Wess-Zumino-Witten

4次元WZW模型と古典解

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（名大多元数理）

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§1 Introduction

[Donaldson '85]
[Losev-Moore-Nekrasov-Shatashvili '96]
[Inami-Kanno-Ueno-Xiong '96]
...

4-dim WZW (WZW$_4$) model

- analogue of 2-dim WZW model
- EOM = Yang's eq ≡ Anti-Self-Dual Yang-Mills eq.
  (ASD)
- In the split signature $\underset{\sim\sim\sim\sim}{(-,-,+,+)}$, ← Today we focus on
  SFT action of $N=2$ string theory [Ooguri-Vafa '91]

↑ implication
application

We discuss classical soliton sols. of it

Original Motivation:

(NC) Ward's conjecture

We've made it!

(τ-fcn?)

4-dim (NC) ASDYM ↔ twistor theory

(so far, no Wronskian sol.)

(−,−,+,+) ↓ reduction

(NC) ↔ bkg. B-field

↓ reduction

NC Toda, NC KdV, NC NLS, … ↔ Sato's theory

(τ-fcn ≡ Wronskian sol.)

[Ward '85, Mason-Woodhouse], …

(NC) [Hamanaka '06, …]

Reduction to KdV from ASDYM $(G=SL(2,\mathbb{C}))$

ASDYM : $F_{zw}=0,\ F_{\tilde{z}\tilde{w}}=0,\ F_{z\tilde{z}}-F_{w\tilde{w}}=0$

① $\partial_w-\partial_{\tilde{w}}=0,\ \partial_{\tilde{z}}=0$ (dim. reduction)

② $A_{\tilde{w}}=\begin{pmatrix}0&0\\ \frac{u}{2}&0\end{pmatrix},\ A_{\tilde{z}}=\begin{pmatrix}0&0\\ 1&0\end{pmatrix},\ A_w=\begin{pmatrix}0&-1\\ u&0\end{pmatrix}$

$$A_z=\frac{1}{4}\begin{pmatrix}u'&-2u\\ u''+2u^2&-u'\end{pmatrix}$$

$u=u(z,\underset{\sim}{x})$ $\underset{\sim}{w+\tilde{w}}$

$u'=\partial_x u$

$$\underset{\substack{\Uparrow\\ t}}{u_z}-u_{xxx}-\frac{3}{2}uu_x=0$$: KdV eq.!

cf. P14

(t,x) are real ⇒ not $(++++)$ but $(++--)$

Plan of Talk (simple discussion)

§1 Introduction (5 min)

§2 Soliton Solutions of KdV (skip)

§3 Soliton Solutions of KP (skip)

§4 Soliton Solutions of Yang's eq (5 min)

§5 4dim WZW model (10 min)

§6 Conclusion & Discussion (5 min)

§ 2 Soliton Solutions of KdV eq.

Korteweg-de Vries (KdV) eq.

$$u_t + u_{xxx} + 6uu_x = 0$$

- describes shallow water wave
- first observed by Scott Russel (1834)

union canal
in Edinburgh (2005)

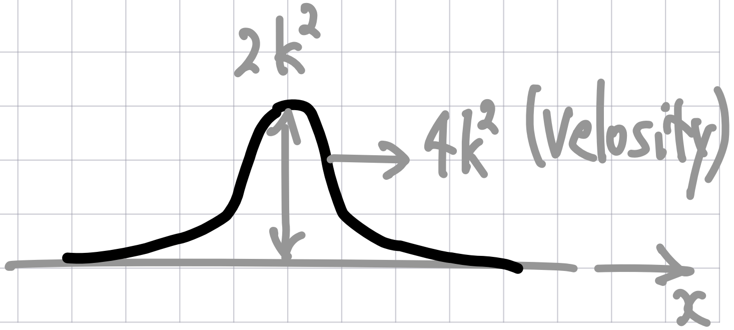
1-soliton sol.

$\swarrow$ $\operatorname{sech} x := \dfrac{1}{\cosh x}$

$$u(t,x) = 2k^2 \operatorname{sech}^2(kx - 4k^3 t)$$

$\uparrow$

height of water surface



Lax representation

spectral parameter

$$\begin{cases} Lf = 0 \\ Mf = 0 \end{cases} \cdots ① \quad \text{w/} \quad \begin{cases} L := \partial_x^2 + u - \zeta \\ M := \partial_t + 4\partial_x^3 + 6u\,\partial_x + 3u_x \end{cases}$$

- Compatible condition of ① $[L, M] = 0 \Leftrightarrow$ KdV eq.

Darboux transformation

$\zeta = \lambda$ (fix)

$\theta = f(\lambda)$

Prepare a special sol. of ① : $\begin{cases} L\theta = 0 \\ M\theta = 0 \end{cases}$

The following trf. leaves ① as it is:

$$(D) \quad \begin{cases} L \mapsto \tilde{L} = G_\theta L G_\theta^{-1} \\ M \mapsto \tilde{M} = G_\theta M G_\theta^{-1} \\ f \mapsto \tilde{f} = G_\theta f \end{cases}$$

$$G_\theta := \theta\,\partial_x\,\theta^{-1} \cdots ②$$
$$= \partial_x - \theta_x\,\theta^{-1}$$

The Darboux trf (D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$

n-iterations of (D) from a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\theta_k = 0 \end{cases}$$

$$k \in \{1, \cdots, n\}$$

$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{\underline{(n-1)}} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

derivative

Quasideterminant (see. Appendix)

squares

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := d - CA^{-1}B \xrightarrow[d:1\times 1]{\text{commutative limit}} \frac{\begin{vmatrix} A & B \\ C & d \end{vmatrix}}{|A|}$$

(Schur complement)

The Darboux trf (D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$

n-iterations of (D) from a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\theta_k = 0 \end{cases}$$

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$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$= 2\partial_x \frac{\partial_x Wr(\theta_1, \cdots, \theta_n)}{\underbrace{Wr(\theta_1, \cdots, \theta_n)}_{\text{det}\,\square}} = 2\partial_x^2 \log \overbrace{Wr(\theta_1, \cdots, \theta_n)}^{\tau\text{-fcn}}$$

Hirota trf.

1-soliton sol.

$$u_1 = 2\partial_x^2 \log \theta_1$$

$$= 2k^2 \mathrm{sech}^2(kx - 4k^3t)$$

(previous one!)

$(k = \sqrt{\lambda_1})$

a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\theta_k = 0 \end{cases}$$

$$k \in \{1, \cdots, n\}$$

$$\theta_k = e^{X_k} + e^{-X_k}$$

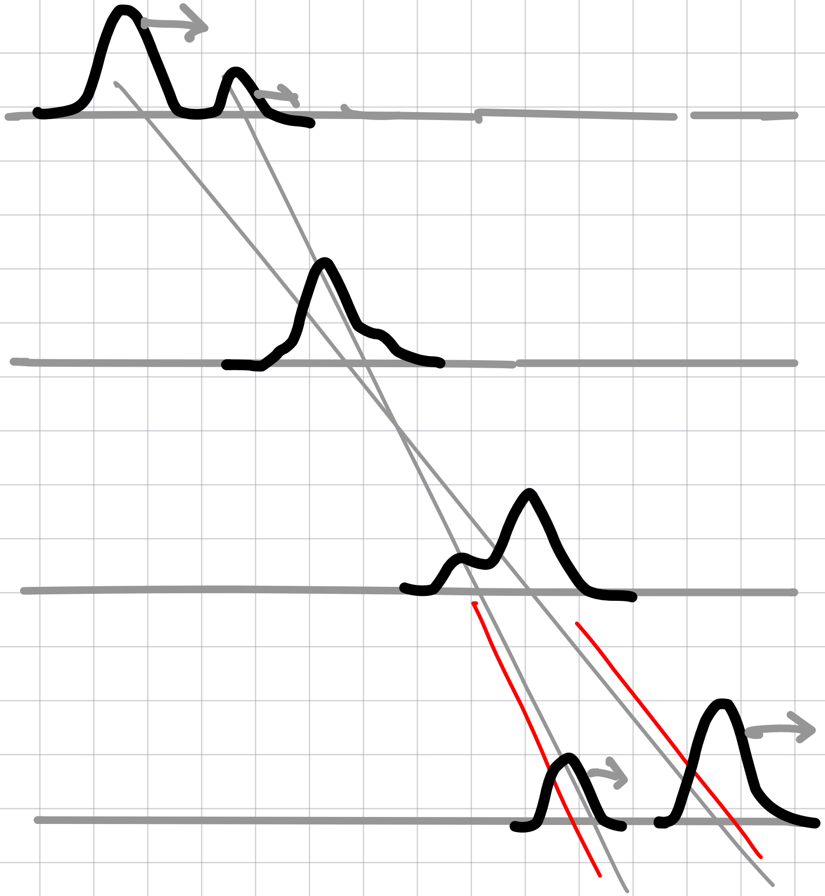
choice ↗
for solitons

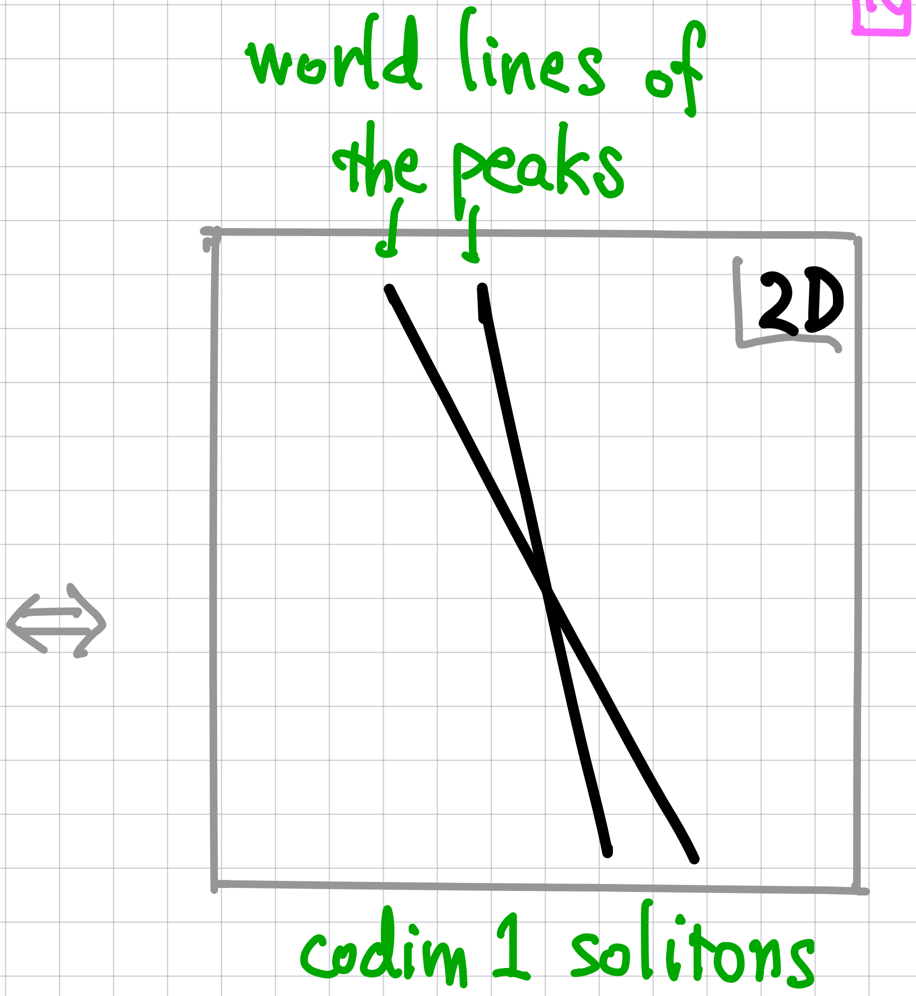
$$X_k := \lambda_k^{\frac{1}{2}}(x - \lambda_k^2 t)$$

2-soliton sol.

$$u_2 = 2\partial_x^2 \log \begin{vmatrix} \theta_1 & \theta_2 \\ \theta_1' & \theta_2' \end{vmatrix}$$

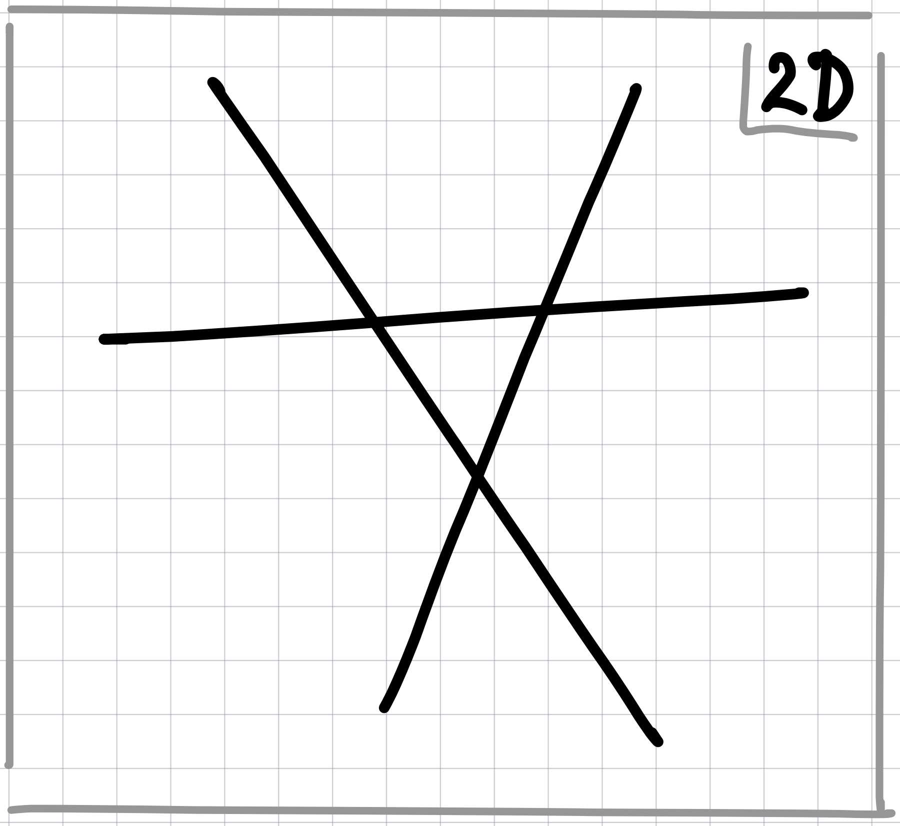
2-soliton scattering





- shapes and velocities preserved
- positions are a little bit shifted (phase shift)

n- soliton sol. = "non-linear superposition" of n one-solitons



intersecting n- lines

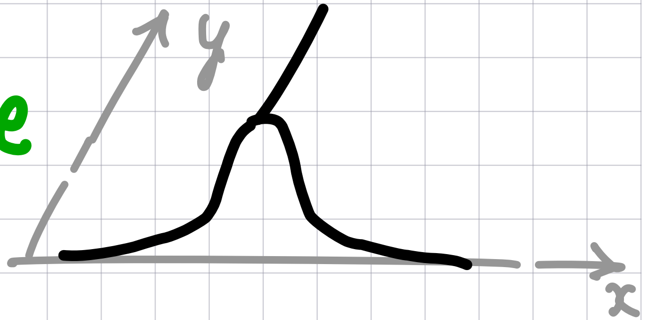
(with phase shifts)

§3 Soliton Solutions of KP eq.

Kadomtsev-Petviashvili (KP) eq.

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

- describes shallow water wave in (2+1)-dims.



$$\Big\downarrow \quad u = 2\partial_x^2 \log \tau$$

$$(\tau_{xxxx} - 4\tau_{tx} + 3\tau_{yy})\tau - 4(\tau_{xxx} - \tau_t)\tau_x + 3(\tau_{xx}^2 - \tau_y^2) = 0$$

Hirota's bilinear eqs. $\underset{\uparrow}{\equiv}$ **Plücker ids** $\rightsquigarrow$ **$Gr(\frac{\infty}{2}, \infty)$**

Sato

n- soliton sol. = "non-linear superposition" of n one solitons

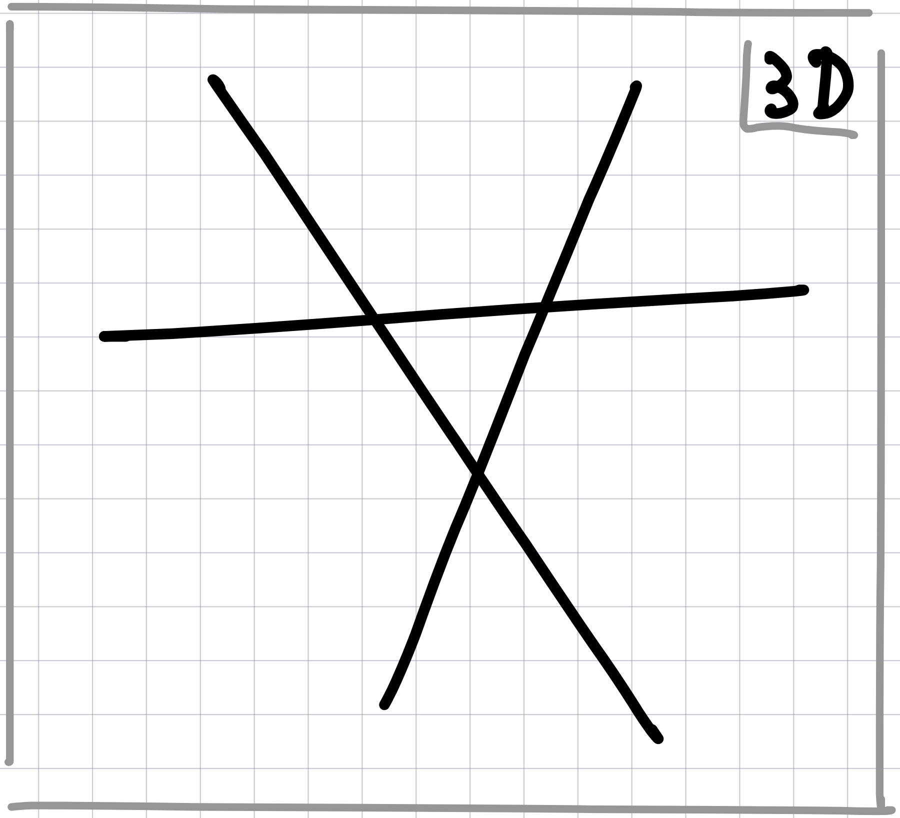
$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k = e^{X_k} + e^{-X_k}$$

$$X_k = \alpha_k x + \alpha_k^2 y - 4\alpha_k^3 t$$

$$k \in \{1, \cdots, n\}$$

e.g. [Matveev - Salla]



intersecting n planes

(with phase shifts)

§4 Soliton Solutions of Yang's eq

Yang's eq. (on $\mathbb{C}^4$: complexified space-time)

$$\partial_{\tilde{z}}\left((\partial_z \sigma)\sigma^{-1}\right) - \partial_{\tilde{w}}\left((\partial_w \underbrace{\sigma}_{\in\, G = GL(N,\mathbb{C})})\sigma^{-1}\right) = 0$$

※ Real slice

$(z, w, \tilde{z}, \tilde{w}) \in \mathbb{C}^4$, $\quad ds^2 = dz\,d\tilde{z} - dw\,d\tilde{w}$

① $\downarrow$ $z = x^1 + x^3,\ w = x^2 + x^4$

$\tilde{z} = x^1 - x^3,\ \tilde{w} = x^4 - x^2$

$\mathbb{R}^4(+,+,-,-)$

Ultrahyperbolic sp. $\mathbb{U}$

② $\downarrow$ $z = x^1 + i x^2,\ w = x^3 + i x^4$

$\tilde{z} = \bar{z},\ \tilde{w} = -\bar{w}$

$\mathbb{R}^4(++++)$

Euclid sp. $\mathbb{E}$

Lax representation :

N×N const matrix

$$(*)\begin{cases} Lf = \sigma\,\partial_w(\sigma^{-1}f) - (\partial_{\tilde z} f)\underset{\sim}{\zeta} = 0 & \begin{pmatrix}\text{right}\\ \text{action}\end{pmatrix} \\ Mf = \sigma\,\partial_z(\sigma^{-1}f) - (\partial_{\tilde w} f)\underset{\sim}{\zeta} = 0 \end{cases}$$

compatible condition ⇒ Yang's eq.

$$L(M\phi) - M(L\phi) \doteq 0$$

Darboux trf.

[Nimmo-Gilson-Ohta '00] [Gilson-H-Huang-Nimmo '20]

$$(D)\begin{cases} \tilde f = f\zeta - \theta\Lambda\theta^{-1}f \\ \tilde\sigma = -\theta\Lambda\theta^{-1}\sigma \end{cases}$$

$\underset{N\times N}{\theta}$: special sol. for $\underset{\sim}{\Lambda}$

special value

Under the Darboux trf. (*) is form invariant $\left(\text{i.e. } \begin{matrix}\tilde L\tilde f = 0 \\ \tilde M\tilde f = 0\end{matrix}\right.$

n-iterations of (D) from a trivial seed sol.

$(\sigma = 1)$

$$\underset{N\times N}{\sigma_n} = \begin{vmatrix} \overset{N\times N}{\theta_1} & \cdots & \theta_n & \overset{N\times N}{1} \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k^{(l)} := \theta_k \Lambda_k^l$$

$$(\theta_i, \Lambda_i) : \partial_w \theta_i = \partial_{\tilde{z}} \theta_i \Lambda_i$$

$$\partial_z \theta_i = \partial_{\tilde{w}} \theta_i \Lambda_i$$

Wronskian-type!

Quasideterminant (see. Appendix)

squares

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := \underset{N\times N}{d} - C A^{-1} B \quad \text{(Schur complement)}$$

n-soliton sols. for $G = SL(2,\mathbb{C})$:

[H-Huang '20]

$$\sigma_n = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k = \begin{pmatrix} e^{L_k} & e^{-\bar{L}_k} \\ -e^{-L_k} & e^{\bar{L}_k} \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

$$L_k = \lambda_k \alpha_k z + \beta_j \tilde{z} + \lambda_j \beta_j w + \alpha_j \tilde{w}$$

(linear in space-time coord)

Rmk (U) $\mu_k = \bar{\lambda}_k$, $|\mu_k| = 1$

$\Rightarrow G = SU(2)$

(E) $\mu_k = -1/\bar{\lambda}_k$, $|\mu_k| = 1$

$\Rightarrow G = U(2)$

Non-abelian system

&

Calculate the WZW action density of them

§5 4-dim WZW model

$\sigma(x) \in G$

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$$S_\sigma = \frac{i}{4\pi} \int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1} \wedge (\tilde{\partial}\sigma)\sigma^{-1} \right]$$

$$S_{WZ} = -\frac{i}{12\pi} \int_{M_4} A \wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1} \right]^3$$

$(z, w, \tilde{z}, \tilde{w})$: local coords of M_4

w/ $\omega = dA$: Kähler form of M_4

$$\omega = \frac{i}{2}(dz \wedge d\tilde{z} - dw \wedge d\tilde{w})$$

M_4 : flat 4-dim space-time

$$d = \partial + \tilde{\partial}, \quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \bar{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

EOM: $\tilde{\partial}(\omega \wedge (\partial\sigma)\sigma^{-1}) = 0 \Leftrightarrow$ Yang's eq.

$\mathcal{N}=2$ string theory

# WS SUSY	Name	Target sp.	field contents
$\mathcal{N}=0$	Bosonic String	$(1+25)$ dim	$g_{\mu\nu}, B_{\mu\nu}, \phi, \cdots$
$\mathcal{N}=1$	Superstring	$(1+9)$ dim	〃 〃
$\mathcal{N}=2$	$\mathcal{N}=2$ string	$(2+2)$ dim	massless scalar only!

open $\mathcal{N}=2$ string

$$\sigma = e^{\varphi} \longleftarrow \text{massless scalar}$$

[Ooguri-Vafa, '91]

$$S_{WZW_4} = (\text{in terms of } \varphi) \rightsquigarrow \text{n-pt. fcn of } \varphi$$

|||

$S_{\mathcal{N}=2 \text{ string}}$ (SFT)

(coincides with WS calculations)

One soliton (on Ⅲ)

※ $\lambda = \bar{\lambda} \Rightarrow \mathcal{L}_a \equiv 0$

$$\sigma = -\theta \Lambda \theta^{-1}, \quad \theta = \begin{pmatrix} e^{L} & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix}, \quad \Lambda = \underset{\sim\sim\sim}{\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}}$$

$$\Downarrow$$

$$\mathcal{L}_a = \frac{1}{8\pi} \underset{\sim}{d_{11}} \operatorname{sech}^2 X \quad \propto \underset{\sim\sim\sim}{(\lambda - \bar{\lambda})^3}$$

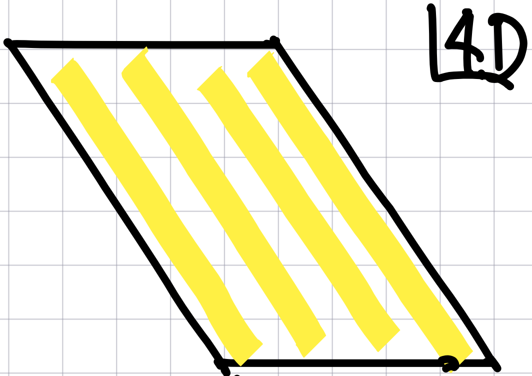
$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$X := L + \bar{L}$: linear in x^μ

$\mathcal{L}_{WZ} \equiv 0$ (identically)

peak →

$X = 0$



3-dim hyperplane
(codim 1)

not instanton!

Similar!

cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^{X} + e^{-X}\right) \propto \operatorname{sech}^2 \underset{\sim}{X}$$

linear in t, x, y

Two Soliton

$X_k = L_k + \bar{L}_k, \quad \Theta_{12} = \Theta_1 - \Theta_2$

$i\Theta_k = L_k - \bar{L}_k$

$$\mathcal{L}_a = \frac{\left[A\cosh^2 X_1 + B\cosh^2 X_2 + C_\pm \cosh^2\left(\frac{X_1+X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1-X_2 \pm i\Theta_{12}}{2}\right)\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right)^2}$$

non-singular

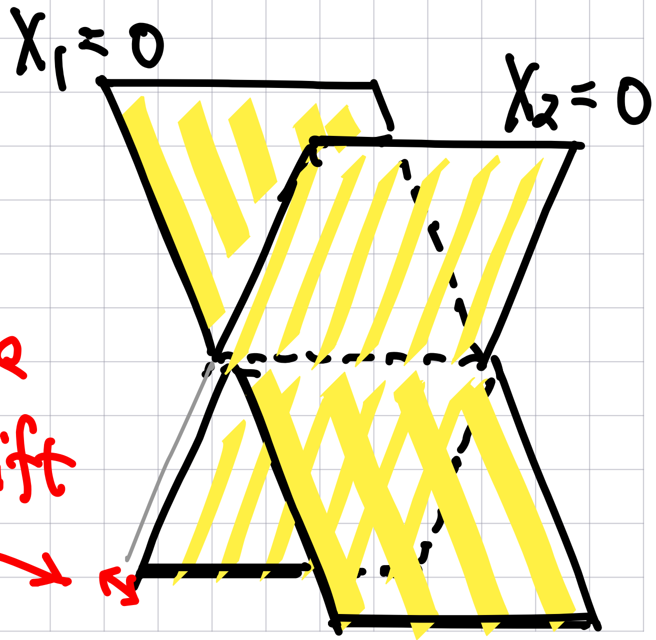
$$\xrightarrow[X_1 : \text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_1 \pm \underset{\sim}{\delta_1})$$

$$\xrightarrow[X_2 : \text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_2 \pm \underset{\sim}{\delta_2})$$

$$\xrightarrow[\text{otherwise}]{r\to\infty} \quad 0$$

phase shift

(non-linear effect)



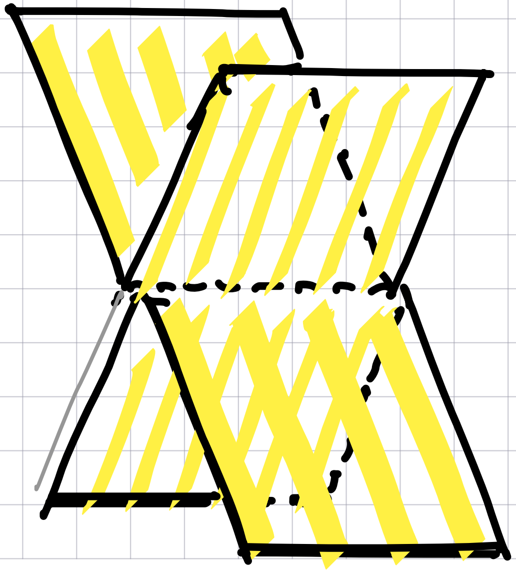
Two Soliton

$\mathcal{L}_{WZ}$ = (very long many terms) non-singular

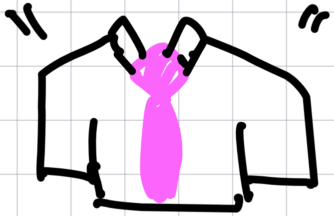
$\xrightarrow{r\to\infty}$ 0 (in any direction)

$\mathcal{L}_{total} = \mathcal{L}_a$ + ("dressing" in the middle region)

=



+



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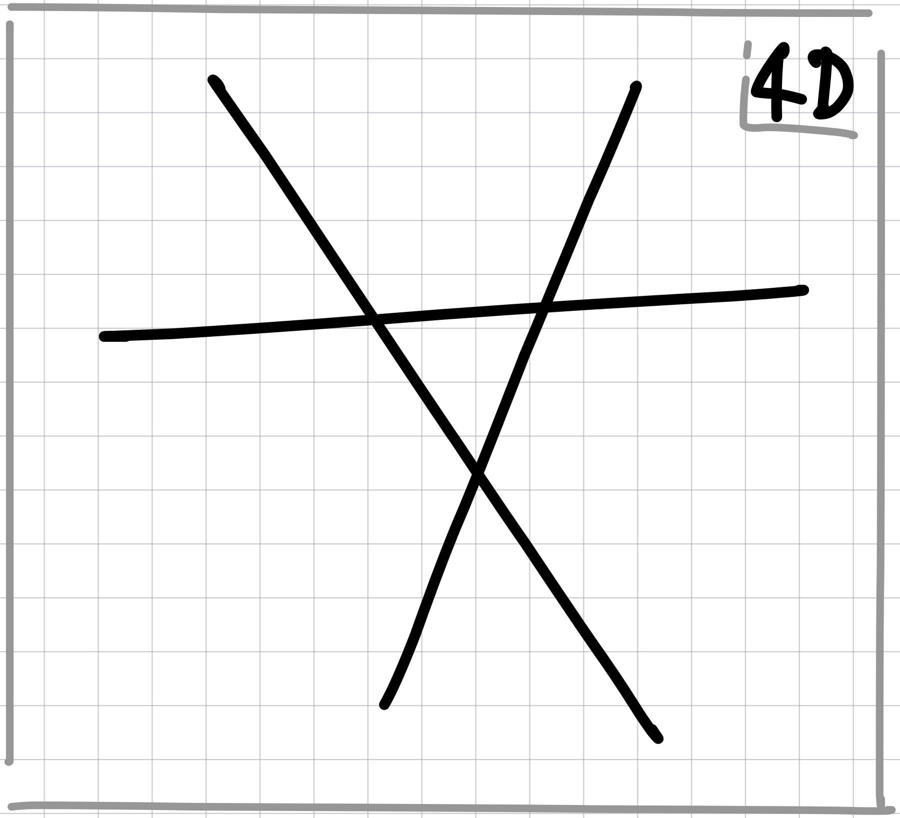
n- soliton sol. = "non-linear superposition" of n one solitons [H-Huang '22]



intersecting n hyperplanes (with phase shifts)

Rmk 1 Reduction to (1+2) dim.

Consider $(x^1, x^2, x^3, x^4) \longrightarrow (\overset{t\ (\text{time})}{\overset{\shortparallel}{x^1}}, x^3, x^4)$

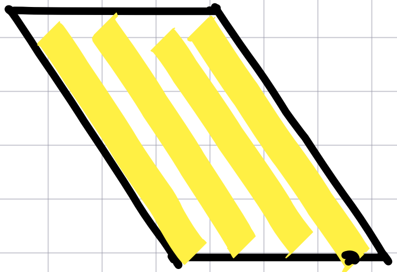
The soliton sol. $\sigma(\alpha_k = \lambda_k \beta_k)$ solves EoM in (1+2)d

$$\Updownarrow$$

∵ $L_k = (\lambda_k \alpha_k + \beta_k) x^1 + \underbrace{(\lambda_k \beta_k - \alpha_k)}_{0 =} x^2 + \cdots$ ▨

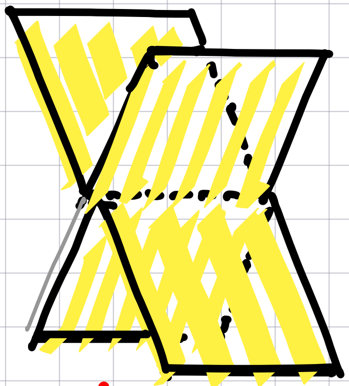
Hamiltonian $\mathcal{H} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} \partial_t \phi_i - \mathcal{L}$ $\quad (\mathcal{H}_{WZ} \equiv 0\,!)$

One soliton



Two soliton

(no dressing)



Energy density has the same peaks as action density.

Rmk 2 Euclidean case $\mathbb{E}$

The soliton sols. : almost the same as in $\mathbb{U}$

Instanton solution (well-known in YM)

(Ex) $G_{YM} = SU(2)$ 't Hooft 1-instanton

$$\mathcal{L}_a \propto \frac{(z\bar{z} + w\bar{w})^3}{\utilde{(z\bar{z} w\bar{w})}^2 (1 + z\bar{z} + w\bar{w})^2}$$

$$\mathcal{L}_{wz} \propto \frac{(z\bar{z} + w\bar{w})(z\bar{z} - w\bar{w})^2}{\utilde{(z\bar{z} w\bar{w})}^2 (1 + z\bar{z} + w\bar{w})^4}$$

singular

⤵ localized at the origin (codim 4)

§6 Conclusion and Discussion

We constructed new-type of codim 1 solitons and calculated action densities of WZW$_4$ model.

$\rightsquigarrow$ intersecting 3-branes in the $N=2$ string

(new branes)

Solitonic properties in the open $N=2$ string

charge, mass, moduli, description of φ, ...

Classification of the "soliton planes" cf. [Kodama-Williams '14]

Resonance solutions $\rightsquigarrow$ 3-brane reconnections

Anather Motivation : Unified theory of integrable systems

6d meromorphic Chern-Simons (CS)

↙ [Costello] [Bittleston-Skinner] ↘

4d CS ← duality? → **4d WZW**

↓ [Costello-Yamazaki (-Witten)]

various solvable models [Delduc-Lacroix-Magro-Vicedo], [Yoshida (K), Sakamoto, Fukushima, …] …

(spin chains, PCM, …)

[Ward] ↓ [Mason-Woodhouse]

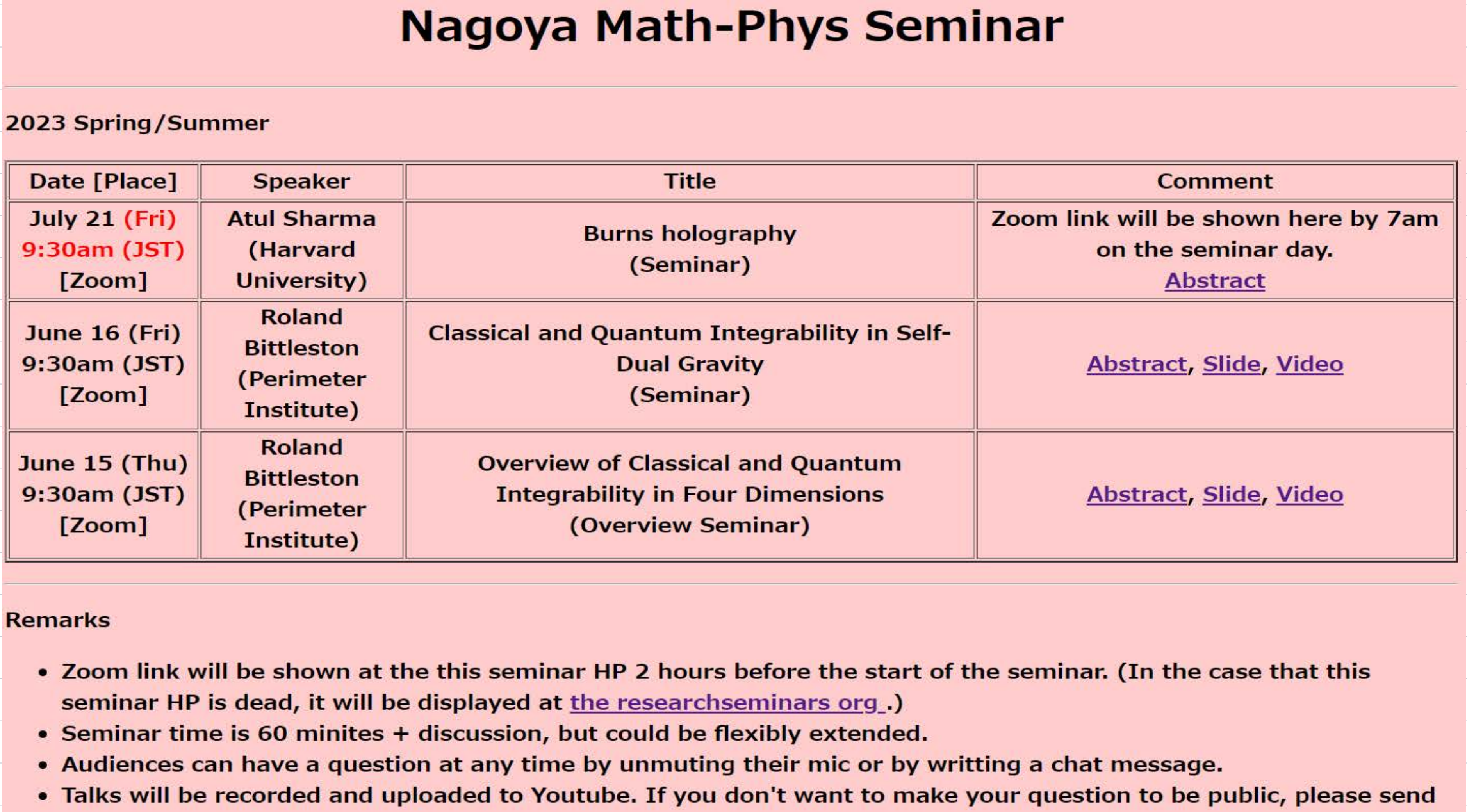
various integrable eqs.

(KdV NLS, Toda, …)

Nagoya Math-Phys Seminar Online has start!

(welcome to join!) worldwide!

announced in the researchseminars.org



Nagoya Math-Phys Seminar

2023 Spring/Summer

Date [Place]	Speaker	Title	Comment
July 21 (Fri) 9:30am (JST) [Zoom]	Atul Sharma (Harvard University)	Burns holography (Seminar)	Zoom link will be shown here by 7am on the seminar day. Abstract
June 16 (Fri) 9:30am (JST) [Zoom]	Roland Bittleston (Perimeter Institute)	Classical and Quantum Integrability in Self-Dual Gravity (Seminar)	Abstract, Slide, Video
June 15 (Thu) 9:30am (JST) [Zoom]	Roland Bittleston (Perimeter Institute)	Overview of Classical and Quantum Integrability in Four Dimensions (Overview Seminar)	Abstract, Slide, Video

Remarks

- Zoom link will be shown at the this seminar HP 2 hours before the start of the seminar. (In the case that this seminar HP is dead, it will be displayed at the researchseminars org .)
- Seminar time is 60 minites + discussion, but could be flexibly extended.
- Audiences can have a question at any time by unmuting their mic or by writting a chat message.
- Talks will be recorded and uploaded to Youtube. If you don't want to make your question to be public, please send