A proof of A_n AGT conjecture at β=1

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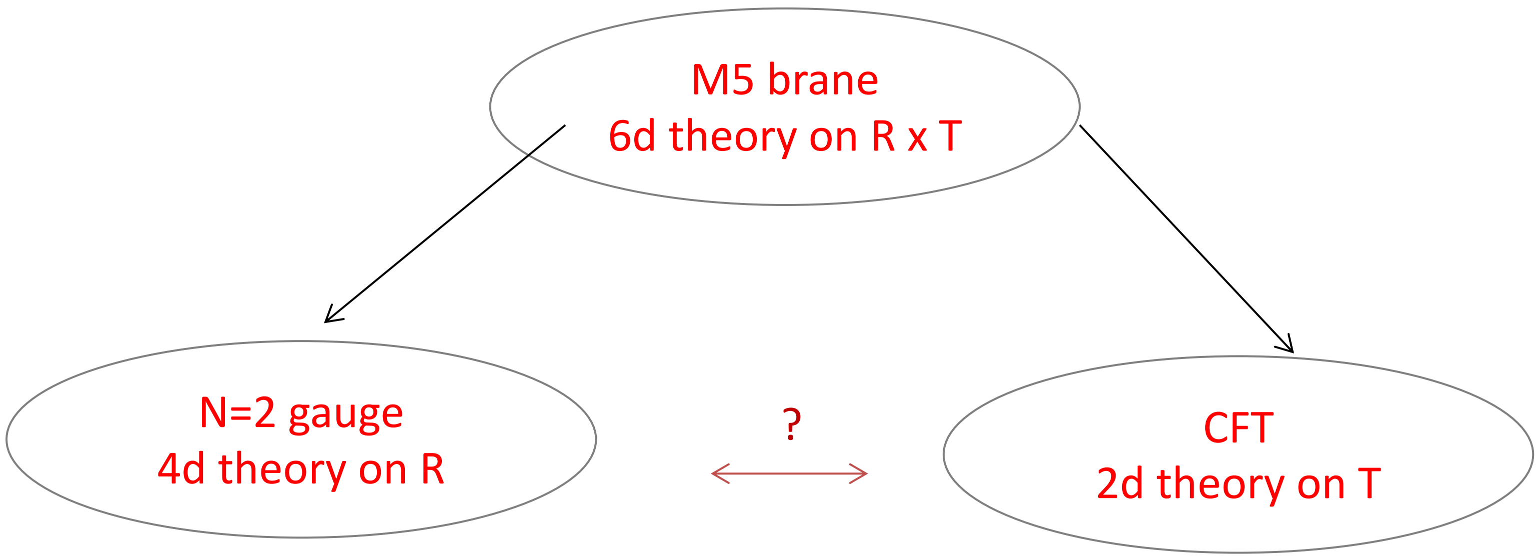
Qing-Jie Yuan, Shao-Ping Hu, Zi-Hao Huang and KZ, arXiv:2305.11839.

Introduction

ABSTRACT: AGT conjecture reveals a connection between 4D $\mathcal{N}=2$ gauge theory and 2D conformal field theory. Though some special instances have been proven, others remain elusive and the attempts on its full proof never stop. When the Ω background parameters satisfy $-\epsilon_1/\epsilon_2 \equiv \beta = 1$, the story simplifies a bit. A proof of the correspondence in the case of A_1 gauge group was given in 2010 by Mironov et al., while the A_n extension is verified by Matsuo and Zhang in 2011, with an assumption on the Selberg integral of $n+1$ Schur polynomials. Then in 2020, Albion et al. obtained the rigorous result of this formula. In this paper, we show that the conjecture on the Selberg integral of Schur polynomials is formally equivalent to their result, after applying a more complicated complex contour, thus leading to the proof of the A_n case at $\beta = 1$. To perform a double check, we also directly start from this formula, and manage to show the identification between the two sides of AGT correspondence.

Introduction

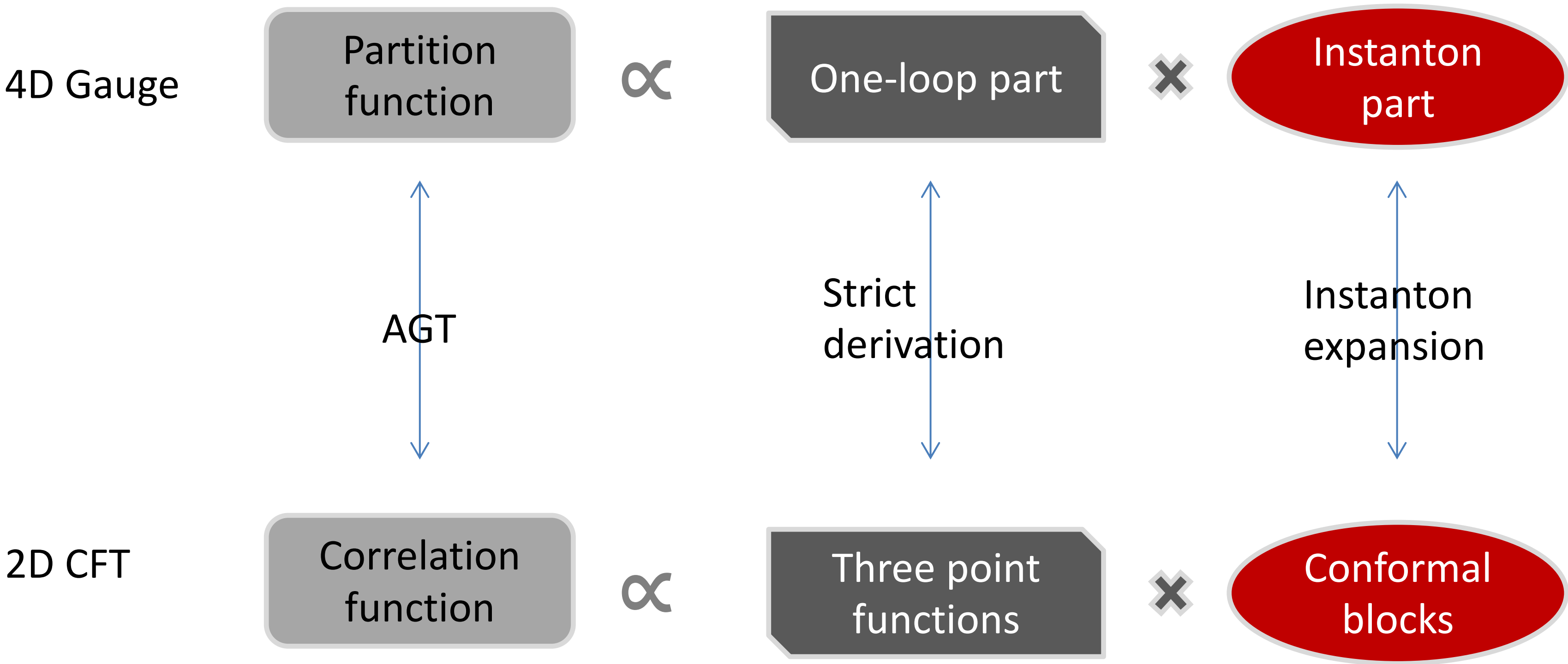
In 2002, Nekrsov performed a technique called Ω deformation in the reduction from 6D $\mathcal{N}=1$ gauge theory to 4D $\mathcal{N}=2$ gauge theory, and implied its connection with 2D conformal theory.



He found exact formulae of the partition function (Nekrasov partition function) of the $\mathcal{N}=2$ gauge theory, and showed that it reproduces the prepotential as determined by the Seiberg–Witten curve.

Introduction

AGT conjecture



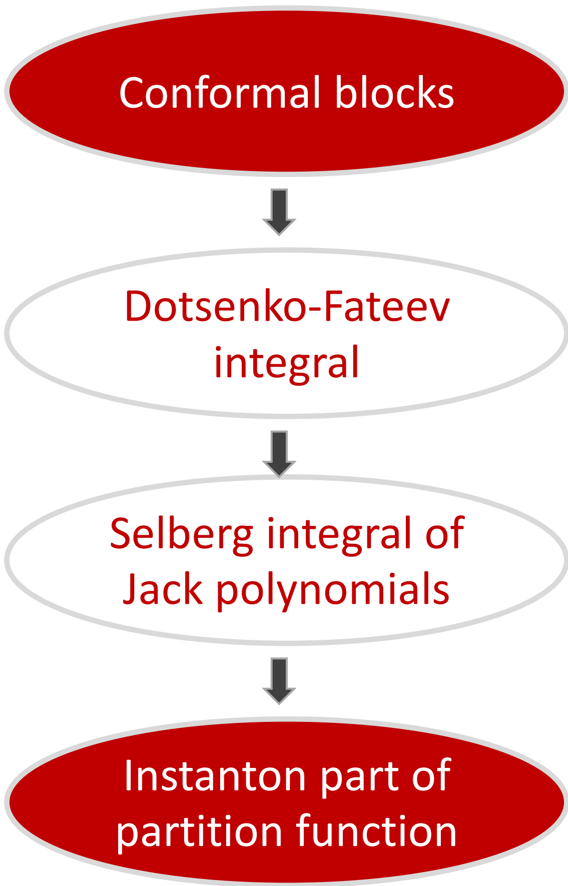
SU(2) [Alday–Gaiotto–Tachikawa '09]

SU(N) [Wyllard '09]

Introduction

We have been working on the proof of AGT conjecture through two different ways.

Direct approach



We calculated the conformal block in the form of Dotsenko−Fateev integral and reduce it in the form of Selberg integral of N Jack polynomials.

We found a formula for such Selberg average which satisfies some nontrivial consistency conditions and showed that it reproduces the SU(N) version of AGT conjecture.

$\beta = 1$ SU(2) [A. Mironov et. al. '10]

$\beta = 1$ SU(N) [Zhang Matsuo '11]

Introduction

Recursive approach

2D CFT

4D Gauge

Conformal blocks

AGT conjecture

Partition function

$$\langle\!\langle \vec{Y}|V|\vec{W}\rangle\!\rangle = Z$$

Satisfy Ward identity

trivial

nontrivial

Constrained by a recursion Relation

$$\sum \hat{O}\langle\!\langle \vec{Y}|V|\vec{W}\rangle\!\rangle = 0$$

$$\sum \hat{O} Z = 0$$

$\beta = 1$ SU(N) [Kanno–Matsuo–Zhang '12]

arbitrary β SU(N) [Kanno–Matsuo–Zhang '13]

A brief review of AGT conjecture and Nekrasov formula

Nekrasov partition function

Consider lifting the $\mathcal{N}$= 2 four dimensional theory to $\mathcal{N}$= (1, 0) six dimensional theory, and then compactify the six dimensional $\mathcal{N}$= 1 SUSY gauge theory on the manifold with the topology **T2 × *R4*** with the metric :

$$ds^2 = r^2 dz d\bar{z} + g_{\mu\nu}\left(dx^\mu + V^\mu dz + \bar{V}^\mu d\bar{z}\right)\left(dx^\nu + V^\nu dz + \bar{V}^\nu d\bar{z}\right)$$

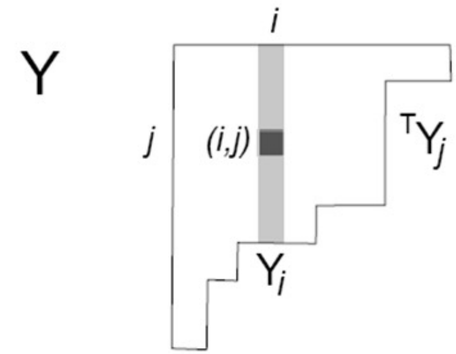
where $V^\mu = \Omega^\mu_\nu x^\nu,\ \bar{V}^\mu = \bar{\Omega}^\mu_\nu x^\nu,$ and

$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{\mu\nu} = \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\epsilon}_2 \\ 0 & 0 & -\bar{\epsilon}_2 & 0 \end{pmatrix}$$

The action of the four dimensional theory in the limit *r → 0 is not that of the pure supersymmetric Yang–Mills* theory on *R4*. Rather, it is a deformation of the latter by the Ω−dependent terms. It is called an *$\mathcal{N}$= 2 theory in the Ω-background.*

$$\beta = -\epsilon_1/\epsilon_2$$

Nekrasov partition function



With his idea of the Ω−background, Nekrasov calculated the following partition function

$$Z(\tau, a, m, \epsilon) = \int_{\phi(\infty)=a} D\Phi D A D\lambda \ldots e^{-S(\Omega)}$$

It has the important property that it gives the prepotential of the Seiberg-Witten theory in the limit $\epsilon_1 = -\epsilon_2 = \hbar \to 0$

$$F(\tau, a, m) = \lim_{\hbar \to 0} \hbar^2 \log Z_{\text{full}}(\tau, a, m; \hbar, -\hbar)$$

$$Z_{\text{full}}(q; a, m; \epsilon) = Z_{\text{tree}} Z_{1\text{loop}} Z_{\text{inst}}, \quad Z_{\text{inst}}(q; a, m; \epsilon) = \sum_{\mathbf{Y}} \mathbf{q}^{\mathbf{Y}} z(\mathbf{Y}, a, m)$$

where the instanton is labeled by a N-tuple of Young diagrams: $\mathbf{Y} := (\vec{Y}^{(1)}, \cdots, \vec{Y}^{(n)})$, (Fig. 1). The parameter a (resp. m) represents the diagonalized VEV of vector multiplets (resp. mass of hypermultiplets) whereas $q_i = e^{\pi i \tau_i}$ is the instanton expansion parameter for ith gauge group $SU(N_i)$, $\mathbf{q}^{\mathbf{Y}} := \prod_{i=1}^{n} q_i^{|\vec{Y}^{(i)}|}$. The total partition function is decomposed into a product of the contributions of the perturbative parts Z_{tree}, $Z_{1-\text{loop}}$ and non-perturbative instanton correction Z_{inst}. The latter is further decomposed into a sum of sets of Young diagrams. $\vec{Y}^{(i)} = (Y_1^{(i)}, \cdots, Y_{N_i}^{(i)})$ is a collection of N_i Young diagram which parameterizes the fixed points of instanton moduli space for i th gauge group $U(N_i)$.

Nekrasov partition function

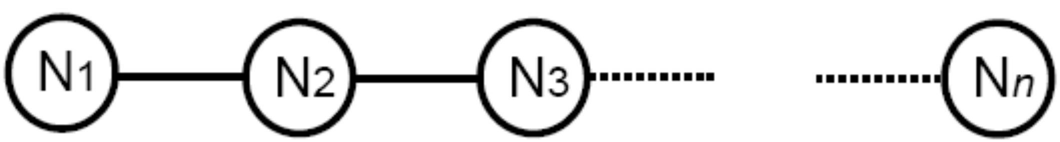
Single gauge group case

$$Z_{\rm full}(q;a,\mu;\epsilon) = Z_{\rm tree} Z_{\rm 1loop} Z_{\rm inst}, \quad Z_{\rm inst}(q;a,m;\epsilon) = \sum_{\vec{Y}} q^{|\vec{Y}|} N_{\vec{Y}}^{\rm inst}(a,\mu),$$

$$N_{\vec{Y}}^{\rm inst}(a,\mu) = z_{\rm vect}(\vec{Y},a) \prod_{i=1}^{2N} z_{\rm fund}(\vec{Y},\mu_i) = \frac{\prod_{s=1}^{N}\prod_{k=1}^{2N} f_{Y_s}(\mu_k + a_s)}{\prod_{t,s=1}^{N} g_{Y_t,Y_s}(a_t - a_s)},$$

linear quiver gauge case

with gauge group SU(N_1)$\times\cdots\times$SU(N_n).



$$Z^{\rm Nek} = \sum_{\vec{Y}^{(1)},\cdots,\vec{Y}^{(n)}} q_i^{|\vec{Y}^{(i)}|} \bar{V}_{\vec{Y}^{(1)}} \cdot Z_{\vec{Y}^{(1)}\vec{Y}^{(2)}} \cdots Z_{\vec{Y}^{(n-1)}\vec{Y}^{(n)}} \cdot V_{\vec{Y}^{(n)}}$$

$$\begin{aligned} Z_{\vec{Y}^{(i)}\vec{Y}^{(i+1)}} &= Z(\vec{a}^{(i)}, \vec{Y}^{(i)}; \vec{a}^{(i+1)}, \vec{Y}^{(i+1)}; \mu^{(i)}), \\ \bar{V}_{\vec{Y}^{(1)}} &= Z(\vec{\lambda}, \vec{\emptyset}; \vec{a}^{(1)}, \vec{Y}^{(1)}; \mu^{(0)}), \\ V_{\vec{Y}^{(n)}} &= Z(\vec{a}^{(n)}, \vec{Y}^{(n)}; \vec{\lambda}', \vec{\emptyset}; \mu^{(n)}), \end{aligned}$$

AGT conjecture

For a Liouville theory on a sphere, the four–point correlation function of *V at* positions *∞, 1, q, 0 is*

$$\langle V_{\beta_0}(\infty)V_{m_0}(1)V_{m_1}(q)V_{\beta_1}(0)\rangle = \int \frac{d\beta}{2\pi} C(\beta_0^*, m_0, \beta)C(\beta^*, m_1, \beta_1)\left|q^{\Delta_\beta - \Delta_{m_1} - \Delta_{\beta_1}} \mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)\right|^2$$

$$\propto \int a^2 da \left|Z_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)\right|^2$$

Where C(β 1, β 2, β 3) is the three point function given by the DOZZ formula.

The function *F* carries the coordinate (*q*) dependence and reflects the contributions of the conformal descendants. It is called conformal block.

$$Z_{\text{inst}}^{U(2),N_f=4}(a, m_0, \tilde{m}_0, m_1, \tilde{m}_1) = (1-q)^{2m_0(Q-m_1)} \mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)$$

it is Checked $\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)$ is the conformal block of a virasoro algebra with central charge c = 1 + 6Q2 at position ∞, 1, q, 0,

Up to order q11

AGT conjecture

SU(N) generalization

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{\alpha_2}(q)V_{\alpha_1}(0)\rangle$$

The conformal block of this
correlation function is written in the form,

$$\mathcal{F}_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} N_{\vec{Y}}^{\text{Toda}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

Correlation functions of Toda theory and Selberg Formula

SU(N) Toda

Bosons $$\phi(z) = (\phi_1(z), \cdots, \phi_N(z))$$

$$\phi_i(z)\phi_j(0) \sim \delta_{ij} \ln(z)$$

Symmetry: W_N algebra

$$R_N =: \prod_{m=1}^{N} \left(Q\frac{d}{dz} - i(h_m, \partial_z \varphi) \right) := \sum_k W^{(k)}(z) \left(Q\frac{d}{dz} \right)^{N-k}$$

$$c = (N-1)(1 + N(N+1)Q^2)$$

$W^{(k)}$ satisfies W_N algebra (a nonlinear algebra) with central charge

$$V_{\vec{\alpha}}(z) =: e^{(\alpha, \phi(z))} :$$

$$Q_j^{(\pm)} = \int \frac{dz}{2\pi i} V_j^{(\pm)}(z) = \int \frac{dz}{2\pi i} : e^{\alpha_\pm (e_j, \phi(z))} :$$

Dotsenko-Fateev integral

$$Z_{\mathrm{DF}}(q) =$$
$$\left\langle\!\!\left\langle : e^{(\tilde{\alpha_1},\phi(0))} :: e^{(\tilde{\alpha_2},\phi(q))} :: e^{(\tilde{\alpha_3},\phi(1))} :: e^{(\tilde{\alpha_4},\phi(\infty))} : \prod_{a=1}^{N-1}\left(\int_0^q : e^{b(e_a,\phi(z))} : dz\right)^{N_a}\left(\int_1^\infty : e^{b(e_a,\phi(z))} : dz\right)^{\tilde{N}_a}\right\rangle\!\!\right\rangle$$

We apply *Wick's theorem* to evaluate the correlator

$$\left\langle\left\langle : e^{(\tilde{\alpha}_1,\phi(z_1))} : \ \ldots \ : e^{(\tilde{\alpha}_n,\phi(z_n))} : \right\rangle\right\rangle = \prod_{1\le i<j\le n} (z_j - z_i)^{(\tilde{\alpha}_i,\tilde{\alpha}_j)}$$

$$Z_{\mathrm{DF}}(q) = q^{(\alpha_1,\alpha_2)/\beta}(1-q)^{(\alpha_2,\alpha_3)/\beta} \prod_{a=1}^{N-1}\prod_{I=1}^{N_a}\int_0^q dz_I^{(a)} \prod_{J=N_a+1}^{N_a+\tilde{N}_a}\int_1^\infty dz_J^{(a)} \prod_{i<j}^{N_a+\tilde{N}_a}(z_j^{(a)}-z_i^{(a)})^{2\beta} \quad \times$$
$$\times \prod_i^{N_a+\tilde{N}_a}(z_i^{(a)})^{(\alpha_1,e_a)}(z_i^{(a)}-q)^{(\alpha_2,e_a)}(z_i^{(a)}-1)^{(\alpha_3,e_a)} \prod_{a=1}^{N-2}\prod_i^{N_a+\tilde{N}_a}\prod_j^{N_{a+1}+\tilde{N}_{a+1}}(z_j^{(a+1)}-z_i^{(a)})^{-\beta} \quad .$$

Selberg integral

$$\int_{[0,1]^k} |\Delta(x)|^{2\gamma} \prod_{i=1}^{k} x_i^{\alpha-1}(1-x_i)^{\beta-1}\,\mathrm{d}x = \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma+1)}$$

When $k = 1$ the Selberg integral simplifies to the Euler beta integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha) > 0,\ \Re(\beta) > 0,$$

Here we consider its $AN-1$ extension ($AN-1$ Selberg integral):

$$S_{\vec{u},\vec{v},\beta} = \int dx \prod_{a=1}^{N-1} \left[\left|\Delta(x^{(a)})\right|^{2\beta} \prod_{i=1}^{N_a} (x_i^{(a)})^{u_a} (1-x_i^{(a)})^{v_a} \right] \prod_{a=1}^{N-2} \left|\Delta(x^{(a)}, x^{(a+1)})\right|^{-\beta}$$

Selberg integral

$$I^{A_n}_{N_1,\ldots,N_n}(\mathcal{O};u_1,\ldots,u_n,v;\beta)$$

$$\equiv \int\limits_{C_\beta^{N_1,\ldots,N_n}[0,1]} \mathcal{O}(x^{(1)},\ldots,x^{(n)}) \prod_{r=1}^{n}\prod_{i=1}^{N_r} (x_i^{(r)})^{u_r}(1-x_i^{(r)})^{v_r}$$

$$\times \prod_{r=1}^{n} |\Delta(x^{(r)})|^{2\beta} \prod_{r=1}^{n-1} |\Delta(x^{(r)},x^{(r+1)})|^{-\beta}\, \mathrm{d}x^{(1)}\cdots\mathrm{d}x^{(n)}$$

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For β = 1

$$C^{N_1,\ldots,N_n} = C_1^{N_1}\times\cdots\times C_n^{N_n}, \quad \text{where} \quad C_r^{N_r} = \underbrace{C_r\times\cdots\times C_r}_{N_r \text{ times}}.$$

$$I^{A_n}_{N_1,\ldots,N_n}(\mathscr{O};u_1,\ldots,u_n,v)$$

$$\equiv \frac{1}{(2\pi\mathrm{i})^{N_1+\cdots+N_n}} \int\limits_{C^{N_1,\ldots,N_n}} \mathscr{O}(x^{(1)},\ldots,x^{(n)}) \prod_{r=1}^{n}\prod_{i=1}^{N_r} (x_i^{(r)})^{u_r}(x_i^{(r)}-1)^{v_r}$$

$$\times \prod_{r=1}^{n} \Delta^2(x^{(r)}) \prod_{r=1}^{n-1} \Delta^{-1}(x^{(r)},x^{(r+1)})\, \mathrm{d}x^{(1)}\cdots\mathrm{d}x^{(n)}$$

Reduction to Selberg integral

$$Z_{DF}(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} \left\langle \prod_{a=1}^{N} j_{Y_a}^{(\beta)}(-r_k^{(a)} - \frac{v'_{a+}}{\beta}) \right\rangle_+ \left\langle \prod_{a=1}^{N} j_{Y_a}^{(\beta)}(\tilde{r}_k^{(a)} + \frac{v'_{a-}}{\beta}) \right\rangle_-$$

$j_{Y_a}^{(\beta)}$: Jack polynomials

$\langle \cdots \rangle_\pm$: Selberg average

$$p_k^{(a)} := \sum_i (x_i^{(a)})^k$$

Cauchy−Stanley identity

$$\exp(\beta \sum_{k=1}^{\infty} \frac{1}{k} p_k p'_k) = \sum_R j_R^{(\beta)}(p) j_R^{(\beta)}(p')$$

Jack polynomials

Jack polynomials are characterized by the fact that they are the eigenfunctions of Calogero-Sutherland Hamiltonian written in the form,

$$\mathcal{H} = \sum_{i=1}^{M} D_i^2 + \beta \sum_{i<h} \frac{z_i + z_j}{z_i - z_j}(D_i - D_j), \quad D_i := z_i \frac{\partial}{\partial z_i} .$$

The explicit form of low level ones are listed below;

$$J_{[1]}^{(\beta)}(p_k) = p_1 ,$$

$$J_{[2]}^{(\beta)}(p_k) = \frac{p_2 + \beta p_1^2}{\beta + 1}, \quad J_{[11]}^{(\beta)}(p_k) = \frac{1}{2}(p_1^2 - p_2) ,$$

$$J_{[3]}^{(\beta)}(p_k) = \frac{2p_3 + 3\beta p_1 p_2 + \beta^2 p_1^3}{(\beta+1)(\beta+2)}, \quad J_{[21]}^{(\beta)}(p_k) = \frac{(1-\beta)p_1 p_2 - p_3 + \beta p_1^3}{(\beta+1)(\beta+2)}, \quad J_{[111]}^{(\beta)}(p_k) = \frac{1}{6}p_1^3 - \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3 .$$

Selberg integral

Schur functions,

$$\chi_\lambda(x) = \frac{\det_{1\leqslant i,j\leqslant n}\left(x_i^{\lambda_j+n-j}\right)}{\Delta(x)}$$

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$$\begin{aligned}
&\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}}\left[x^{(r)} - x^{(r-1)}\right] \right\rangle_{u_1,\dots,u_n,v}^{N_1,\dots,N_n} \\
&\quad = \prod_{r=1}^{n+1} \prod_{1\leqslant i<j\leqslant \ell_r} \frac{Y_i^{(r)} - Y_j^{(r)} + j - i}{j-i} \prod_{r,s=1}^{n+1} \prod_{i=1}^{\ell_r} \frac{(A_{r,s} - N_{s-1} + N_s - i + 1)_{Y_i^{(r)}}}{(A_{r,s} + \ell_s - i + 1)_{Y_i^{(r)}}} \\
&\quad\quad \times \prod_{1\leqslant r<s\leqslant n+1} \prod_{i=1}^{\ell_r} \prod_{j=1}^{\ell_s} \frac{Y_i^{(r)} - Y_j^{(s)} + A_{r,s} + j - i}{A_{r,s} + j - i},
\end{aligned}$$

AGT conjecture from Selberg integral

Reduction to Selberg integral

Before we go to the details conclusion first

$$Z_{\text{inst}}(q) = Z_{\text{DF}}(q)$$

$$N^{\text{inst}}_{\vec{Y}} = N^{\text{Toda}}_{\vec{Y}}$$

$$N^{\text{inst}}_{\vec{Y}} \equiv N^{\text{inst}}_{\vec{Y}+} N^{\text{inst}}_{\vec{Y}-}, \quad N^{\text{Toda}}_{\vec{Y}} \equiv N^{\text{Toda}}_{\vec{Y}+} N^{\text{Toda}}_{\vec{Y}-},$$

$$N^{\text{inst}}_{\vec{Y}+} \equiv \frac{\prod_{s=1}^{n+1}\prod_{k=1}^{n+1} f_{Y_s}(\mu_k + a_s)}{\prod_{t,s=1}^{n+1} G_{Y_t,Y_s}(a_t - a_s)} \prod_{s=1}^{n+1} \left\{ (-1)^{|Y_s|} \sqrt{\frac{G_{Y_s,Y_s}(0)}{G_{Y_s,Y_s}(1-\beta)}} \right\},$$

$$N^{\text{Toda}}_{\vec{Y}\pm} \equiv \left\langle \prod_{a=1}^{n+1} j^{(\beta)}_{Y_a}(-r^{(a)}_k - \frac{v'_{a\pm}}{\beta}) \right\rangle_{\pm} = \prod_{a=1}^{n+1} \sqrt{\frac{G_{Y_a,Y_a}(0)}{G_{Y_a,Y_a}(1-\beta)}} \left\langle \prod_{a=1}^{n+1} J^{(\beta)}_{Y_a}(-r^{(a)}_k - \frac{v'_{a\pm}}{\beta}) \right\rangle_{\pm}$$

Known results on Selberg average

SU(2) **case:** The relevant Selberg averages for one and two Jack polynomials were obtained by Kadell.

$$\left\langle J_Y^{(\beta)}(p) \right\rangle_{u,v,\beta}^{SU(2)} \qquad \left\langle J_A^{(\beta)}(p+w) J_B^{(\beta)}(p) \right\rangle^{SU(2)}$$

SU(*n*+1) **case:**

The one-Jack Selberg integral for *SU*(*n*+1) could be calculated by the formula offered by Warnaar.

$$\left\langle J_B^{(\beta)}(p_k^{(n)}) \right\rangle_{\vec{u},\vec{v},\beta}^{SU(n+1)}$$

He also gives A_2 two Jack integral $\left\langle J_R^{(\beta)}(p_k^{(1)}) J_B^{(\beta)}(p_k^{(2)}) \right\rangle_{u,v,\beta}^{SU(3)}$

A conjecture on Selberg average

To evaluate we need Selberg average of (n + 1) Jack polynomials. While we do not perform the integration so far, we find a formula for β = 1 which reproduces known results and satisfies consistency conditions.

The Jack polynomial for β = 1 is called Schur polynomial. $J_Y^{(\beta)}|_{\beta=1} = \chi_Y.$

Conjecture : *We propose the following formula of Selberg average for n + 1 Schur polynomials,*

$$\begin{aligned}
&\left\langle \chi_{Y_1}(-p_k^{(1)} - v_1') \dots \chi_{Y_r}(p_k^{(r-1)} - p_k^{(r)} - v_r') \dots \chi_{Y_{n+1}}(p_k^{(n)}) \right\rangle_{\vec{u},\vec{v},\beta=1}^{SU(n+1)} \\
&= \prod_{s=1}^{n} \left\{ (-1)^{|Y_s|} \times \frac{[v_s + N_s - N_{s-1}]_{Y_s'}}{[N_s + N_{s-1}]_{Y_s'}} \times \prod_{1\le i<j\le N_{s-1}+N_s} \frac{(j-i+1)_{Y_{si}'-Y_{sj}'}}{(j-i)_{Y_{si}'-Y_{sj}'}} \right\} \times \prod_{1\le i<j\le N_n} \frac{(j-i+1)_{Y_{(n+1)i}-Y_{(n+1)j}}}{(j-i)_{Y_{(n+1)i}-Y_{(n+1)j}}} \\
&\times \prod_{1\le t<s\le n+1} \left\{ \frac{[v_t + u_t + \dots + u_{s-1} + N_t - N_{t-1}]_{Y_t'}}{[v_t - v_s + u_t + \dots + u_{s-1} + N_t - N_{t-1} - N_s]_{Y_t'}} \times \frac{[-v_s + u_t + \dots + u_{s-1} - N_s + N_{s-1}]_{Y_s}}{[v_t - v_s + u_t + \dots + u_{s-1} - N_{t-1} - N_s + N_{s-1}]_{Y_s}} \right. \\
&\left. \times \prod_{i=1}^{N_t} \prod_{j=1}^{N_{s-1}} \frac{v_t - v_s + u_t + \dots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} + 1 - (i+j)}{v_t - v_s + u_t + \dots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} + 1 + Y_{ti}' + Y_{sj} - (i+j)} \right\},
\end{aligned}$$

Proof

$$\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}} \left[x^{(r-1)} - x^{(r)} \right] \right\rangle_+ = \left. \frac{\prod_{s=1}^{n+1} (-1)^{|Y^{(s)}|} \prod_{r=1}^{n+1} \prod_{s=1}^{n+1} f_{Y^{(s)}}(\mu_r + a_s)}{\prod_{r,s=1}^{n+1} G_{Y^{(r)},Y^{(s)}}(a_r - a_s)} \right|_{\beta=1},$$

and the second part,

$$\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}} \left[y^{(r)} - y^{(r-1)} \right] \right\rangle_- = \left. \frac{\prod_{s=1}^{n+1} (-1)^{|Y^{(s)}|} \prod_{r=n+2}^{2n+2} \prod_{s=1}^{n+1} f_{Y^{(s)}}(\mu_r + a_s)}{\prod_{r,s=1}^{n+1} G_{Y^{(r)},Y^{(s)}}(a_r - a_s)} \right|_{\beta=1}.$$

Proof

Lemma 1

$$\prod_{1\leqslant i<j\leqslant l_Y}\frac{(Y_i-Y_j+\beta(j-i))_\beta}{(\beta(j-i))_\beta}\prod_{i=1}^{l_Y}\frac{1}{(\beta(l_Y-i+1))_{Y_i}}=\frac{1}{G_{Y,Y}(0)}$$

Lemma 2

$$\prod_{i=1}^{l_Y}(-z-\beta i+\beta)_{Y_i}=(-1)^{|Y|}f_Y(z)$$

Lemma 3

$$\prod_{i=1}^{l_Y}\prod_{j=1}^{l_W}\frac{Y_i-W_j-x+j-i}{-x+j-i}\prod_{i=1}^{l_Y}\frac{1}{(-x+l_W-i+1)_{Y_i}}\prod_{i=1}^{l_W}\frac{1}{(x+l_Y-i+1)_{W_i}}$$

$$=\left.\frac{(-1)^{|Y|+|W|}}{G_{Y,W}(x)G_{W,Y}(-x)}\right|_{\beta=1}.$$

Conclusion

In this paper, we complete the direct proof for A_n AGT correspondence in a particular case of $\beta = 1$, following the method proposed in [11] by Mironov et al. with the help of the strict formula for A_n Selberg integral of Schur polynomials proved in [15]. We provide two approaches for this proof. One is to confirm the conjecture given in [14], which can lead to AGT relation. The other is to transform the four-point function to Selberg integral and prove its equality to the Nekrasov partition function directly.

This work can be generalized in several ways. First, the simplest nontrivial case, $G = A_n$, with $N_f = 2n+2$ hypermultiplets in fundamental representation can be generalized to linear quiver gauge theory with gauge group $G = A_{n_1} \times \cdots \times A_{n_k}$. Secondly, for general β, even A_1 AGT correspondence has not been strictly proved because of the necessity of extra poles cancellation. The once term-wise relation of (2.9) for each Young diagram now needs to be summed over at general β. Morozov et al. [12, 13] defined the so-called generalized Jack polynomials to circumvent this issue. However, the Selberg integral for generalized Jack polynomials is still open and we may discover some other methods to avoid the extra poles problem. Besides, since AGT correspondence has an extension to 5D Gauge theory and 2D q-CFT, the proof of this case can also be addressed.