場の理論と弦理論 @ オンライン (Aug. 4, 2023)

複素構造のT双対性変換とT-fold上の世界面インスタントン

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Introduction

- T-duality: physical equivalence between *different spacetimes*
- A well-known example of T-duality is the relation between the *NS5-brane* and the *Taub-NUT*.

T-duality: well-known example

NS5-brane	Taub-NUT
bi-hypercomplex	hyperkähler
$(J_{a,\pm}, \omega_{a,\pm}, g, B)$	(J_a, ω_a, g)

- A well-known example of T-duality is the relation between the NS5-brane and the Taub-NUT.
  - The Taub-NUT space has a hyperkähler structure.
  - It is known that the NS5-brane has a *bi-hypercomplex* structure.

Bi-hypercomplex structure

- Let M be a $4n$-dimensional differentiable manifold.
- The **bi-hypercomplex structure** on M is $(J_{a,\pm},\ \omega_{a,\pm},\ g)$ satisfying the following conditions.
  - Each $J_{a,\pm}$ is an *integrable almost complex structure* on M.
  - Each of $\{J_{a,+}\}$ and $\{J_{a,-}\}$ satisfies a *quaternion algebra*.
  - $J_{a,+}$ and $J_{b,-}$ are *commutative*: $[J_{a,+}, J_{b,-}] = 0$.
  - g is a *Hermitian metric* for each $J_{a,\pm}$.
  - $\omega_{a,\pm}$ is a *fundamental 2-form* satisfying condition $\omega_{a,\pm} = -\,gJ_{a,\pm}$.

T-duality: Taub-NUT and 5^2_2-brane

NS5-brane	**Taub-NUT**	**5^2_2-brane**
bi-hypercomplex	hyperkähler	??
$(J_{a,\pm}, \omega_{a,\pm}, g, B)$	(J_a, ω_a, g)	??

- It is known that making *another isometry* on the Taub-NUT and applying T-duality yields a 5^2_2-brane

- The metric, B-field, and dilaton of the 5^2_2-brane are known, but its *geometric structure* is **not** well-known

- The 5^2_2-brane has strange properties 👈

T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

Taub-NUT space $\qquad ds^2 = H\, dx_{123}^2 + H^{-1}(dx_4^2 + A_i\, dx^i)^2$

T-duality transformation (Buscher rule) along x^3

$$g'_{ij} = g_{ij} - \frac{g_{iy}g_{jy} - B_{iy}B_{jy}}{g_{yy}}, \qquad g'_{iy} = \frac{B_{iy}}{g_{yy}}, \qquad g'_{yy} = \frac{1}{g_{yy}},$$

$$B'_{ij} = B_{ij} - \frac{B_{iy}g_{jy} - g_{iy}B_{jy}}{g_{yy}}, \qquad B'_{iy} = \frac{g_{iy}}{g_{yy}}, \qquad \phi' = \phi - \frac{1}{2}\log g_{yy}\,.$$

5^2_2-brane geometry

$$ds^2 = H\, dx_{12}^2 + \frac{H}{H^2 + A_3^2}\, dx_{34}^2, \quad B = -\frac{A_3}{H^2 + A_3^2}\, dx^3 \wedge dx^4, \quad e^{2\phi} = e^{2\phi_0}\frac{H}{H^2 + A_3^2}\,.$$

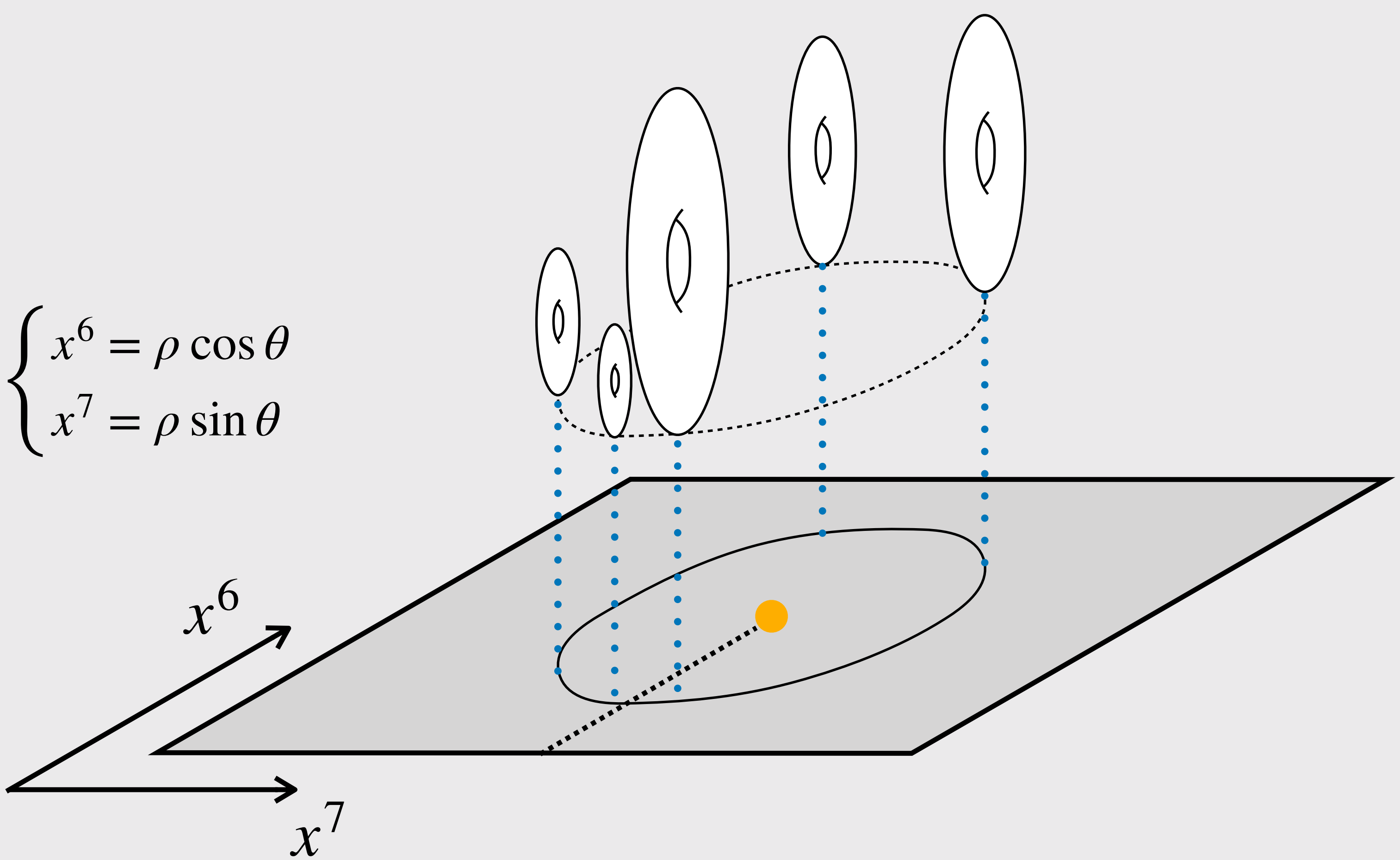
T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

$$\boxed{\begin{aligned} ds^2 &= H\,dx_{12}^2 + \underline{\frac{H}{H^2+A_3^2}\,dx_{34}^2}, \\ H &= H(\rho), \quad A_3 = -\,\sigma\theta, \quad \sigma = \text{const}\,. \end{aligned}}$$

$$\begin{cases} \theta = 0: & \dfrac{H}{H^2+A_3^2} = \dfrac{1}{H} \\ \theta = 2\pi: & \dfrac{H}{H^2+A_3^2} = \dfrac{H}{H^2+(2\pi\sigma)^2} \end{cases}$$

$$\begin{cases} x^6 = \rho\cos\theta \\ x^7 = \rho\sin\theta \end{cases}$$



- The geometry of 5^2_2-brane is torus fibered
- The torus radii do **not** match at $\theta = 0$ and 2π
- This geometry has a *monodromy*
- This monodromy is neither a diffeo. nor a B-field gauge transformation

T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

- The 5^2_2 monodromy is clearly evaluated in $O(D,D)$ covariant form

- The metric and B-fields are combined into an $O(D,D)$ covariant form called the *generalized metric*:

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

  - generalized metric for 5^2_2 at $\theta = 0$:

$$\mathcal{H}(\theta = 0) = \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H\delta \end{pmatrix}$$

  - generalized metric for 5^2_2 at $\theta = 2\pi$:

$$\mathcal{H}(\theta = 2\pi) = \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 2\pi\sigma H^{-1}\epsilon \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & H^{-1}\delta & 0 & (H + (2\pi\sigma)^2 H^{-1})\delta \end{pmatrix}$$

T-fold: 5^2_2-brane

[de Boer-Shigemori, 1004.2521, 1209.6056]

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  - generalized metric for 5^2_2 at $\theta = 2\pi$:

$$\mathcal{H}(\theta = 2\pi) = \underbrace{\begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & -2\pi\sigma\epsilon & 0 & \delta \end{pmatrix}} \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H\delta \end{pmatrix} \underbrace{\begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 2\pi\sigma\epsilon \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}}$$

$O(D,D)$ matrix known as the β-shift

T-fold: 5^2_2-brane

- The 5^2_2-brane monodromy is an $O(D,D)$ transformation
  - the charts are glued together by *T-duality* $\rightarrow$ **T-fold** [Hull '04]
- The 5^2_2 should have a *bi-hypercomplex structure* as required by SUSY
- It is expected that the bi-hypercomplex structure also has a monodromy
- Combining these geometric structures into $O(D,D)$ covariant form
  - generalized (hyper)Kähler structure

T-duality b/w Kähler and bi-Hermitian

T-duality b/w (J, ω) and $(J_\pm, \omega_\pm)$

- Here, we derive the T-duality transformation rule for geometric structures
- We focus on the Kähler and bi-Hermitian structures
  - The relation b/w the hyperkähler and bi-hypercomplex structures can be considered in the same way as in the following discussion
- $O(D,D)$ covariant form: *generalized Kähler structure*

$$\mathcal{J}_\pm = \frac{1}{2} e^B \Big(\mathcal{J}_{J_+} \pm \mathcal{J}_{J_-} + \mathcal{J}_{\omega_+} \mp \mathcal{J}_{\omega_-} \Big) e^{-B}$$

D-dimensional section

Kähler structure

bi-Hermitian structure

Depending on how to take the D-dimensional sections, a Kähler or bi-Hermitian structures can be obtained.

T-duality b/w (J,ω) and $(J_\pm,\omega_\pm)$

- Using the generalized Kähler structure, we derive the T-duality transformation rule for geometric structures

T-duality transformation from the Kähler (J,ω) to the bi-hermitian structure $(J_\pm,\omega_\pm)$:

[Hassan '95][Kimura-Sasaki-KS '22]
[Blair-Hulik-Sevrin-Thompson '22]

$$(J'_\pm)^i{}_j = J^i{}_j - \frac{J^i{}_y g_{yj}}{g_{yy}}, \qquad (J'_\pm)^i{}_y = \mp \frac{J^i{}_y}{g_{yy}}, \qquad (J'_\pm)^y{}_j = \pm\,\omega_{yj}, \qquad (J'_\pm)^y{}_y = 0,$$

$$(\omega'_\pm)_{ij} = \omega_{ij} - \frac{\omega_{iy}g_{yj} + g_{iy}\omega_{yj}}{g_{yy}}, \qquad (\omega'_\pm)_{iy} = \mp \frac{\omega_{iy}}{g_{yy}}.$$

- The above transformation rules have been studied previously using the supersymmetric sigma models, but we can *systematically* derive them by using our $O(D,D)$ covariant formulation

- In general, a bi-Hermitian structure maps to a bi-Hermitian structure by T-duality; the Kähler structure is a special case where the J_+ and J_- are *degenerate*

(Almost) bi-hypercomplex structure of 5^2_2

- Using the analogue of the Buscher rule, we can derive **the 5^2_2 (almost) bi-hypercomplex structure** from the hyperkähler structure of the Taub-NUT

[Kimura-Sasaki-KS, to appear]

bi-hypercomplex structure

$$
\left\{
\begin{aligned}
J_{1,+} &= \begin{pmatrix} 0 & 0 & A_8K^{-1} & -HK^{-1} \\ 0 & 0 & HK^{-1} & A_8K^{-1} \\ -A_8 & -H & 0 & 0 \\ H & -A_8 & 0 & 0 \end{pmatrix}, \quad
J_{1,-} = \begin{pmatrix} 0 & 0 & -A_8K^{-1} & -HK^{-1} \\ 0 & 0 & -HK^{-1} & A_8K^{-1} \\ A_8 & H & 0 & 0 \\ H & -A_8 & 0 & 0 \end{pmatrix}, \\
J_{2,+} &= \begin{pmatrix} 0 & 0 & -HK^{-1} & -A_8K^{-1} \\ 0 & 0 & A_8K^{-1} & -HK^{-1} \\ H & -A_8 & 0 & 0 \\ A_8 & H & 0 & 0 \end{pmatrix}, \quad
J_{2,-} = \begin{pmatrix} 0 & 0 & HK^{-1} & -A_8K^{-1} \\ 0 & 0 & -A_8K^{-1} & -HK^{-1} \\ -H & A_8 & 0 & 0 \\ A_8 & H & 0 & 0 \end{pmatrix}, \\
J_{3,+} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad
J_{3,-} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\right.
$$

$$K = H^2 + A_8^2$$

corresponding fundamental 2-forms

$$
\left\{
\begin{aligned}
\omega_{1,+} &= HK^{-1} \begin{pmatrix} 0 & 0 & -A_8 & H \\ 0 & 0 & -H & -A_8 \\ A_8 & H & 0 & 0 \\ -H & A_8 & 0 & 0 \end{pmatrix}, \quad
\omega_{1,-} = HK^{-1} \begin{pmatrix} 0 & 0 & A_8 & H \\ 0 & 0 & H & -A_8 \\ -A_8 & -H & 0 & 0 \\ -H & A_8 & 0 & 0 \end{pmatrix}, \\
\omega_{2,+} &= HK^{-1} \begin{pmatrix} 0 & 0 & H & A_8 \\ 0 & 0 & -A_8 & H \\ -H & A_8 & 0 & 0 \\ -A_8 & -H & 0 & 0 \end{pmatrix}, \quad
\omega_{2,-} = HK^{-1} \begin{pmatrix} 0 & 0 & -H & A_8 \\ 0 & 0 & A_8 & H \\ H & -A_8 & 0 & 0 \\ -A_8 & -H & 0 & 0 \end{pmatrix}, \\
\omega_{3,+} &= \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & -HK^{-1} \\ 0 & 0 & HK^{-1} & 0 \end{pmatrix}, \quad
\omega_{3,-} = \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & HK^{-1} \\ 0 & 0 & -HK^{-1} & 0 \end{pmatrix}.
\end{aligned}
\right.
$$

Monodromy of $(J_{a,\pm}, \omega_{a,\pm})$ in 5^2_2-brane

- It is expected that the bi-hypercomplex structure of the 5^2_2-brane also has a monodromy, as well as the metric and B-fields

- We examine the monodromy using a *generalized hyperkähler structure* that combines the bi-hypercomplex structure into the $O(D,D)$ covariant form

- A result was obtained as follows, explicitly showing that the bi-hypercomplex structure of the 5^2_2-brane has a *monodromy*

[Kimura-Sasaki-KS, to appear]

$$
\begin{aligned}
&\mathcal{J}^{(2\pi)}_{1,+} = \Omega_{-2\pi}\mathcal{J}^{(0)}_{1,+}\Omega_{2\pi}, \quad \mathcal{J}^{(2\pi)}_{2,+} = \Omega_{-2\pi}\mathcal{J}^{(0)}_{2,+}\Omega_{2\pi}, \quad \mathcal{J}^{(2\pi)}_{3,+} = \Omega_{-2\pi}\mathcal{J}^{(0)}_{3,+}\Omega_{2\pi}, \\
&\mathcal{J}^{(2\pi)}_{1,-} = \Omega_{-2\pi}\mathcal{J}^{(0)}_{1,-}\Omega_{2\pi}, \quad \mathcal{J}^{(2\pi)}_{2,-} = \Omega_{-2\pi}\mathcal{J}^{(0)}_{2,-}\Omega_{2\pi}, \quad \mathcal{J}^{(2\pi)}_{3,-} = \mathcal{J}^{(0)}_{3,-}.
\end{aligned}
$$

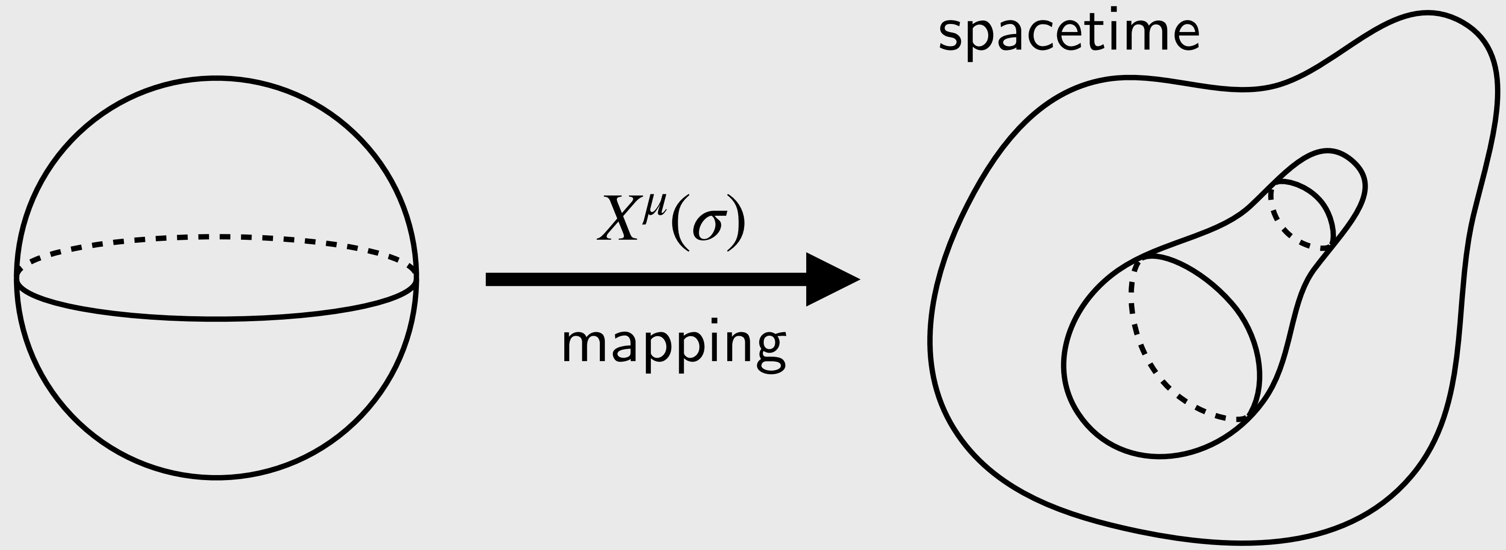
$$
\Omega_{2\pi} = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi\sigma \\ 0 & 0 & -2\pi\sigma & 0 \end{pmatrix}
$$

Application: Worldsheet Instantons

Worldsheet instantons

[Wen-Witten '86]

- A worldsheet instanton is a mapping from a worldsheet with S^2 topology to a 2-cycle in the target space



S^2
string worldsheet

$C_2 \simeq S^2$
2-cycle in spacetime

- This map is classified by the homotopy group $\pi_2(S^2) = \mathbb{Z}$

- This map satisfies the worldsheet instanton eq.: $dX^\mu \pm J^\mu{}_\nu * dX^\nu = 0$
  - $J^2 = -1$: complex structure of spacetime

- The worldsheet instantons contribute to the string scattering amplitude as non-perturbative effects of α' $\leftarrow$ *a "stringy" nature of spacetime*

Worldsheet instantons in T-fold

- The geometry of T-fold is not well understood

- In order to evaluate the worldsheet instantons appropriately, a complex structure is required

- The complex structures of T-fold have a *monodromy*, so the worldsheet instantons will be *multivalued* ⇒ **ill-defined**

$$\boxed{dX^\mu \pm J^\mu{}_\nu * dX^\nu = 0} \neq \boxed{dX^\mu \pm J'^\mu{}_\nu * dX^\nu = 0} \quad \begin{cases} \theta = 0 & : \quad J^\mu{}_\nu \\ \theta = 2\pi & : \quad J'^\mu{}_\nu \end{cases}$$

- If we use the $O(D,D)$ covariant description, the worldsheet instantons are well-defined

  - consider the **Born geometry**

Born sigma model

- A 2-dim. sigma model in which the target space is a *2D-dim. Born geometry* : a **Born sigma model** [Tseytlin '90][Hull '07][Copland '11][Arvanitakis-Blair '18][Sakatani-Uehara '20][Marotta-Szabo '22] &c.

$$S = \frac{1}{4}\int_\Sigma \Big(\underbrace{\mathcal{H}_{MN}}_{\text{generalized metric}} d\mathbb{X}^M \wedge * d\mathbb{X}^N - \underbrace{\Omega_{MN}\, d\mathbb{X}^M \wedge d\mathbb{X}^N}_{\text{topological term}} \Big) \qquad \mathbb{X}^M = (X^\mu,\ \tilde{X}_\mu)$$

By imposing the *chiral condition*, a **D**-dim. subspace of the **2D**-dim. target space is selected

$$d\mathbb{X}^M \pm \big(\eta^{MP}\mathcal{H}_{PN}\big) * d\mathbb{X}^N = 0$$

chiral condition

$$\mathbb{X}^M = (X^\mu,\ \cancel{\tilde{X}_\mu})$$

choosing T-dual frame

- The Born sigma model is then reduced to a *string sigma model*

$$S = \frac{1}{2}\int \Big(g_{\mu\nu}\, dX^\mu \wedge * dX^\nu + B_{\mu\nu}\, dX^\mu \wedge dX^\nu \Big)$$

Instantons in Born sigma model

[Kimura-Sasaki-KS '22]

The Bogomol'nyi completion of the Born sigma model action is as follows.

generalized (hyper)Kähler strc.

$$
\begin{aligned}
S_{\mathrm{E}} &= \frac{1}{8}\int \mathcal{H}_{MN}\Big(\underline{d\mathbb{X}^M \pm \mathcal{J}_{\pm}^M{}_P * d\mathbb{X}^P}\Big) \wedge * \Big(\underline{d\mathbb{X}^N \pm \mathcal{J}_{\pm}^N{}_Q * d\mathbb{X}^Q}\Big) \\
&\quad \pm 2\int (\omega_{\pm})_{MN}\, d\mathbb{X}^M \wedge d\mathbb{X}^N \\
&\geq \pm 2\int (\omega_{\pm})_{MN}\, d\mathbb{X}^M \wedge d\mathbb{X}^N
\end{aligned}
$$

The following instanton eq. is obtained as a cond. for saturating this bound.

$$
\underbrace{d\mathbb{X}^M \pm \mathcal{J}_{\pm}^M{}_P * d\mathbb{X}^P = 0}_{\text{doubled instanton equation}}
$$

Since the Born sigma model is an $O(D,D)$ covariant formulation, this instanton eq. is also *T-duality covariant*.

Consistency check

$$d\mathbb{X}^M \pm \mathcal{J}_{\pm}^{M}{}_{P} * d\mathbb{X}^P = 0$$

doubled instantons

$$dX^\mu \pm J^\mu{}_\nu * dX^\nu = 0$$

worldsheet instantons

$$S_{\text{inst.}}^{\text{Born}} = \frac{1}{4}\left|\int \omega_\pm\right| + \frac{i}{4}\int \Omega$$

action bound

$$\Longrightarrow$$

chiral constraint

choosing a polarization

$$S_{\text{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega\right| + \frac{i}{2}\int_{C_2} B$$

action bound

Non-wrapping inst. as doubled inst.

TN polarization

$$S^{\mathrm{TN}}_{\mathrm{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega_{\mu\nu} dX^\mu \wedge dX^\nu\right| + \frac{i}{2}\int_{C_2} B_{\mu\nu} dX^\mu \wedge dX^\nu$$

2-cycle in physical space — well-defined

$$\uparrow$$

$$S^{\mathrm{Born}}_{\mathrm{inst.}} = \frac{1}{4}\left|\int \omega_\pm\right| + \frac{i}{4}\int \Omega$$

$$\downarrow$$

$$S^{\mathrm{NS5}}_{\mathrm{inst.}} = \frac{1}{2}\left|\int_{C_2} \omega'_{\mu\nu} dX'^\mu \wedge dX'^\nu\right| + \frac{i}{2}\int_{C_2} B'_{\mu\nu} dX'^\mu \wedge dX'^\nu$$

NS5 polarization

2-cycle does not lie in physical space

partition function

$$Z = \int \mathcal{D}\tilde{X} \underline{\int \mathcal{D}X\ e^{-S^{\mathrm{TN}}(X)}}$$

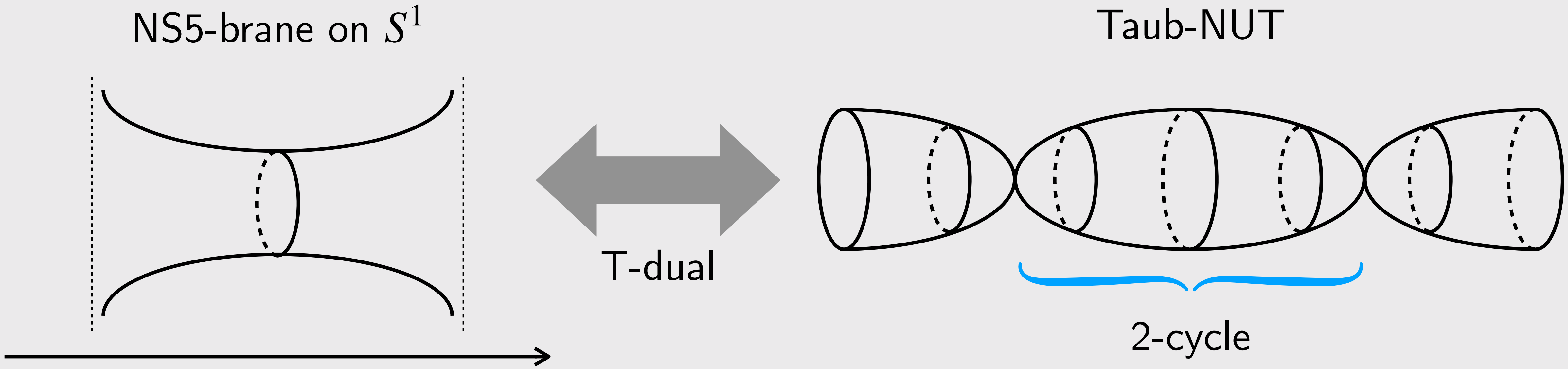
$$\uparrow$$

$$Z = \int \mathcal{D}\mathbb{X}\ e^{-S^{\mathrm{Born}}(\mathbb{X})}$$

$$\downarrow$$

$$Z = \underbrace{\int \mathcal{D}\tilde{X}'}_{\text{vol.}} \underbrace{\int \mathcal{D}X'\ e^{-S^{\mathrm{NS5}}(X')}}_{\text{D-dim.}}$$

Non-wrapping inst. as doubled inst.



no 2-cycle in physical space

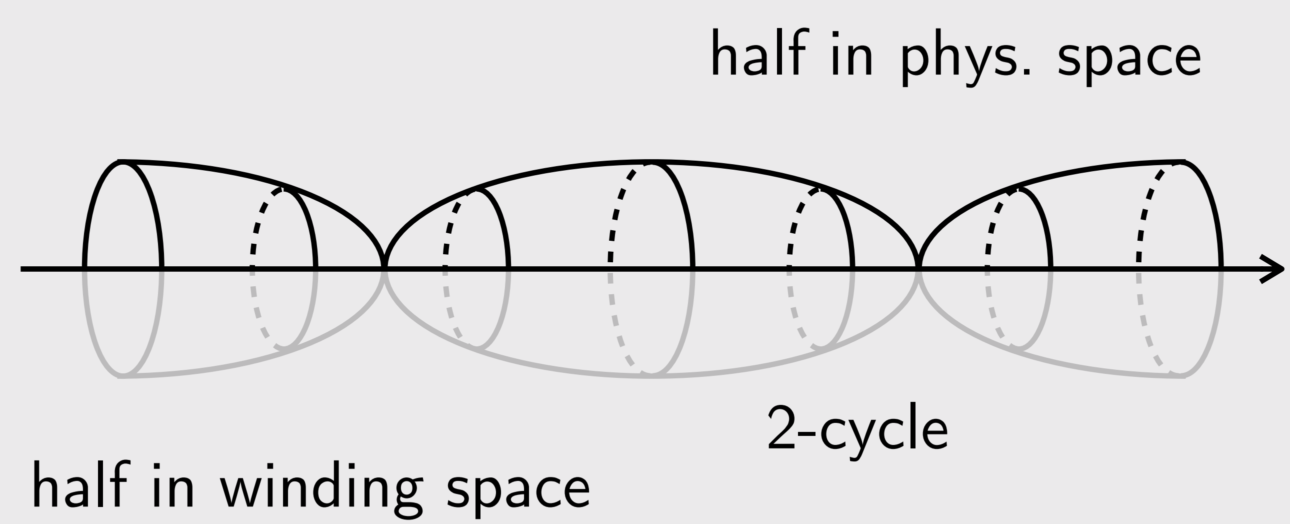
but it exists in doubled space

instantons are well-defined

worldsheet wraps in dual winding direction

an interpretation of “non-wrapping” inst.
(“point-like inst.”)

Doubled space



Instantons in T-fold

doubled inst. action

$$S^{\rm Born}_{\rm inst.} = \frac{1}{4}\left|\int \omega_{\pm}\right| + \frac{i}{4}\int \Omega$$

$\downarrow$ 5^2_2 polarization

$$S^{5^2_2}_{\rm inst.} = \frac{1}{2}\left|\int_{C_2} \omega''_{\mu\nu} dX''^{\mu} \wedge dX''^{\nu}\right| + \frac{i}{2}\int_{C_2} B''_{\mu\nu} dX''^{\mu} \wedge dX''^{\nu}$$

2-cycles?? T-fold is non-geometric

partition function

$$Z = \int \mathcal{D}\mathbb{X}\; e^{-S^{\rm Born}(\mathbb{X})}$$

$\downarrow$

$$Z = \underbrace{\int \mathcal{D}\tilde{X}''}_{\text{vol.}} \underbrace{\int \mathcal{D}X''\; e^{-S^{5^2_2}(X'')}}_{\text{D-dim.}}$$

however, **2-cycle still exist in doubled space**

(2D-dim. Born geometry with gen. hyperKähler strc.)

T-duality covariant instanton

- The T-fold spacetime can be realized as a 2D-dim. Born geometry with a generalized (hyper)Kähler structure

- In the Born sigma model, the worldsheet instanton eq. is **well-defined** because it can be transformed as follows

$$d\mathbb{X}'^M \pm \mathcal{J}'^M_{\pm}{}_P * d\mathbb{X}'^P = 0 \qquad (\mathcal{J}_\pm \text{ is at } \theta = 0 \text{ and } \mathcal{J}'_\pm \text{ is at } \theta = 2\pi)$$

$$\Leftrightarrow \quad (\Omega_{-2\pi})^M{}_N d\mathbb{X}^N \pm (\Omega_{-2\pi})^M{}_N \mathcal{J}^N_{\pm}{}_K (\Omega_{2\pi})^K{}_P * (\Omega_{-2\pi})^P{}_Q d\mathbb{X}^Q = 0$$

$$\Leftrightarrow \quad (\Omega_{-2\pi})^M{}_N \Big(\underbrace{d\mathbb{X}^N \pm \mathcal{J}_{\pm}{}^N{}_K * d\mathbb{X}^K}_{\text{well-defined!}}\Big) = 0$$

$\Rightarrow$ The worldsheet instantons in T-fold have to be treated in an $O(D,D)$ covariant doubled formalism

Summary

Summary

- The T-duality relates a Kähler (hyperkähler) manifold to a bi-Hermitian (bi-hypercomplex) manifold

- We *systematically* derived the T-duality transformation rules for complex structures and fundamental 2-forms by using the $O(D, D)$ covariant form

- We also found the local geometric structure $(J_{a,\pm}, \omega_{a,\pm})$ of the 5^2_2-brane known as a T-fold, and explicitly showed that not only metric and B-fields, but also *they have monodromy*

- The worldsheet instantons in T-fold also have the monodromy, so the $O(D, D)$ covariant formulation is a good description to study the physics of T-fold

Future directions

- T-fold geometry (in detail)
- Worldsheet instanton effects in T-folds (in detail)
- T-duality of integrability conditions for geometric structures
- U-duality relations of geometric structures
- Membrane instantons and U-duality
- &c.

Thank you

Backup

Generalized Kähler structure

Bi-Hermitian manifold $(M, J_\pm, \omega_\pm)$

$$J_\pm : TM \to TM \qquad\qquad \omega_\pm : TM \to T^*M$$

$O(D,D)$ covariant formulation of structure [Gualtieri '04]

Gualtieri map $$\mathcal{J}_{J_\pm} = \begin{pmatrix} J_\pm & 0 \\ 0 & -J_\pm^* \end{pmatrix} \qquad \mathcal{J}_{\omega_\pm} = \begin{pmatrix} 0 & -\omega_\pm^{-1} \\ \omega_\pm & 0 \end{pmatrix}$$

generalized Kähler structure

$$\mathcal{J}_\pm = \frac{1}{2}\Big(\mathcal{J}_{J_+} \pm \mathcal{J}_{J_-} + \mathcal{J}_{\omega_+} \mp \mathcal{J}_{\omega_-}\Big)$$

$$\mathcal{J}_\pm^2 = -1 \qquad\qquad [\mathcal{J}_+, \mathcal{J}_-] = 0$$

$$\mathcal{J}_\pm : TM \oplus T^*M \to TM \oplus T^*M$$

Born geometry

Born structure $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ on 2D-dimensional manifold $\mathcal{M}^{2D}$

para-quaternion algebra

$$\mathcal{I}^2 = -\mathcal{J}^2 = -\mathcal{K}^2 = -1 \qquad \mathcal{I}\mathcal{J}\mathcal{K} = -1$$

$$\{\mathcal{I}, \mathcal{J}\} = \{\mathcal{J}, \mathcal{K}\} = \{\mathcal{K}, \mathcal{I}\} = 0$$

$\mathcal{I}$: almost complex structure $\quad \mathcal{I} = \mathcal{H}^{-1}\Omega = -\Omega^{-1}\mathcal{H}$

$\mathcal{J}$: chiral structure $\quad \mathcal{J} = \eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta$

$\mathcal{K}$: almost para-complex structure $\quad \mathcal{K} = \eta^{-1}\Omega = \Omega^{-1}\eta$

metrics in Born geometry

$\mathcal{H}$: generalized metric

η : $O(D,D)$ invariant metric

Ω : fundamental two-form

DFT quantities

Born and generalized Kähler

bi-quaternion geometry

$$(\mathcal{M}^{2D}, \mathcal{J}_J, \mathcal{J}_\omega, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{P}, \mathcal{Q})$$

$$\mathcal{P} = \mathcal{K}\mathcal{J}_\omega \qquad \mathcal{Q} = \mathcal{K}\mathcal{J}_J$$

Born structure

$$(\mathcal{I}, \mathcal{J}, \mathcal{K})$$

generalized Kähler structure

$$(\mathcal{J}_J, \mathcal{J}_\omega, \mathcal{J})$$

metrics in Born

$$(\mathcal{H}, \eta, \Omega)$$

metrics in gen. Kähler

$$(\mathcal{H}, \omega_+, \omega_-)$$

Monodromy of codim 2 branes

defect NS5-brane

$$\mathcal{H} = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & A_3\epsilon \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H^{-1}\delta \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & -A_3\epsilon & 0 & \delta \end{pmatrix}$$

$O(D,D)$ matrix known as the B-shift (gauge symmetry)

KK-vortex

$$\mathcal{H} = \begin{pmatrix} \Lambda^{\mathsf{T}} & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} g_0 & 0 \\ 0 & g_0^{-1} \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-\mathsf{T}} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_3 & 1 \end{pmatrix} \qquad g_0 = \begin{pmatrix} H\delta & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}$$

$O(D,D)$ matrix corresponding to diffeomorphism

5^2_2-brane

$$\mathcal{H}(\theta = 2\pi) = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & -2\pi\sigma\epsilon & 0 & \delta \end{pmatrix} \begin{pmatrix} H\delta & 0 & 0 & 0 \\ 0 & H^{-1}\delta & 0 & 0 \\ 0 & 0 & H^{-1}\delta & 0 \\ 0 & 0 & 0 & H\delta \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 2\pi\sigma\epsilon \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

$O(D,D)$ matrix known as the β-shift

non-geometry!