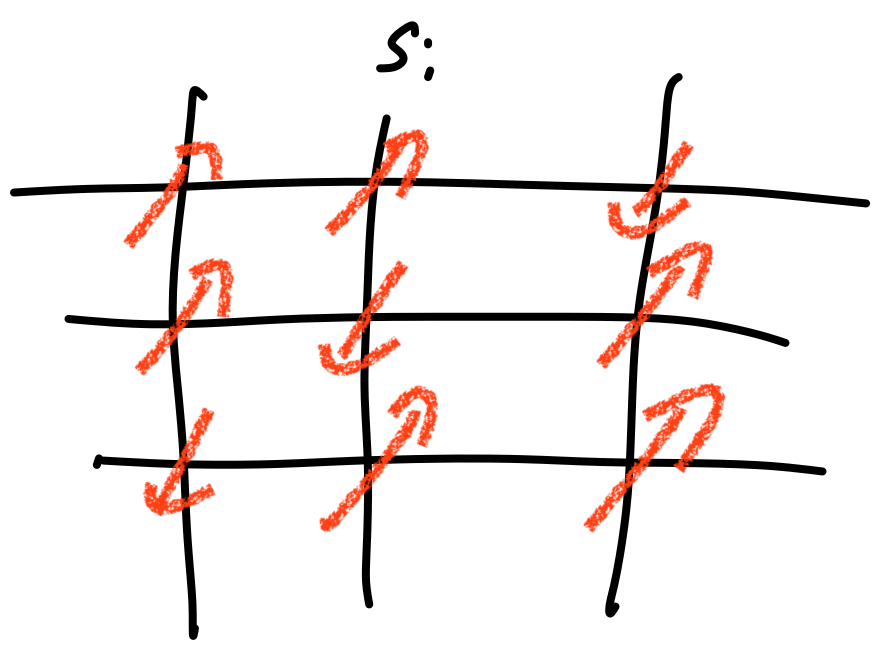
More general fusion rule $\Rightarrow$ "noninvertible" (or "categorical") symmetry

$a \quad b \quad \Longrightarrow \quad \sum_c N_{ab}^c$



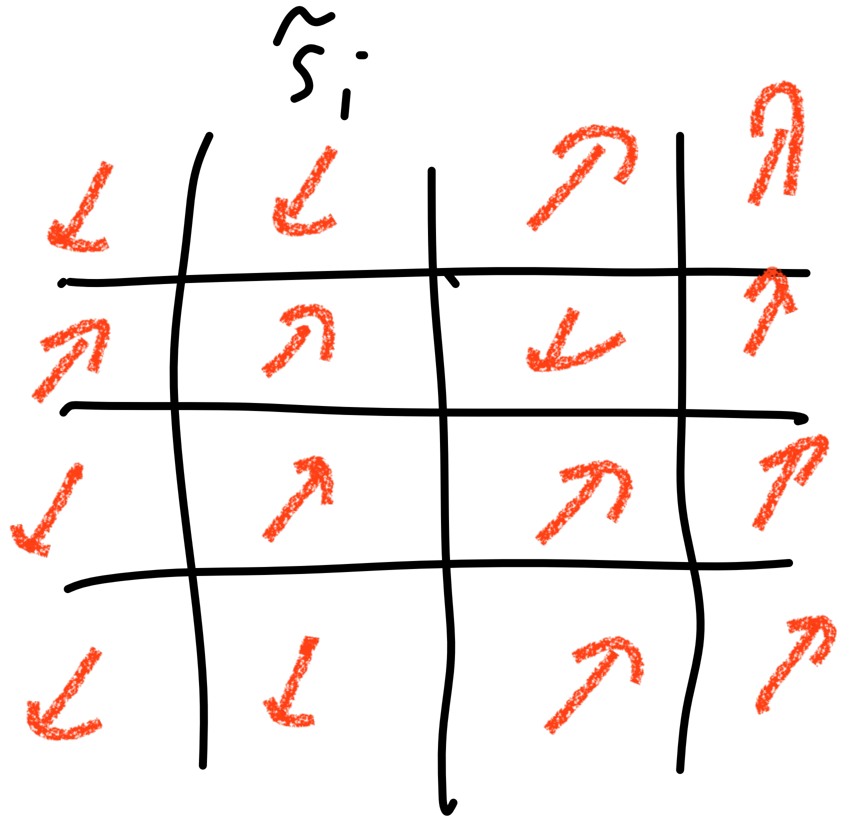
$c$

# Kramers - Wannier duality of 2d Ising model




Dual

$\longleftrightarrow$




$$e^{J \sum s_i s_j} \qquad\qquad e^{\tilde{J} \sum \tilde{s}_i \tilde{s}_j}$$

This exchanges the original spin & dual spin.

$$s(x) \longleftrightarrow \tilde{s}(x).$$

Not mutually local

what's this trans?

A. Non-invertible symmetry! (See the talks in the following slot!)

# Summary

Take-home message

$$\boxed{\text{Symmetry} = \text{Topological defect operators}}$$

$\Rightarrow$ New aspects of strongly-coupled QFTs.