Foliated-Exotic Duality in Fractonic BF Theories

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Overview

- We consider a new type of continuum QFT called a **fracton system**, which has been getting attention in condensed matter physics.

  $\Rightarrow$ **A generalization of QFT**

- We show **a new type of duality** between fractonic QFTs.

Contents

Fracton

Fracton phase

A fracton phase is a new kind of phase of matter that has sub-dimensional excitations:

fracton : immobile; cannot move in space

lineon : move along a one-dimensional line

planon : move on a two-dimensional plane

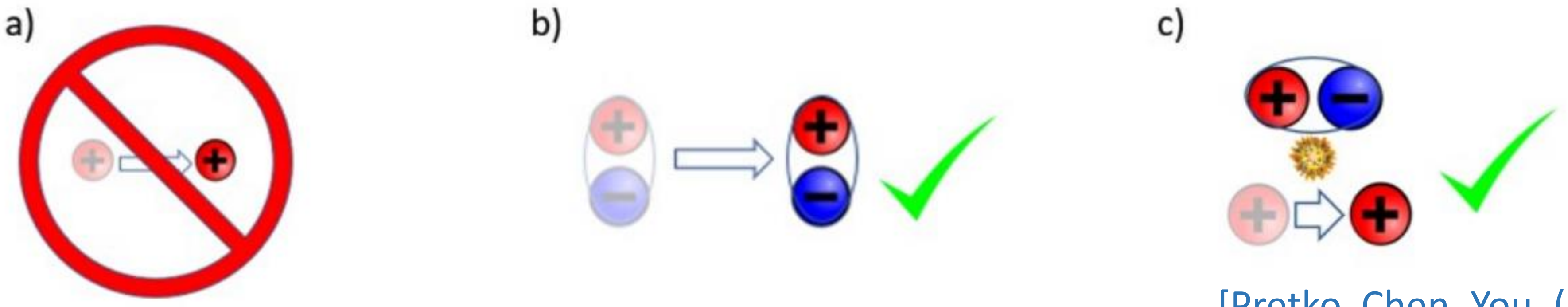
These excitations with restricted mobility are theoretically novel.

For example, on fractons:

a) A single fracton **cannot move** in any direction.

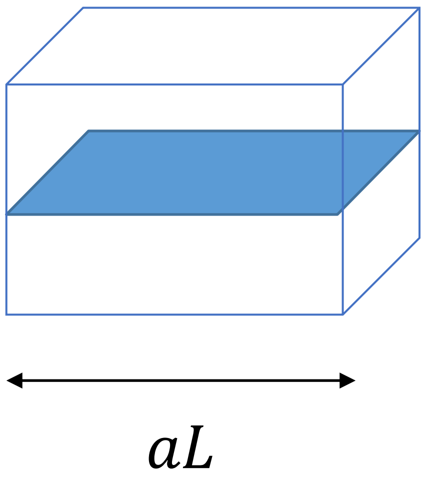
b) **A dipole of fractons** can move along a line or plane orthogonal to the direction of the dipole moment **like a lineon or planon**.

c) A single fracton can move at the cost of creating a new dipole.



[Pretko, Chen, You, (2020)]

Fracton phases are described by fracton systems (fracton models).



Fracton systems have the following new features:

- **Subsystem symmetry**: a global symmetry supported on a rigid submanifold like a point, line or plane.

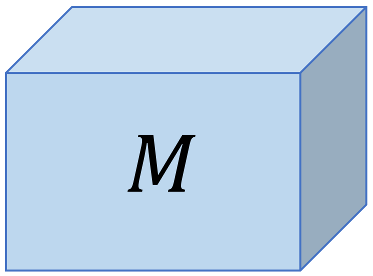
- **log (ground state degeneracy) ∝ (the size of the system)** $\log \mathrm{GSD} \sim L$

Theoretically interesting and expected to be applied to quantum information.

Subsystem symmetry

Ordinary global symmetry: the corresponding symmetry operator is supported on **the whole space *M***.

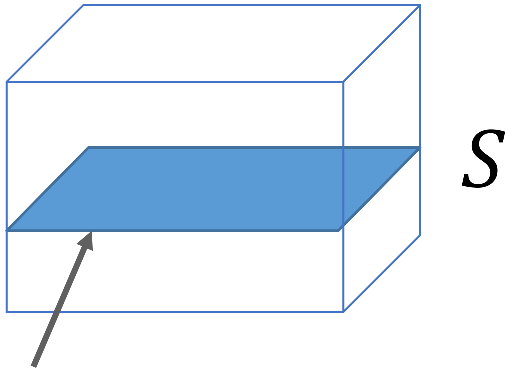
The operator is **topological** (M can be smoothly deformed in spacetime).



$$U(M) = \exp\left[i \int_M j^0 \right]$$

Subsystem symmetry: the corresponding symmetry operator is supported on **a rigid submanifold like a point, line or plane**.

The operator is **not completely topological**.

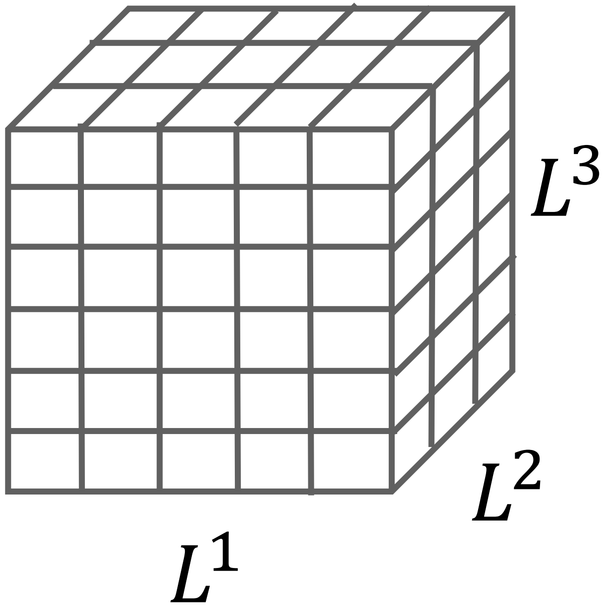


$$U_{\mathrm{sub}}(S)$$

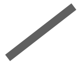
An example: The X-cube model

The X-cube model (The $\mathbb{Z}_2$ X-cube model) [Vijay, Haah, Fu (2016)] is a gapped fracton model in 3+1d, which has fracton and lineon excitations.

Now we consider the model on a simple cubic lattice with periodic boundary condition.



We assign a two-dim Hilbert space and the Pauli operators on each edge.



edge e

$$\mathcal{H}_e = \{ |0\rangle_e , |1\rangle_e \}$$

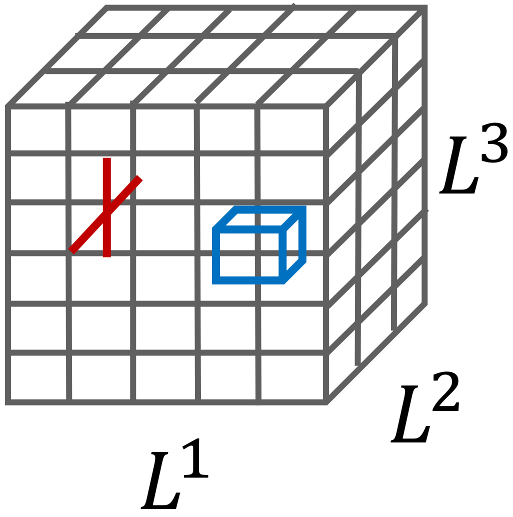
$$Z_e |0\rangle_e = |0\rangle_e , \ Z_e |1\rangle_e = - |1\rangle_e$$

$$X_e |0\rangle_e = |1\rangle_e , \ X_e |1\rangle_e = |0\rangle_e$$

$$X_e Z_e = - Z_e X_e$$

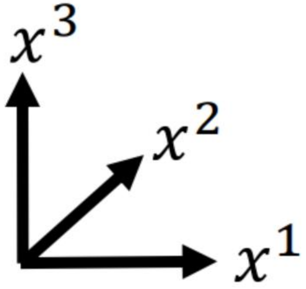
Then the total Hilbert space is $\mathcal{H} = \bigotimes_e \mathcal{H}_e$.

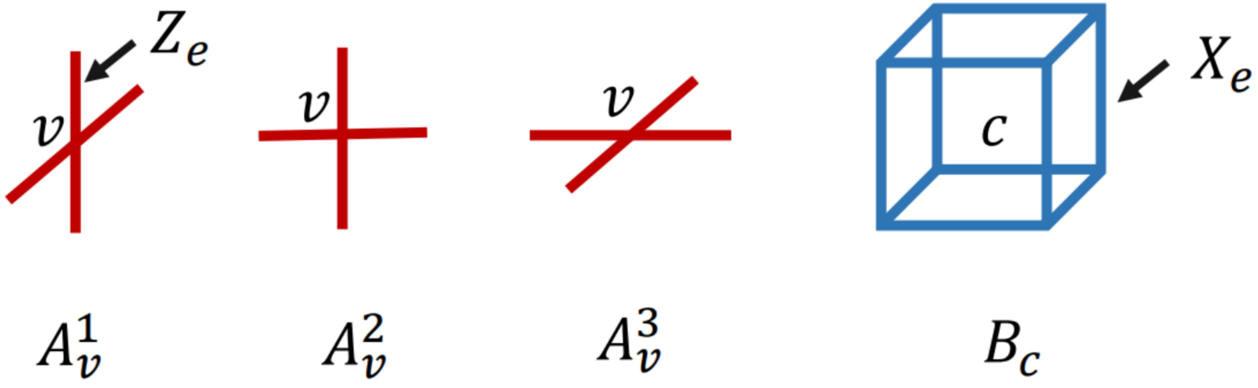
The number of edges is $3L^1L^2L^3$: $\dim \mathcal{H} = 2^{3L^1L^2L^3}$



The Hamiltonian of the X-cube model is

$$H = -\kappa^2 \sum_{v:\text{sites}} (A_v^1 + A_v^2 + A_v^3) - \gamma^2 \sum_{c:\text{cubes}} B_c$$





Ground state degeneracy

The total Hilbert space: $\dim \mathcal{H} = 2^{3L^1L^2L^3}$

The ground state conditions :
$$B_c \left|GS\right\rangle = \left|GS\right\rangle$$
$$A_v^k \left|GS\right\rangle = \left|GS\right\rangle$$

The ground state degeneracy $= \mathbf{2^{2L^1+2L^2+2L^3-3}}$

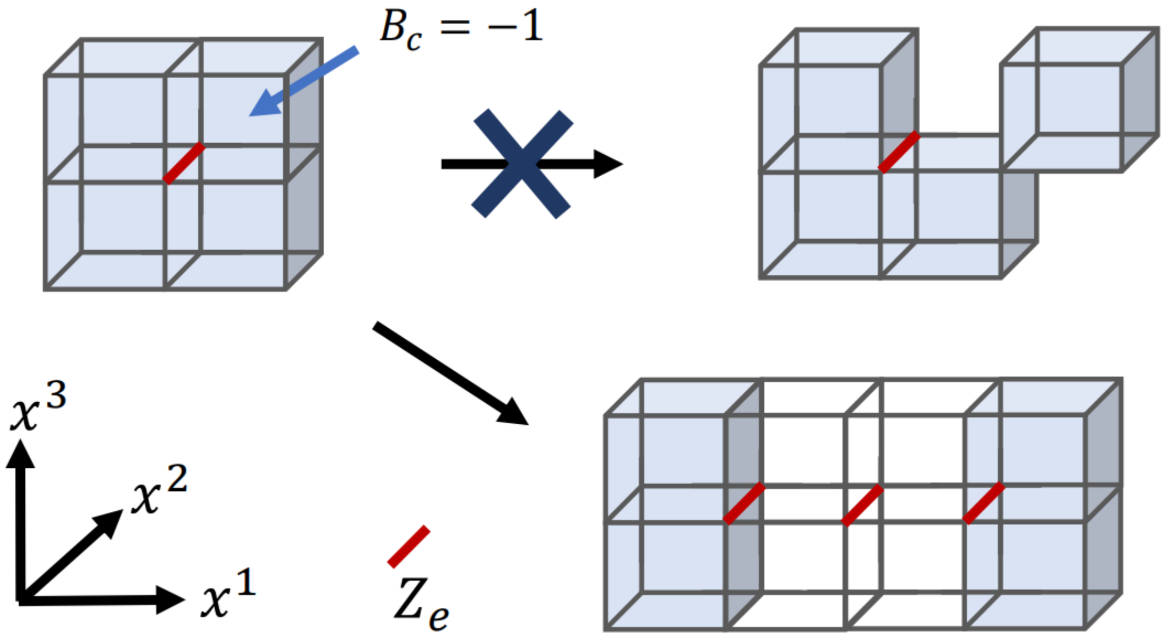
log (ground state degeneracy) $\propto$ (the size of the system)

Fracton

Fracton excitations are created by **a Z operator acting on some state**.

At four cubes around the Z_e, the eigenvalues of each B_c are -1.

Then the energy increases.
→ Some gapped excitations



This excitation **cannot move alone** by any Pauli operators,

but a dipole can move like a planon by Z operators.

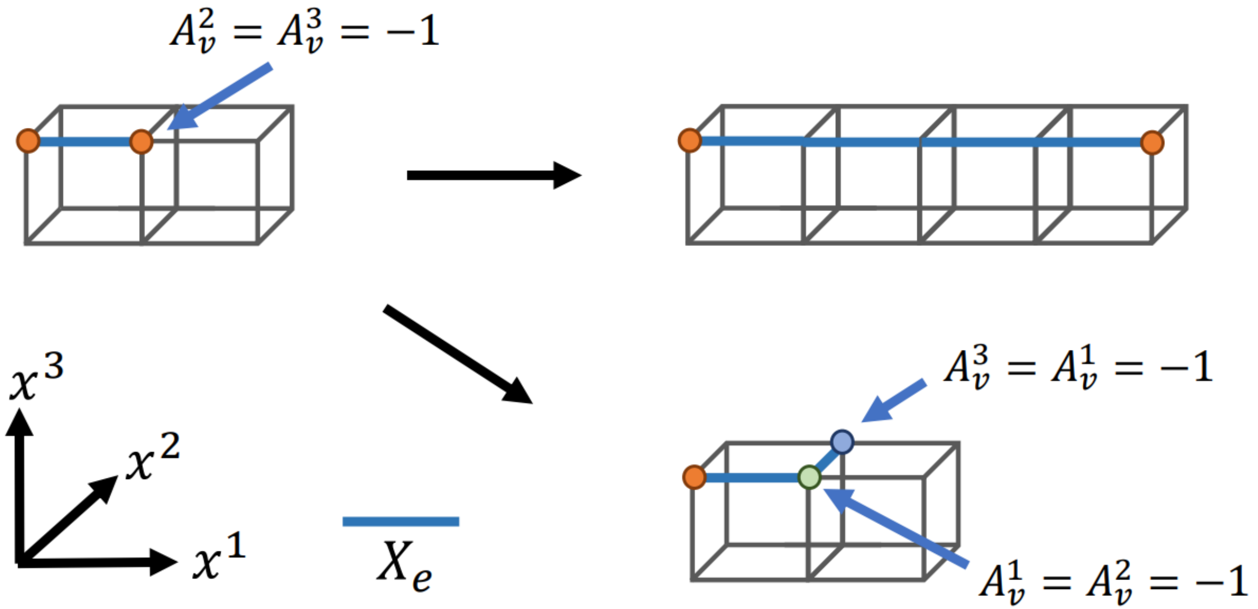
Lineon

Another excitation is a lineon.

x^k -lineon that **can move along the x^k direction** is created by **a X operator acting on some state**.

x^k -lineon at v is characterized by the eigenvalues $A_v^k = 1$ and $A_v^i = -1$ $(i \neq k)$.

Lineon excitations cannot turn without creating another lineon.



Subsystem symmetry

The $\mathbb{Z}_2$ X-cube model has two $\mathbb{Z}_2$ subsystem symmetries.

- **The $\mathbb{Z}_2$ dipole global symmetries**

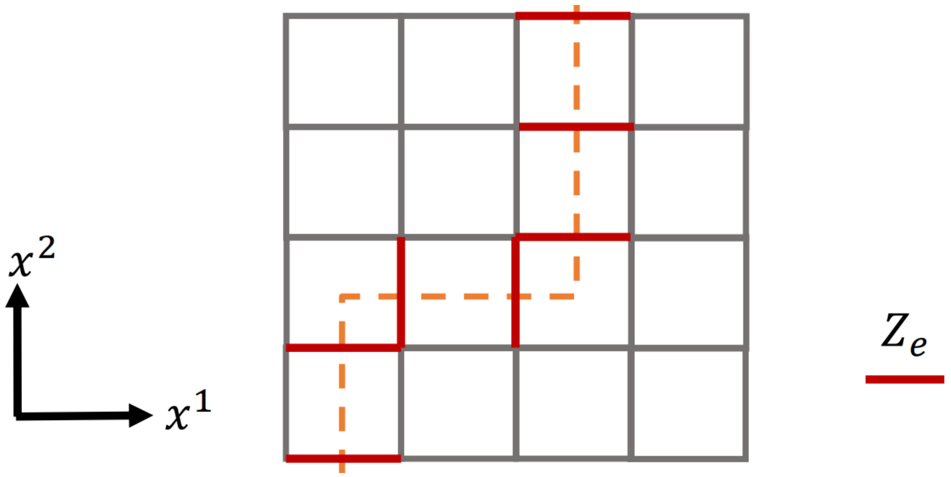
  The symmetry operators

$$W_3(C^{12}) = \prod_{e \in C^{12}} Z_e$$

(and $W_1(C^{23}), W_2(C^{31})$)

C^{12}

: closed zigzag line



Subsystem symmetry

C^3

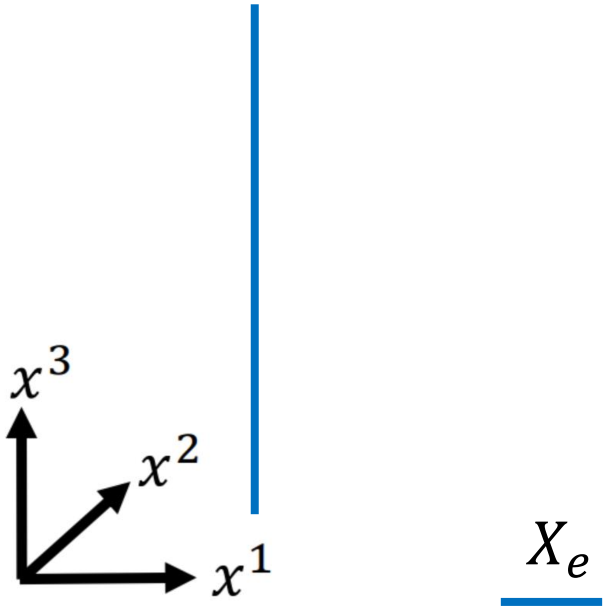
: closed straight line

- **The $\mathbb{Z}_2$ tensor global symmetries**

  The symmetry operators

$$L_3(C^3) = \prod_{e \in C^3} X_e$$

(and $L_1(C^1), L_2(C^2)$)



These operators **cannot be deformed freely**. → subsystem symmetry

Continuum QFT?

- The X-cube model is a lattice model. What is the low-energy continuum description ?

$\longrightarrow$ **Fractonic quantum field theory (QFT)**

Two descriptions $\left\{ \begin{array}{l} \text{Foliated QFTs} \\ \text{Exotic QFTs (tensor gauge)} \end{array} \right.$

They are not Lorentz invariant and not fully rotational invariant.

Fractonic QFT has **the discrete rotational symmetry** that the corresponding lattice model has.

Contents

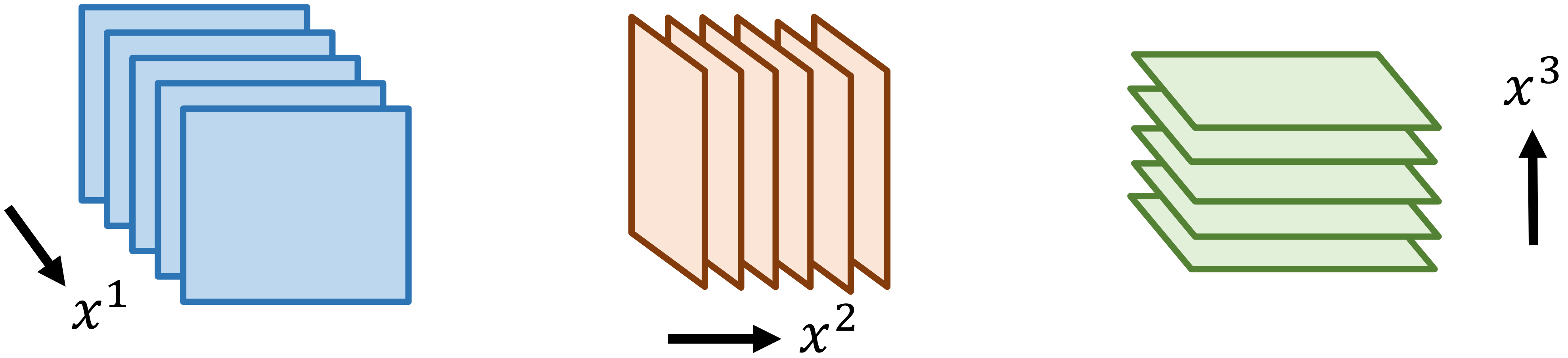
- **Fracton**
- **Foliated QFT**
- **Exotic QFT**
- **Correspondences**

Foliated QFT

Foliation

A decomposition of d-dimensional manifold to a stack of an infinite of $(d-1)$-dimensional submanifolds: leaves.

We consider three simultaneous foliations x^1, x^2, x^3.



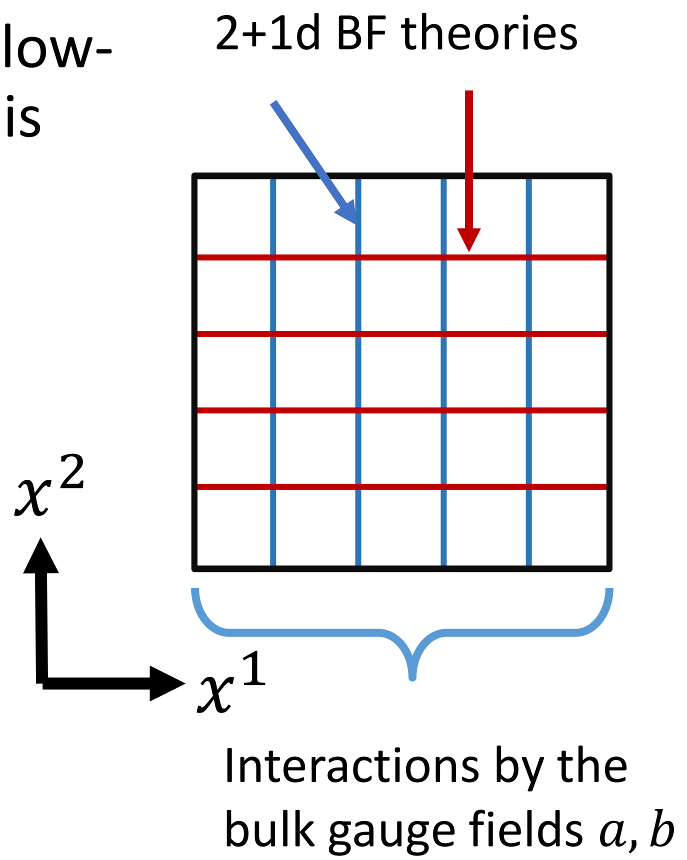
Foliated BF theory

We consider **the foliated BF theory** in 3+1d [Slagle, 2021] that is a low-energy continuum QFT of the $\mathbb{Z}_N$ X-cube model. The Lagrangian is

$$\mathcal{L}_f = \sum_{k=1}^{3} \frac{iN}{2\pi}(dB^k + b) \wedge A^k \wedge dx^k + \frac{iN}{2\pi} b \wedge da$$

A^k, B^k are **the foliated gauge fields** that is the gauge fields on planes orthogonal to the k direction.

They have discontinuities in the k direction.

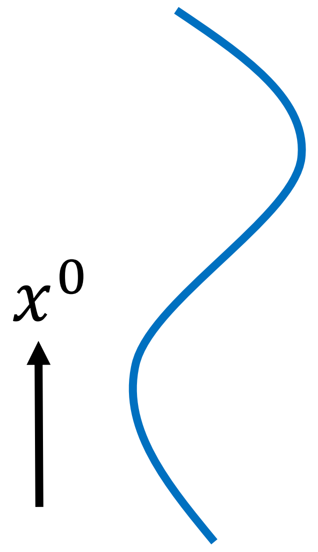


Time-like operators

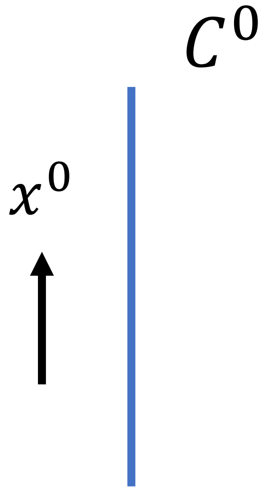
How can we express fractons ?

In the low-energy theory, gapped excitations are described by **the gauge-invariant operators extending in the time-like direction**.

The manifold on which the operator is defined is **the trajectory of the excitation in the spacetime**.



Time-like operators



The fracton operator $$F[C^0] = \exp\left[i \oint_{C^0} a\right]$$

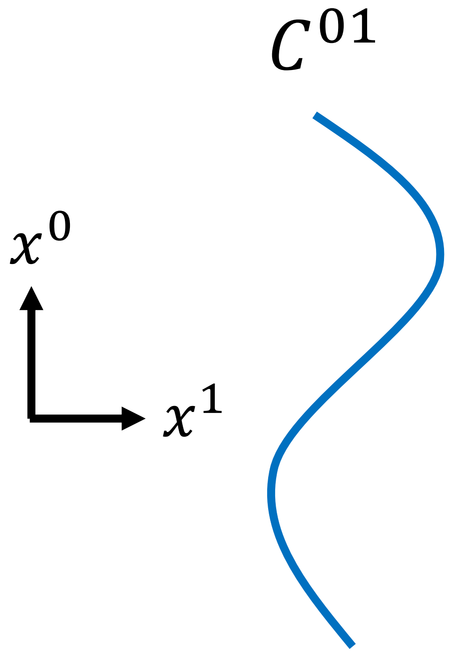
where C^0 is a one-dimensional line along the time direction.

The deformation C^0 to the space would break the gauge invariance.

$\Rightarrow$ **a particle that cannot move in space: fracton**

<u>This operator is specific to fractonic QFT.</u>

Time-like operators



The lineon operators $$L_1[C^{01}] = \exp\left[i \oint_{C^{01}} (B^2 - B^3) \right]$$

and $L_2[C^{02}], L_3[C^{03}]$

where C^{0k} is a one-dimensional loop on a (x^0, x^k)-plane.

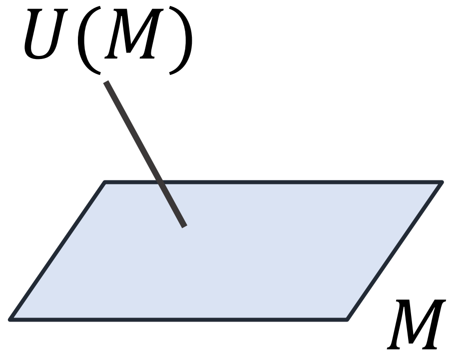
⇨ **a particle that can only move in the x^k-direction : x^k-lineon**

Subsystem symmetry

Global symmetry → gauge invariant operators defined in space.

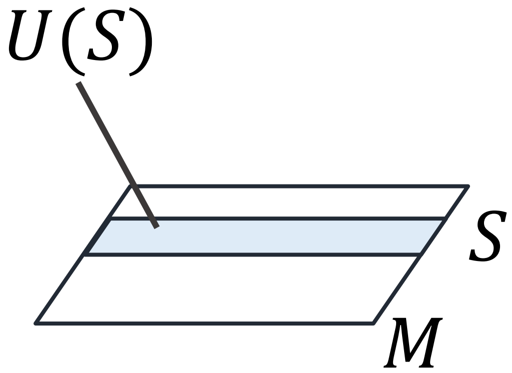
Ordinary QFT ・・・ Noether charge $Q(M) = \int_M j^0$

symmetry operator $U(M) = e^{iQ}$



Fractonic QFT ・・・ A symmetry operator $U(S)$ is defined on partially topological submanifold S.

⇨ **Subsystem symmetry**

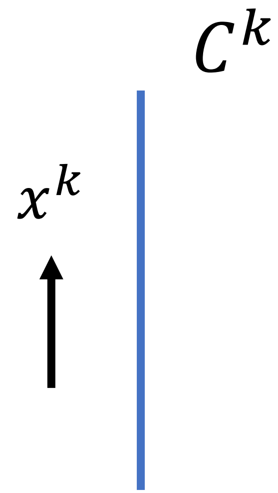


Subsystem symmetry

- $\mathbb{Z}_N$ tensor global symmetries

$$L_k[C^k] = \exp\left[i\oint_{C^k}(B^i - B^j)\right] \qquad (i,j,k) = (1,2,3),(2,3,1),(3,1,2)$$

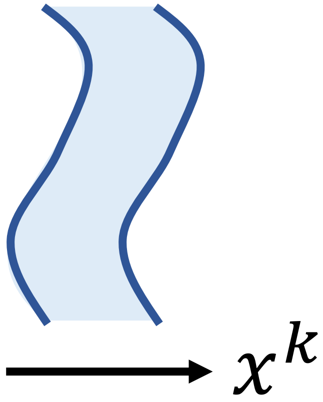
subsystem symmetries on one-dimensional lines



- $\mathbb{Z}_N$ dipole global symmetries

$$W_k[S^k] = \exp\left[i\oint_{S^k} A^k \wedge dx^k + d(a_k dx^k)\right] \qquad (k = 1,2,3)$$

subsystem symmetries on two-dimensional strips



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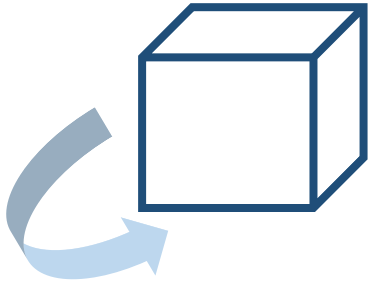
Exotic QFT (Tensor gauge theory)

Second description of fractonic QFT is **exotic QFT** that contains **tensor gauge fields**.

As in the foliated case, we consider **the exotic BF theory** in 3+1d .[Seiberg-Shao, 2021] that is a low-energy continuum QFT of the $\mathbb{Z}_N$ X-cube model.

Each tensor gauge field is in a representation of S_4 (90-degree rotations).



Exotic BF theory

The exotic BF theory in 3+1d contains

A tensor gauge field : (A_0, A_{ij}) , $A_{ij} = A_{ji}$, $i \neq j,\ i,j = 1,2,3$

$\hat{A}$ tensor gauge field : $(\hat{A}_0^{k(ij)}$, $\hat{A}^{ij}$) ,

$\hat{A}_0^{k(ij)} = \hat{A}_0^{k(ji)},\ i \neq j \neq k$, $i,j,k = 1,2,3$, the sum = 0

$\hat{A}^{ij} = \hat{A}^{ji}$, $i \neq j,\ i,j = 1,2,3$

Exotic BF theory

The Lagrangian is

$$\mathcal{L}_e = \frac{iN}{2\pi} \sum_{i,j,k=1,2,3} \left(\frac{1}{2} A_{ij} \left(\partial_0 \hat{A}^{ij} - \partial_k \hat{A}_0^{k(ij)} \right) + \frac{1}{2} A_0 \partial_i \partial_j \hat{A}^{ij} \right)$$

The gauge transformations are

$$\begin{aligned} A_0 &\to A_0 + \partial_0 \alpha \,, \\ A_{ij} &\to A_{ij} + \partial_i \partial_j \alpha \,, \\ \hat{A}_0^{i(jk)} &\to \hat{A}_0^{i(jk)} + \partial_0 \hat{\alpha}^{i(jk)} \,, \\ \hat{A}^{ij} &\to \hat{A}^{ij} + \partial_k \hat{\alpha}^{k(ij)} \end{aligned}$$

Gauge-invariant operators

- The fracton operator $$\tilde{F}[C^0] = \exp\left[i \oint_{C^0} dx^0 A_0\right]$$
- The lineon operators $$\tilde{L}_k[C^{0k}] = \exp\left[i \oint_{C^{0k}} (dx^0 \hat{A}_0^{k(ij)} + dx^k \hat{A}^{ij})\right] \quad (k = 1,2,3)$$
- $\mathbb{Z}_N$ tensor symmetries $$\tilde{L}_k[C^k] = \exp\left[i \oint_{C^k} dx^k \hat{A}^{ij}\right] \quad (k = 1,2,3)$$
- $\mathbb{Z}_N$ dipole symmetries $$\tilde{W}_k\,[S^k] = \exp\left[i \oint_{S^k} (dx^0 dx^k \partial_k A_0 + dx^i dx^k A_{ki} + dx^j dx^k A_{jk})\right]$$ $$(i,j,k) = (1,2,3),(2,3,1),(3,1,2)$$

→ **The exotic BF theory is a fractonic QFT that has the same types of operators as the foliated BF thoery**

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Correspondences

- Both the foliated BF theory and the exotic BF theory are the continuum QFTs of the X-cube model and have the same types of gauge-invariant operators.

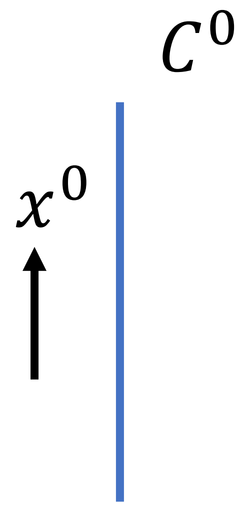
- By matching them, we can find **the explicit correspondences of the gauge fields and parameters**.

- We call this correspondences **the foliated-exotic duality** [Ohmori-Shimamura, 2022].

Field correspondences

For example, the fracton operators are extended along the time direction.

By matching them,



$$\exp\left[i\oint_{C^0} a\right] \simeq \exp\left[i\oint_{C^0} dx^0 A_0\right]$$

Foliated Exotic

$$\boxed{a_0 \simeq A_0}$$

Field correspondences

By matching the other operators, we can obtain the field correspondences

$$\boxed{\begin{aligned} &A_i^k + \partial_i a_k \simeq A_{ki}\,, \quad k \neq i \quad k,i \in \{1,2,3\}\,, \\ &B_0^i - B_0^j \simeq \hat{A}_0^{k(ij)} \quad (i,j,k) = (1,2,3),(2,3,1),(3,1,2)\,, \\ &B_k^i - B_k^j \simeq \hat{A}^{ij} \quad (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \end{aligned}}$$

Foliated Exotic

The foliated gauge fields correspond to the tensor gauge fields nontrivially.

Also, we can show correspondences among the gauge parameters.

Lagrangian correspondence

Integrating out b on the foliated side and using the field correspondences, we can derive the exotic BF Lagraingian from the foliated Lagrangian.

⇨ The foliated BF theory can be said to be equivalent to the exotic BF theory.

A duality in fractonic QFT

Conclusion

- **Fracton phase** is a new kind of phase of matter that has new characteristic features. Ex) the X-cube model

- There are two types of continuum QFTs of the X-cube model: **the foliated BF theory** and **the exotic BF theory**.

- We showed **the explicit correspondences of the gauge fields** between the foliated BF theory and the exotic BF theory.