Holographic くりこみ群 と Weyl anomaly

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"A Note on the Weyl Anomaly in the Holographic RG"
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§1. Introduction

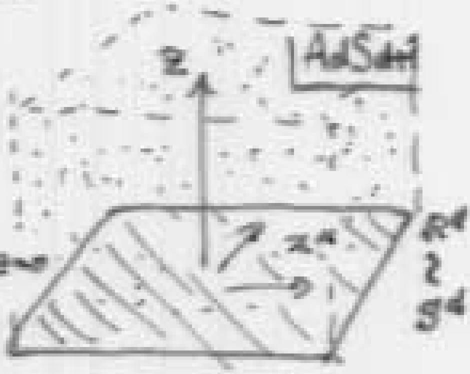
- <u>String theory</u>

  重力を含む統一理論の候補

- <u>AdS/CFT 対応</u> [Maldacena (hep-th/9711200), Gubser-Klebanov-Polyakov (th/9802109), Witten (th/9802150)]

  "string theory の consistency check"

[復習]



- <u>AdS_{d+1} 上の (super) gravity</u>

<u>metric</u>

$$ds^2_{d+1} = \frac{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}$$

$$= G_{MN}\, dx^M dx^N$$

$$\left(x^M = (x^\mu, z) \quad : \mu = 0, 1, \cdots, d-1 \right)$$

<u>action</u>

スカラー場

$$S_{d+1}[\phi_i(x,z)] = \int d^dx\, dz \sqrt{G} \left[-\frac{1}{2} G^{MN} \partial_M \phi_i \partial_N \phi_i - \frac{m_i^2}{2} \phi_i^2 - \cdots \right]$$

<u>classical soln</u>

$$\bar{\Phi}^i(x,z) \quad \text{with B.C.} \quad \bar{\Phi}^i(x, z=0) = \phi_0^i(x)$$

- <u>AdS_{d+1} / CFT_d 対応</u>

scaling op.

$$e^{-S_{d+1}[\bar{\Phi}(x,z)]} = e^{-S[\phi_0(x)]} = \left\langle e^{\int d^dx\, \phi_0^i(x) \mathcal{O}_i(x)} \right\rangle_{CFT}$$

classical (super) gravity on AdS_{d+1} $\longleftrightarrow$ CFT_d

(例: classical IIB SUGRA on $AdS_5 \times S^5$ $\longleftrightarrow$ $\mathcal{N}=4$ SYM_4)

[essence]

"asymptotic に AdS であれば 何でもよい"

実際,

$$\mathrm{AdS}_{d+1}: \quad ds^2 = G_{MN}\, dx^M dx^N$$

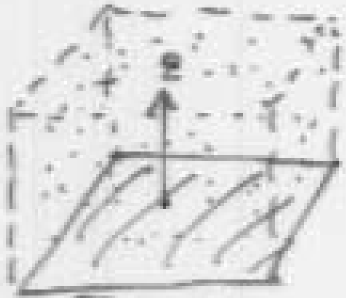
$$= \frac{1}{z^2}\left(\eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2 \right)$$

$$\downarrow$$

M_{d+1} (ただし asymptotic に AdS_{d+1}):

$$ds^2 = G_{\mu\nu}(x,z)\, dx^\mu dx^\nu + 2G_{\mu z}(x,z)\, dx^\mu dz + G_{zz}(x,z)\, dz^2$$

$$\text{with} \begin{cases} G_{\mu\nu}(x,z) = \dfrac{\eta_{\mu\nu}}{z^2} + O(z^0) \\ G_{\mu z}(x,z) = O(z) \\ G_{zz}(x,z) = \dfrac{1}{z^2} + O(z^0) \end{cases}$$



と指定されれば.

M_{d+1} 上の <u>diffeo</u> は 次の形:

$$x^M \longrightarrow x^M + \epsilon^M(x)$$

$\Sigma_d \to \Sigma_d$

$$\begin{cases} \epsilon^\mu(x,z) = \xi^\mu(x) + z^2 f^\mu(x) + O(z^4) \\ \epsilon^z(x,z) = z \cdot h(x) + O(z^3) \end{cases}$$

<u>isometry</u> は

$$\delta_\epsilon G_{\mu\nu} = \frac{1}{z^2}\left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2h\,\eta_{\mu\nu}\right) + O(z^0) \quad \equiv O(z^0)$$

$$\delta_\epsilon G_{\mu z} = \frac{1}{z}\left(\partial_\mu h + 2 f_\mu\right) + O(z) \quad \equiv O(z)$$

$$\delta_\epsilon G_{zz} = O(z^0) \quad \equiv O(z^0)$$

$$(\xi_\mu \equiv \eta_{\mu\nu}\,\xi^\nu)$$

∴ Killing eq for $(M_{d+1}, G_{\mu\nu})$ は

$$\begin{cases} \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2h\,\eta_{\mu\nu} = \frac{2}{d}(\partial_\lambda \xi^\lambda)\,\eta_{\mu\nu} & \leftarrow \text{conformal Killing for } \Sigma_d = R^d \\ f_\mu = -\frac{1}{2}\partial_\mu h \end{cases}$$

$(\xi_\mu = \eta_{\mu\nu}\xi^\nu)$

つまり

> $(M_{d+1}, G_{\mu\nu})$ を、asymptotic に AdS_{d+1} の空間とすると、
>
> $f : M_{d+1} \to M_{d+1}$ が isometry
>
> $\Rightarrow \varphi = f|_{\Sigma_d} : \Sigma_d \to \Sigma_d$ は conformal trsf.

[AdS_{d+1}/CFT_d 対応の"証明"]

今、$S_{d+1}[G(x,z), \phi(x,z)]$ を次の2つ性質をみたすものとする:

(1) S_{d+1} は $(d+1)$次元 diffeo で invariant:

$$S_{d+1}[f^*G(x,z), f^*\phi(x,z)] = S_{d+1}[G(x,z), \phi(x,z)]$$

(2) functional の値が 初期値のみによるような path が存在する:

$$\exists\, \bar{G}_{\mu\nu}(x,z),\ \bar{\Phi}(x,z)$$

$$\text{s.t. } S_{d+1}[\bar{G}(x,z), \bar{\Phi}(x,z)] = S[\bar{G}(x,z=0), \bar{\Phi}(x,z=0)]$$

(注: gravity の action は これをみたす. ただし 必ずしも local でなくてよい)

さらに、ここに M_{d+1} 上の isometry $f: M_{d+1} \to M_{d+1}$ について

$$S_{d+1}[\![f^*\bar{G}(x,z),\ f^*\bar{\Phi}(x,z)]\!] = S_{d+1}[\![\bar{G}(x,z),\ \bar{\Phi}(x,z)]\!]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$S_{d+1}[\![\bar{G}(x,z),\ f^*\bar{\Phi}(x,z)]\!]$$

$$\|$$

$$S[G(x),\ \rho^*\phi(x)] \qquad = \quad S[G(x),\ \phi(x)]$$

$$\left(G_{\mu\nu}(x) \sim \eta_{\mu\nu} \right)$$

ここで

$$\rho = f|_{\Sigma_d} \text{ は } \Sigma_d \text{ の conf. trsf.}$$

∴

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta^n S}{\delta\phi(x_1)\cdots\delta\phi(x_n)}$$

とすれば

$$\langle \rho^*\mathcal{O}(x_1) \cdots \rho^*\mathcal{O}(x_n) \rangle = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle$$

つまり $\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle$ は CFT の相関関数 //

Subtle points

(1) bulk の積分は $z=0$ で発散

→ $z \geq \epsilon$ に制限 (IR regulator)

(2) $\phi(x) \neq 0$ としても asymptotic な構造を AdS に保つ

→ relevant operator に制限

(3) $\bar{\Phi}(x,z)$ は $\phi(x) = \bar{\Phi}(x, z=0)$ だけでは一般に決まらないが、$z \to +\infty$ での正則性を仮定すれば一意に定まる。

> これらの systematic な解析 ⇒ holographic RG.

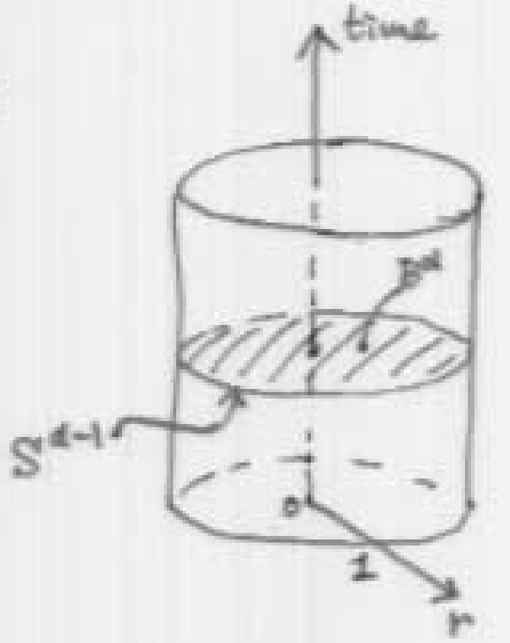
§2. Holography と UV/IR 対応

○ holographic principle

（時間1次元，空間 d 次元，$D=d+1$ 次元時空）

(1) 重力を含む $(d+1)$ 次元理論において
d 次元の空間領域は，
$(d-1)$ 次元の境界で記述可能

(2) その境界の理論は，
1 Planck 面積あたり 1 自由度の
情報を超えない。



（例）(static) BH の entropy

通常の system ： $S = k_B \ln W \propto V_d$ （示量性）

重力 〃 ： $S \propto A_{d-1}$ （正確には $S = \frac{1}{4G_{d+1}} A_{d-1}$）

○ AdS_{d+1} での holography ［Susskind-Witten (hep-th/9805114)］

$$ds^2_{d+1} = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i^2 + dz^2\right) \qquad (x^\mu, z) = (t, x_i, z)$$

$\downarrow$ $\qquad (t, r, \theta_i)$, $\theta_i \in S^{d-1}$ （$(d-1)$ 個）

$$ds^2_{d+1} = \ell^2\left[-\left(\frac{1+r^2}{1-r^2}\right)dt^2 + \frac{4}{(1-r^2)^2}\left(dr^2 + r^2 d\Omega^2_{d-1}\right)\right]$$

∴ 各時刻 t において

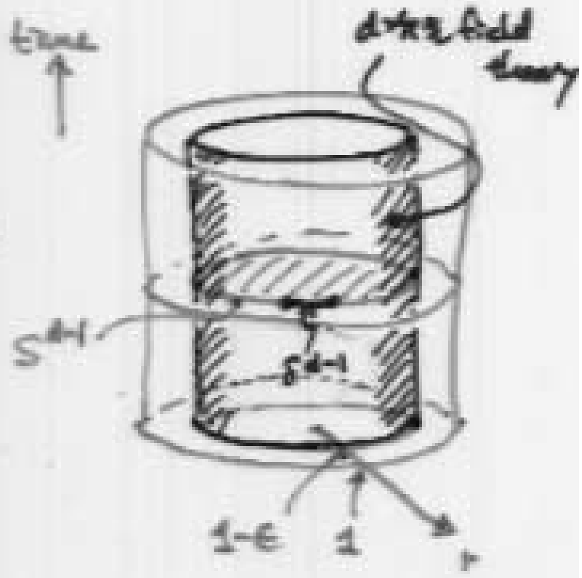
$\begin{cases} \text{space} \sim B^d \\ \text{boundary} \sim S^{d-1} \quad (r=1) \end{cases}$

ここで boundary の面積は発散

↓

自由度の counting をするために有限化

(bulk の IR 正則化)



これより

$$A_{d-1}(\epsilon) = \int_{r=1-\epsilon} d\Omega_{d-1}^{d-1} \left(\frac{4r^2}{(1-r^2)^2} \right)^{\frac{d-1}{2}}$$

$$\sim \left(\frac{\ell}{\epsilon} \right)^{d-1}$$

特に $d=4$ とし

$$A_3(\epsilon) \sim \left(\frac{\ell}{\epsilon} \right)^3$$

一方、4次元 SU(N) YM に UV cut off $\Lambda_0 = 1/\delta$ を入れれば。

entropy は　　　　(δ^3 毎に N^2 の自由度)

$$S_{YM} \sim N^2 \times \frac{1}{\delta^3}$$

これを

$$= \frac{1}{G_5} A_3(\epsilon)$$

とおくと、

$$\frac{1}{G_5} A_3(\epsilon) = \frac{\ell^5}{\kappa_0^2} \cdot \left(\frac{\ell}{\epsilon} \right)^3$$

$$= \frac{\ell^5}{g_{st}^2 \ell_s^8} \cdot \frac{\ell^3}{\epsilon^3}$$

$$= \frac{1}{g_{st}^2 \epsilon^3} \left(\frac{\ell}{\ell_s} \right)^8$$

$$\ell = (g_{st} N)^{1/4} \cdot \ell_s$$

$$= \frac{N^2}{\epsilon^3}$$

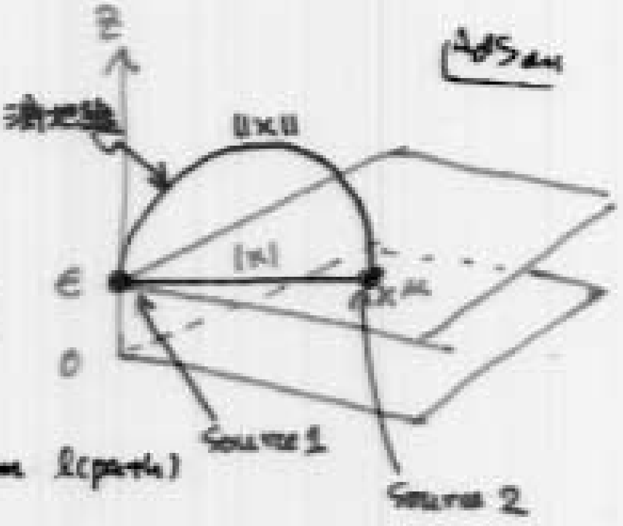
$\therefore \ \delta \sim \epsilon.$　　$\therefore$ **bulk の IR cut off = boundary の UV cut off**

【別の見方】

2点 $X_1^\mu=(0,\epsilon)$ と $X_2^\mu=(x^\mu,\epsilon)$ を結ぶ AdS_{d+1} 内での測地距離:

$$\|X\| = 2\ln\frac{1}{2\epsilon}\left(|x|+\sqrt{|x|^2+4\epsilon^2}\right)$$

$$\left(|x| = \eta_{\mu\nu}x^\mu x^\nu\right)$$



$$\therefore \quad \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_\epsilon \longleftrightarrow \sum_{\substack{\text{path}\\ \text{connecting}\\ X_1 \text{ and } X_2}} e^{-m\, l(\text{path})}$$

$$= e^{-m\|X\|} + \cdots$$

$$\sim \left(\frac{2\epsilon}{|x|+\sqrt{|x|^2+4\epsilon^2}}\right)^{2m}$$

$$\left(\sim\left(\frac{1}{|x|}\right)^{2m} \quad \text{when } |x| \gg \epsilon\right)$$

$\therefore$ ϵ は scaling を 破る[illegible] length-scale

$\therefore$ UV cut-off $\Lambda_0 \sim 1/\epsilon$.

§3. Hamilton-Jacobi constraint と flow equation

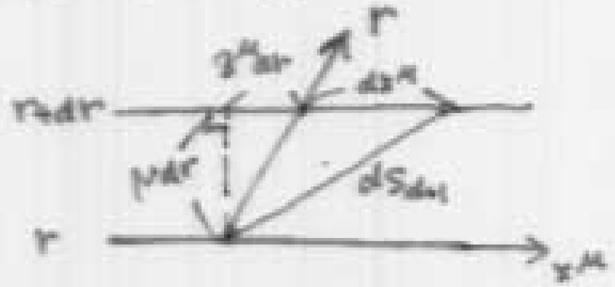
- 準備 1 : ADM分解 (Euclidian)

metric

$$ds^2_{d+1} = G_{MN}(x)\, dX^M dX^N \qquad (z = e^r)$$

$$= N^2 dr^2 + G_{\mu\nu}(dx^\mu + \lambda^\mu dr)(dx^\nu + \lambda^\nu dr)$$

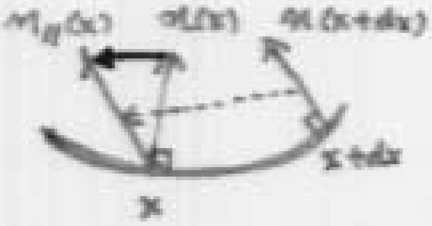
(N : lapse 関数
λ^μ : shift 〃)



このとき

$$\mathbb{R} = R - K_{\mu\nu}K^{\mu\nu} + K^2 - \nabla_L V^L$$

($\mathbb{R}$ ← G_{MN}, R ← $G_{\mu\nu}$; $\nabla_L V^L$: boundary term と cancel)



ここで

- $K_{\mu\nu} \equiv \nabla_\mu n_\nu$

$$= \frac{1}{2N}(\dot{G}_{\mu\nu} - \nabla_\mu \lambda_\nu - \nabla_\nu \lambda_\mu)$$

$$K^{\mu\nu} \equiv G^{\mu\lambda}G^{\nu\kappa}K_{\lambda\kappa}$$

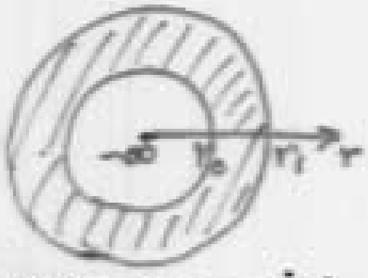
$$K \equiv G^{\mu\nu}K_{\mu\nu}$$

$$n^L = (n^r, n^\mu) = \left(\frac{1}{N}, -\frac{\lambda^\mu}{N}\right)$$

$$n_L = (n_r, n_\mu) = (N, 0)$$

- $V^L = 2(n^L \nabla_M n^M - n^M \nabla_M n^L)$

とくに $V^r = \dfrac{2K}{N}$



action

$$S_{d+1} = \int d^dx \int_{r_0}^{r_1} dr \sqrt{G}\left[V(\phi) - \mathbb{R} + \frac{1}{2}L_{ij}(\phi)G^{MN}\partial_M\phi^i\partial_N\phi^j\right]$$

$$-2\int_{r=r_0} d^dx \sqrt{G}\,K + 2\int_{r=r_1} d^dx \sqrt{G}\,K$$

$$= \int d^d x \int_{r_0}^{r_1} dr \sqrt{G} \Big[N \left(V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \right) + \frac{N}{4} K_{\mu\nu}^2 - \frac{1}{4} K^2 + \frac{1}{2N} L_{ij}(\phi) (\dot\phi^i - \lambda^\mu \partial_\mu \phi^i)(\dot\phi^j - \lambda^\mu \partial_\mu \phi^j) \Big]$$

(注：2階ビブンが入っていない ⇒ Dirichlet B.C. が課せる)

1st order form への書き換え：

$$S_{d+1}[G_{\mu\nu}, \phi^i ; \pi^{\mu\nu}, \pi_i ; N, \lambda^\mu]$$

$$= \int d^d x \int_{r_0}^{r_1} dr \sqrt{G} \left[\pi^{\mu\nu} \dot G_{\mu\nu} + \pi_i \dot\phi^i + N \mathcal{H} + \lambda^\mu \mathcal{P}_\mu \right]$$

with
$$\begin{cases} \mathcal{H} = \frac{1}{d-1} (\pi^\mu_\mu)^2 - \pi_{\mu\nu}^2 - \frac{1}{2} L^{ij}(\phi) \pi_i \pi_j + V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \\ \mathcal{P}_\mu = 2 \nabla^\nu \pi_{\mu\nu} - \pi_i \partial_\mu \phi^i \end{cases}$$

実際、$\pi^{\mu\nu}$, π_i についてのEOM

$$\to \begin{cases} \pi_{\mu\nu} = K_{\mu\nu} - K G_{\mu\nu} \\ \pi^i = \frac{1}{N} (\dot\phi^i - \lambda^\mu \partial_\mu \phi^i) \end{cases}$$

→ これらを代入すると元の action を再現

さらに

$$\begin{cases} \{ \pi^{\mu\nu}(x), G_{\lambda\kappa}(y) \} = \frac{1}{2} (\delta^\mu_\lambda \delta^\nu_\kappa + \delta^\mu_\kappa \delta^\nu_\lambda) \delta^d(x-y) \\ \{ \pi_i(x), \phi^j(y) \} = \delta_i^j \delta^d(x-y) \end{cases}$$

$$\Rightarrow \quad \{ \mathcal{H}(x), \mathcal{H}(y) \} = \partial_\mu \mathcal{P}^\mu(x) \delta^d(x-y)$$

$$\{ \mathcal{P}_\mu(x), \mathcal{H}(y) \} = \mathcal{H}(x) \partial_\mu \delta^d(x-y)$$

$$\{ \mathcal{P}_\mu(x), \mathcal{P}_\nu(y) \} = (\mathcal{P}_\mu(y) \partial_\nu - \mathcal{P}_\nu(x) \partial_\mu) \delta^d(x-y)$$

∴ <u>1st class constraint</u>

・準備2: Hamilton-Jacobi 方程式 (1st class constraint のあるとき)

・constrained action (1st order form)

$$S[q(t), p(t), \lambda(t)] \equiv \int dt \left[p_c(t)\dot{q}^c(t) - H(q(t), p(t), t) + \lambda^a(t)\Phi_a(q(t), p(t), t) \right]$$

・EOM

$$\delta S \Big|_{q=\bar{q},\ p=\bar{p},\ \lambda=\bar{\lambda}} = 0 \quad \text{より}$$

$$\begin{cases} \dot{\bar{q}}^c(t) = \partial_{p_c} H(\bar{q}(t), \bar{p}(t), t) - \bar{\lambda}^a(t)\partial_{p_c}\Phi_a(\bar{q}(t), \bar{p}(t), t) \\ \dot{\bar{p}}_c(t) = -\partial_{q^c} H(\bar{q}(t), \bar{p}(t), t) + \bar{\lambda}^a(t)\partial_{q^c}\Phi_a(\bar{q}(t), \bar{p}(t), t) \\ \Phi_a(\bar{q}(t), \bar{p}(t), t) = 0 \\ \bar{\lambda}^a(t) : \text{任意} \end{cases}$$

対応表

t	$\longleftrightarrow$	r
$q^c(t)$	$\longleftrightarrow$	$G_{\mu\nu}(x,r),\ \phi^i(x,r)$
$p_c(t)$	$\longleftrightarrow$	$\pi^{\mu\nu}(x,r),\ \pi_i(x,r)$
H	$\longleftrightarrow$	$H=0$
λ^a	$\longleftrightarrow$	$N,\ \lambda^\mu$
Φ_a	$\longleftrightarrow$	$\mathcal{H},\ P_\mu$

・1st class constraint とは:

$$\{H, \Phi_a\} = \sum_b C_{ab}\Phi_b$$

$$\{\Phi_a, \Phi_b\} = \sum_c C_{ab}{}^c\Phi_c$$

(C: const)

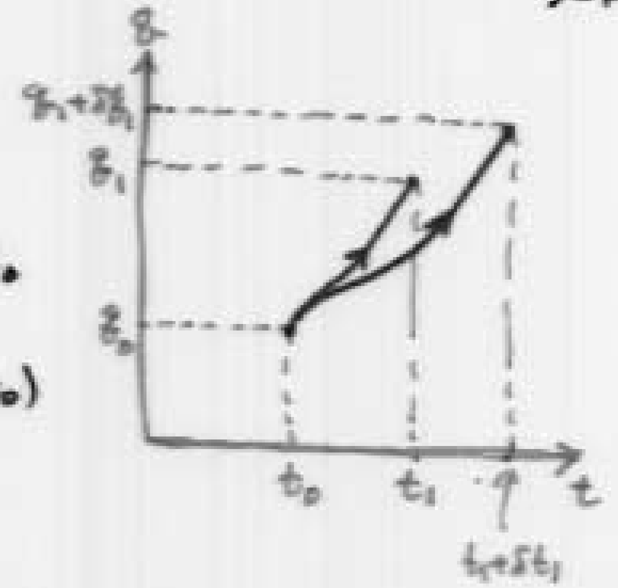
$\Rightarrow$ λ^a の値に関わらず、

up to gauge 変換 で 一意に定まる

$$F(q,p) \sim e^{\xi^a\Phi_a(q,p)}\, F(q,p)\, e^{-\xi^a\Phi_a(q,p)}$$

$$= F(q,p) + \xi^a\{\Phi_a(q,p), F(q,p)\} + \cdots$$

- <u>classical action</u>



cl. soln $\bar{q}(t)$ with the B.C. :

$$\bar{q}(t_1)=q_1,\quad \bar{q}(t_0)=q_0$$

$$\Rightarrow\quad \bar{q}(t)=\bar{q}(t\,;q_1,t_1\,;q_0,t_0)$$

これを S に代入（t_1, t_0 を変数とする）:

$$S[q_1,t_1\,;q_0,t_0]\equiv S[\![\bar{q}(t;q_1,t_1;q_0,t_0),\ \bar{p}(t),\ \bar{\lambda}(t)]\!]$$

↑ B.C.から $\bar{p}$ 決まる (see below)

$$=\int_{t_0}^{t_1}dt\,[\,\bar{p}(t)\dot{\bar{q}}(t)-H(\bar{q}(t),\bar{p}(t),t)\,]$$

$$\left(\Phi_a(\bar{q},\bar{p})=0\ \text{を用いた}\right)$$

すると

$$\delta S(q_1,t_1\,;q_0,t_0)$$

$$=\left(\bar{p}(t_1)\dot{\bar{q}}(t_1)-H(q_1,\bar{p}(t_1),t_1)\right)\delta t_1-\left(\bar{p}(t_0)\dot{\bar{q}}(t_0)-H(q_0,\bar{p}(t_0),t_0)\right)\delta t_0$$

$$+\int_{t_0}^{t_1}dt\left[(\bar{p}(t)\delta\bar{q}(t))^{\cdot}+\delta\bar{p}\,\underset{0}{\underset{\shortparallel}{(\dot{\bar{q}}-\partial_p\bar{H})}}-\delta\bar{q}\,\underset{0}{\underset{\shortparallel}{(\dot{\bar{p}}+\partial_q\bar{H})}}\right]$$

$$=\left(\underline{\bar{p}(t_1)\dot{\bar{q}}(t_1)}-H(q_1,\bar{p}(t_1),t)\right)\delta t_1+\bar{p}(t_1)\,\underline{\delta\bar{q}(t_1)}$$

$$-\begin{pmatrix}t_1\leftrightarrow t_0\\ q_1\leftrightarrow q_0\end{pmatrix}$$

ここで $t\sim t_1$ で

$$\bar{q}(t)=\underset{q_1}{\overset{}{\bar{q}(t_1)}}+\dot{\bar{q}}(t_1)(t-t_1)+O((t-t_1)^2)$$

$$\therefore\ \delta\bar{q}(t)=\delta q_1-\dot{\bar{q}}(t_1)\,\delta t_1+O(t-t_1)$$

$$\therefore\ \underline{\delta\bar{q}(t_1)}=\delta q_1-\underline{\dot{\bar{q}}(t_1)\,\delta t_1}$$

$\therefore$

$$\delta S(q_1,t_1;q_0,t_0) = \bar{p}(t_1)\,\delta q_1 - H(q_1,\bar{p}(t_1),t_1)\,\delta t_1$$
$$- \bar{p}(t_0)\,\delta q_0 + H(q_0,\bar{p}(t_0),t_0)\,\delta t_0$$

$\therefore$

<u>H-J eqn</u>:

$$\begin{cases} \bar{p}(t_1) = \dfrac{\partial S}{\partial q_1}(q_1,t_1;q_0,t_0) \\ \dfrac{\partial S}{\partial t_1}(q_1,t_1;q_0,t_0) = -H(q_1,\bar{p}(t_1),t_1) \\ \qquad = -H\left(q_1,\dfrac{\partial S}{\partial q_1},t_1\right) \\ \Phi_a(q_1,\bar{p}(t_1),t_1) = 0 \end{cases}
\qquad
\begin{cases} \bar{p}(t_0) = -\dfrac{\partial S}{\partial q_0}(q_1,t_1;q_0,t_0) \\ \dfrac{\partial S}{\partial t_0}(q_1,t_1;q_0,t_0) = +H(q_0,\bar{p}(t_0),t_0) \\ \qquad = +H\left(q_0,-\dfrac{\partial S}{\partial q_0},t_0\right) \\ \Phi_a(q_0,\bar{p}(t_0),t_0) = 0 \end{cases}$$

ex: <u>H=0</u> as Σ

$$\frac{\partial S}{\partial t_1} = \frac{\partial S}{\partial t_0} = 0$$

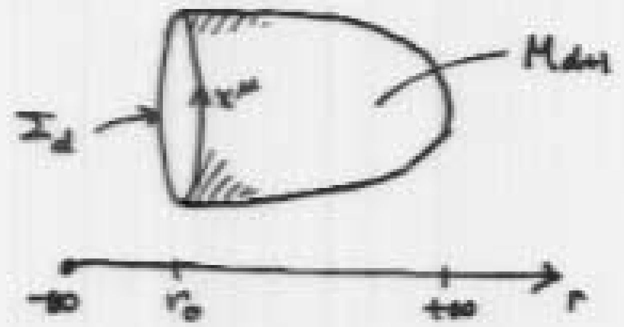
$\therefore$

$$S = S(q_1;q_0)$$

$$\begin{cases} \bar{p}(t_1) = \dfrac{\partial S}{\partial q_1} \\ \Phi_a(q_1,\bar{p}(t_1),t_1) = 0 \end{cases}
\qquad
\begin{cases} \bar{p}(t_0) = -\dfrac{\partial S}{\partial q_0} \\ \Phi_a(q_0,\bar{p}(t_0),t_0) = 0 \end{cases}$$

: Hamilton-Jacobi constraints

• **本題: flow equation** [de Boer-Verlinde-Verlinde (hep-th/9912012)]



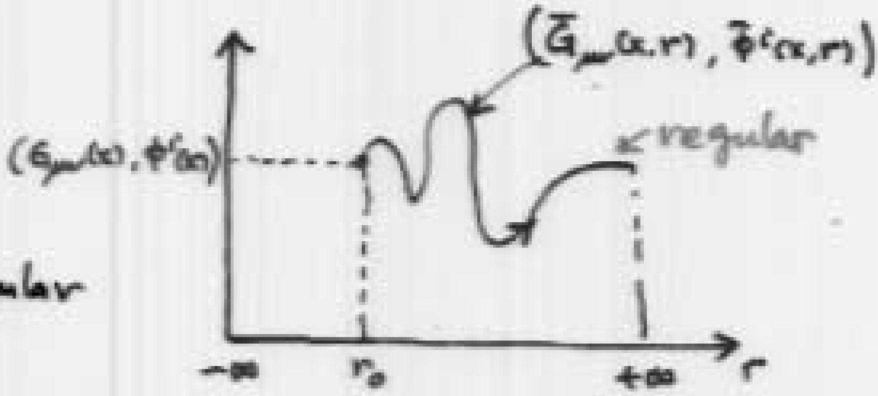
今、古典解

$\bar{G}_{\mu\nu}(x,r)$, $\bar{\phi}^i(x,r)$

に、次の B.C. を課す:

$$(1)\begin{cases}\bar{G}_{\mu\nu}(x,r_0)=G_{\mu\nu}(x)\\ \bar{\phi}^i(x,r_0)=\phi^i(x)\end{cases}$$

(2) $(\bar{G},\bar{\phi})$ は、M_{d+1} の内部で regular



→ 解が、$(G_{\mu\nu}(x),\phi^i(x))$ によって一意に定まる。

⇓

$$\left(\bar{G}_{\mu\nu}(x,r;G(x),r_0)\ ,\ \ \bar{\phi}^i(x,r;\phi(x),r_0)\right)$$

∴ **classical action**

$$S[G_{\mu\nu}(x),\phi^i(x)]\equiv S_{d+1}[\![\bar{G}_{\mu\nu}(x,r;G(x),r_0),\ \bar{\phi}^i(x,r;\phi(x),r_0);r_0]\!]$$

これは次を満たす:

$$\begin{cases}\mathcal{H}=\frac{1}{d-1}(\pi^\mu_\mu)^2-\pi_{\mu\nu}^2-\frac{1}{2}L^{ij}(\phi)\pi_i\pi_j+\mathcal{L}_d\\ P_\mu=2D^\nu\pi_{\mu\nu}-\pi_i\partial_\mu\phi^i \qquad \left(\mathcal{L}_d=V(\phi)-R+\frac{1}{2}L_{ij}(\phi)G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right)\\ \pi^{\mu\nu}(x)=\bar{\pi}^{\mu\nu}(x,r_0)=-\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}(x)}\\ \pi_i(x)=\bar{\pi}_i(x,r_0)=-\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^i(x)}\end{cases}$$

次の記号を導入:

$$\{S_1, S_2\}(x) \equiv \frac{1}{\sqrt{G(x)}}\Big[-\frac{1}{d-1}\, G_{\mu\nu}\frac{\delta S_1}{\delta G_{\mu\nu}(x)} G_{\lambda\kappa}\frac{\delta S_2}{\delta G_{\lambda\kappa}(x)}$$

$$+\, G_{\mu\lambda}G_{\nu\kappa}\frac{\delta S_1}{\delta G_{\mu\nu}(x)}\frac{\delta S_2}{\delta G_{\lambda\kappa}(x)}$$

$$+\,\frac{1}{2}L^{ij}(\phi(x))\frac{\delta S_1}{\delta\phi^i(x)}\frac{\delta S_2}{\delta\phi^j(x)}\Big]$$

すると、

H-J constraint

"flow equation"

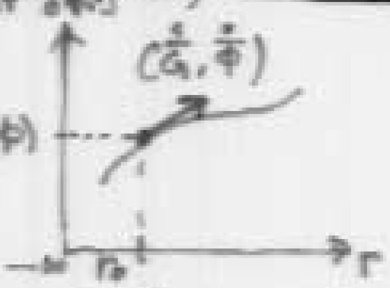
$$\Leftrightarrow \quad \bullet\ \{S, S\}(x) = \sqrt{G(x)}\,\mathcal{L}_d(x) = \sqrt{G(x)}\Big(V(\phi) - R + \frac{1}{2}L_{ij}(\phi)G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\Big)$$

$$\bullet\ 2\nabla_\nu\Big(\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}}\Big) - \frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^i}\nabla^\mu\phi^i = 0$$

$$\Big(\int d^dx\,\frac{\xi}{\sqrt{G}}\Big[(\nabla_\mu G_{\nu\lambda} + \nabla_\nu G_{\mu\lambda})\frac{\delta S}{\delta G_{\mu\nu}} + G^{\mu\nu}\partial_\mu\phi^i\frac{\delta S}{\delta\phi^i}\Big] = 0\Big)$$

さらに "temporal gauge" をとると

$$N \equiv 1, \quad \lambda^\mu \equiv 0$$



$$\Rightarrow \begin{cases} \pi^{\mu\nu}(x) = K^{\mu\nu}(x) - K(x)\,G^{\mu\nu}(x) & \Big(K_{\mu\nu} = \frac{1}{2}\dot G_{\mu\nu}(x, r_0)\Big) \\ \pi_i(x) = L_{ij}\,\dot\phi(x, r_0) \end{cases}$$

$$\therefore \begin{cases} \dot G_{\mu\nu}(x, r_0) = \pi_{\mu\nu}(x) - \frac{1}{d-1}\pi^\lambda_\lambda(x)\,G_{\mu\nu}(x) \\ \dot\phi^i(x, r_0) = L^{ij}(\phi(x))\,\pi_j(x) \end{cases} \Leftrightarrow \text{RG 方程式}$$

with $\pi^{\mu\nu}(x) \equiv -\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}(x)}, \quad \pi_i \equiv -\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^i(x)}$

§4. Holographic RG

- <u>flow equation</u> [dVV]

$$\{S,S\}(x) = \frac{1}{\sqrt{G}}\left[-\frac{1}{d-1}\left(G_{\mu\nu}\frac{\delta S}{\delta G_{\mu\nu}(x)}\right)^2 + \left(\frac{\delta S}{\delta G_{\mu\nu}(x)}\right)^2 + \frac{1}{2}L^{ij}(\phi)\frac{\delta S}{\delta\phi^i(x)}\frac{\delta S}{\delta\phi^j(x)}\right]$$

$$\parallel$$

$$\sqrt{G}\,\mathcal{L}_d(x) = \sqrt{G}\left[V(\phi) - R + \frac{1}{2}L_{ij}(\phi)G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right]$$

- <u>解法</u> ("derivative expansion") [dVV] [M.F.-Matsuura-Sakai]

  - 次の形を仮定

$$S[G(x),\phi(x)] = \underset{\text{local}}{S_{loc}[G(x),\phi(x)]} + \underset{\text{non-local}}{\Gamma[G(x),\phi(x)]}$$

with

$$S_{loc}[G,\phi] = \int d^dx\sqrt{G}\,\mathcal{L}_{loc}(x) = \int d^dx\sqrt{G}\sum_{wt=0,2,4,\cdots}\left[\mathcal{L}_{loc}(x)\right]_{wt}$$

ここで $\sum_{wt}$ は、次で定める "<u>weight</u>" に関する展開：

	wt
$G_{\mu\nu}(x)$, $\phi(x)$, $\Gamma[G(x),\phi(x)]$	0
∂_μ	1
$R_{\mu\nu}$, R, $\partial_\mu\phi^i\partial_\nu\phi^j$, $\cdots$	2
$\frac{\delta\Gamma}{\delta\phi^i(x)}$, $\frac{\delta\Gamma}{\delta G_{\mu\nu}(x)}$, $\cdots$	d

$$\left(\because\ \underset{0}{\delta\Gamma} = \int \underset{-d}{d^dx}\left(\underset{0}{\delta\phi(x)}\underset{+d}{\frac{\delta\Gamma}{\delta\phi(x)}} + \cdots\right)\right)$$

一方.

$$\mathcal{L}_d = \underbrace{V(\phi)}_{wt=0} \underbrace{- R + \frac{1}{2} L_{ij}(\phi)\,\partial\phi^i\partial\phi^j}_{wt=2}$$

これより

flow equation は wt について逐次的に解ける

実際

$$\sqrt{G}\,\mathcal{L}_d(x) = \underbrace{\{S_{loc}, S_{loc}\}(x)}_{wt=0,2,\cdots} + 2\underbrace{\{S_{loc}, \Gamma\}(x)}_{wt=d,d+2,\cdots} + \underbrace{\{\Gamma, \Gamma\}(x)}_{wt=2d}$$

($\sqrt{G}\,\mathcal{L}_d$: $wt=0,2$)

より

$$\sqrt{G}\,\mathcal{L}_d = [\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2 \quad \text{——Ⓐ}$$

$$0 = [\{S_{loc}, S_{loc}\}]_w \quad (w = 4, 6, \cdots, d-2) \quad \text{——Ⓑ}$$

$$0 = [\{S_{loc}, S_{loc}\}]_d + 2[\{S_{loc}, \Gamma\}]_d \quad \text{——Ⓒ}$$

$$\vdots$$

∴

Ⓐ, Ⓑ ⇒ $[\mathcal{L}_{loc}]_0, [\mathcal{L}_{loc}]_2, \cdots, [\mathcal{L}_{loc}]_{d-2}$ が決まる.

Ⓒ ⇒ Γ が決まる

([注] $[\{S_{loc}, S_{loc}\}]_d$ には $[\mathcal{L}_{loc}]_d$ の寄与もあるが, 実際には影響を与えない.)

以下しばらく Ⓐ と Ⓒ に着目する.

Ⓐ :

$$\sqrt{G}\,\mathcal{L}_d = [\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2$$

ここで

$$\mathcal{L}_d = \underbrace{V(\phi)}_{wt=0} \underbrace{- R + \frac{1}{2} L_{ij}(\phi)\,\partial\phi^i \partial\phi^j}_{wt=2}$$

一方

$$[\mathcal{L}_{loc}]_0 = W(\phi)$$

$$[\mathcal{L}_{loc}]_2 = -R\cdot\Phi(\phi) + \frac{1}{2} M_{ij}(\phi)\,\partial\phi^i\partial\phi^j$$

とすると.

$$[\{S_{loc}, S_{loc}\}]_0 = \sqrt{G}\left(\frac{1}{2} L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j W(\phi) - \frac{d}{4(d-1)} W(\phi)^2\right)$$

$$[\{S_{loc}, S_{loc}\}]_2 = \sqrt{G}\Big(R\cdot\Big(\frac{d-2}{2(d-1)} W\cdot\Phi - L^{ij}\,\partial_i W\,\partial_j\Phi\Big)$$

$$+ \Big(-\frac{d-2}{4(d-1)} W M_{ij} - L^{kl}\,\partial_k W\,\Gamma^{(M)}_{l,ij}\Big)\partial\phi^i\partial\phi^j$$

$$- d(d-2)\,W\,\nabla^2\Phi - L^{ij}\,\partial_i W\,M_{jk}\,\nabla^2\phi^k\Big)$$

係数比べて

$$V(\phi) = \frac{1}{2} L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j W(\phi) - \frac{d}{4(d-1)} W(\phi)^2$$

$$-1 = \frac{d-2}{2(d-1)} W(\phi)\,\Phi(\phi) - L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j\Phi(\phi)$$

$$\frac{1}{2} L_{ij}(\phi) = -\frac{d-2}{4(d-1)} W(\phi)\,M_{ij}(\phi) - L^{kl}(\phi)\,\partial_k W(\phi)\,\Gamma^{(M)}_{l,ij}(\phi)$$

$$0 = W(\phi)\,\nabla^2\Phi(\phi) + L^{ij}(\phi)\,\partial_i W(\phi)\,M_{jk}(\phi)\,\nabla^2\phi^k$$

(c) : $$\boxed{0 = [\{S_{loc}, S_{loc}\}]_d + 2\,[\{S_{loc}, P\}]_d}$$

ここで

$$2\,[\{S_{loc}, P\}] = \frac{2}{\sqrt{G}}\Big[-\frac{1}{d-1}\, G_{\lambda\kappa}\frac{\delta S_{loc}}{\delta G_{\lambda\kappa}} \cdot G_{\mu\nu}\frac{\delta P}{\delta G_{\mu\nu}} + G_{\mu\lambda}G_{\nu\kappa}\frac{\delta S_{loc}}{\delta G_{\lambda\kappa}}\frac{\delta P}{\delta G_{\mu\nu}} + \frac{1}{2} L^{ij}(\phi)\frac{\delta S_{loc}}{\delta \phi^j}\frac{\delta P}{\delta \phi^i}\Big]$$

$$= \gamma\Big[-2\,\underset{0}{G_{\mu\nu}}\underset{d}{\frac{\delta P}{\delta G_{\mu\nu}}} + B_{\mu\nu}\underset{d}{\frac{\delta P}{\delta G_{\mu\nu}}} + B^i \underset{d}{\frac{\delta P}{\delta \phi^i}}\Big]$$

with

$$(G^{\mu\nu} B_{\mu\nu} \equiv 0)$$

$$\begin{cases} \gamma \equiv \dfrac{1}{d(d-1)}\dfrac{1}{\sqrt{G}}\, G_{\mu\nu}\dfrac{\delta S_{loc}}{\delta G_{\mu\nu}} \\[2ex] \gamma B_{\mu\nu} \equiv \dfrac{2}{\sqrt{G}}\Big(G_{\mu\lambda}G_{\nu\kappa} - \dfrac{1}{d}G_{\mu\nu}G_{\lambda\kappa}\Big)\dfrac{\delta S_{loc}}{\delta G_{\lambda\kappa}} \\[2ex] \gamma B^i \equiv \dfrac{1}{\sqrt{G}}\, L^{ij}(\phi)\dfrac{\delta S_{loc}}{\delta \phi^j} \end{cases}$$

∴

$$2\,[\{S_{loc}, P\}]_d = [\gamma]_0 \Big(-2\,G_{\mu\nu}\frac{\delta P}{\delta G_{\mu\nu}} + [B_{\mu\nu}]_0 \frac{\delta P}{\delta G_{\mu\nu}} + [B^i]_0 \frac{\delta P}{\delta \phi^i}\Big)$$

ここで

$$[\gamma]_0 = \frac{1}{2(d-1)}\, W(\phi)$$

$$[B_{\mu\nu}]_0 = 0$$

$$[\gamma B^i]_0 = L^{ij}(\phi)\,\partial_j W(\phi) \;\equiv [\gamma]_0 \cdot \beta^i(\phi)$$

従って、Γが満たす式は

$$[T]_0 \times \left(-2 G_{\mu\nu} \frac{\delta \Gamma}{\delta G_{\mu\nu}} + \beta^i(\phi) \frac{\delta \Gamma}{\delta \phi^i} \right) = -\left[\{ S_{loc}, S_{loc} \} \right]_d$$

ここで

$$\begin{cases} [T]_0 = \dfrac{1}{2(d-1)} W(\phi) \\ \beta^i(\phi) = \dfrac{2(d-1)}{W(\phi)} L^{ij}(\phi)\, \partial_j W(\phi) \end{cases}$$

○ くりこみ群方程式としての解釈

$$\begin{cases} G_{\mu\nu}(x) \longrightarrow a^{-2} \delta_{\mu\nu} \\ \phi^i(x) \longrightarrow \phi^i \ (\text{const}) \end{cases}$$

↑ local source　　　↑ finite perturbation

$$\left(\begin{aligned} ds^2 &= dr^2 + G_{\mu\nu}(x)\, dx^\mu dx^\nu \\ &= dr^2 + \frac{1}{a^2} \eta_{\mu\nu}\, dx^\mu dx^\nu \\ \therefore\ & a \to \lambda a \iff x^\mu \to \lambda x^\mu \end{aligned} \right.$$

これより

$$\Gamma[a^{-2} \delta_{\mu\nu}, \phi^i] = \Gamma(\phi^i, a)$$

また

$$\int dx\ 2 G_{\mu\nu} \frac{\delta \Gamma}{\delta G_{\mu\nu}} \bigg|_{G_{\mu\nu} = a^{-2}\delta_{\mu\nu}} = -a \frac{\partial}{\partial a} \Gamma(\phi^i, a)$$

$$\int dx\ \frac{\delta \Gamma}{\delta \phi(x)} \bigg|_{\phi(x) = \phi^i} = \frac{\partial}{\partial \phi^i} \Gamma(\phi^i, a)$$

∴

$$\text{Ⓒ} \iff \left(a\frac{\partial}{\partial a} + \beta^i(\phi)\frac{\partial}{\partial \phi^i} \right) \Gamma(\phi, a) = 0$$

従って 今 ϕ に a-dependence を

$$a\frac{d\phi^i}{da} = \beta^i(\phi)$$

をみたすように持たせれば

$$a\frac{d}{da}\Gamma(\phi(a), a) = 0 \qquad \left(\beta^i(\phi) \text{ は beta function} \right)$$

より一般に.

$$\text{Ⓒ} \Rightarrow \int d^dx \frac{\delta}{\delta\phi^{i_1}(x_1)} \cdots \frac{\delta}{\delta\phi^{i_m}(x_m)} \left(-2G_{\mu\nu}(x)\frac{\delta}{\delta G_{\mu\nu}(x)} + \beta^i \frac{\delta}{\delta\phi^i(x)} \right) \Gamma \Bigg|_{\begin{cases} G_{\mu\nu}(x) = a^{-2}\delta_{\mu\nu} \\ \phi^i(x) = \phi^i \end{cases}}$$

$$= (\text{local}) = 0$$

$$\iff \boxed{\begin{aligned} 0 = {} & \left(a\frac{\partial}{\partial a} + \beta^i(\phi)\frac{\partial}{\partial \phi^i} \right) \langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_m}(x_m) \rangle \\ & + \sum_{s=1}^{m} \gamma_{i_s}{}^{j_s}(\phi) \langle \mathcal{O}_{i_1}(x_1) \cdots \overset{s}{\underset{\vee}{\mathcal{O}_{j_s}(x_s)}} \cdots \mathcal{O}_{i_m}(x_m) \rangle \end{aligned}}$$

ここで

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_m}(x_m) \rangle \equiv \frac{\delta}{\delta\phi^{i_1}(x_1)} \cdots \frac{\delta}{\delta\phi^{i_m}(x_m)} \Gamma \Bigg|_{\substack{G_{\mu\nu}(x) = a^{-2}\delta_{\mu\nu} \\ \phi^i(x) = \phi^i}}$$

$$\gamma_i{}^j(\phi) \equiv \frac{\partial \beta^j(\phi)}{\partial \phi^i}$$

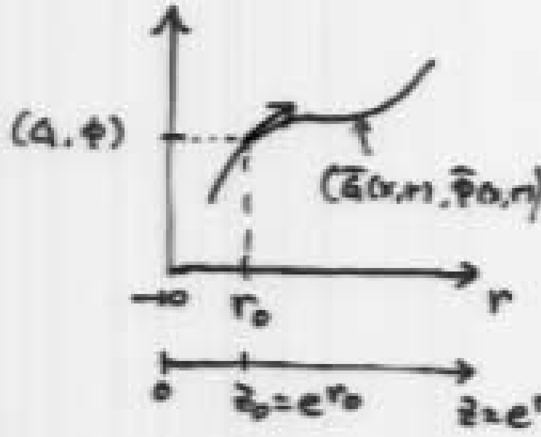
: Callan-Symanzik eqn.

[3章]

RG-flow $\Longleftrightarrow$ classical trajectory

$$\begin{cases} G_{\mu\nu}(x) = \frac{1}{a^2}\delta_{\mu\nu} \\ \phi^i(x) = \phi^i \end{cases}$$

← "classical soln $(\bar G_{\mu\nu}(x,r),\ \bar\phi^i(x,r))$ の $r=r_0$ における値"



一方、$r=r_0$ での「速度」は

- $$\frac{d}{dr}\bar G_{\mu\nu}(x,r;G,r_0)\Big|_{r=r_0}$$

$$= \Pi_{\mu\nu} - \frac{1}{d-1}\Pi^{\lambda}_{\lambda} G_{\mu\nu}$$

$$= -\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}}\Big|_{G_{\mu\nu}=\frac{1}{a^2}\delta_{\mu\nu}} + \frac{1}{d-1}\frac{1}{\sqrt{G}}G_{\lambda\kappa}\frac{\delta S}{\delta G_{\lambda\kappa}}G_{\mu\nu}\Big|_{G_{\mu\nu}=\frac{1}{a^2}\delta_{\mu\nu}}$$

$$= \frac{1}{d-1}W(\phi)\cdot\frac{1}{a^2}\delta_{\mu\nu}$$

- $$\frac{d}{dr}\bar\phi^i(x,r;\phi,r_0)\Big|_{r=r_0} = L^{ij}(\phi)\,\Pi_j$$

$$= -L^{ij}(\phi)\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^j}\Big|_{\phi(x)=\phi}$$

$$= -L^{ij}(\phi)\,\partial_j W(\phi)$$

従って、

$$\begin{cases} \bar G_{\mu\nu}(x,r;G,r_0) = \frac{1}{a(r)^2}\delta_{\mu\nu} \\ \bar\phi^i(x,r;\phi,r_0) = \phi^i(a(r)) \end{cases}$$

とすれば。

$$-\frac{2}{a^3}\,\dot{a}(r) = \frac{W}{d-1}\,\frac{1}{a^2}$$

$$\therefore\ a\frac{dr}{da} = -\frac{2(d-1)}{W(\phi)}$$

$$\therefore\ a\frac{d}{da}\phi^i(a) = a\frac{dr}{da}\,\dot{\phi}^i(r)$$

$$= +\frac{2(d-2)}{W(\phi)}\,L^{ij}(\phi)\,\partial_j W(\phi)$$

$$= \beta^i(a)$$ （場のβ関数と一致）.

- **scaling 次元と 3点関数の決定**

(d+1)次元 bulk action:

$$S_{d+1} = \int d^dx\, dr \sqrt{G} \left[V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \right]$$

↓

$$\mathcal{L}_d^{(0)} = V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j$$

これについて、次の形を仮定:

$$(*) \begin{cases} V(\phi) = 2\Lambda + \frac{1}{2} m_i^2 \phi_i^2 + g_{ijk} \phi_i \phi_j \phi_k + \cdots \\ 2\Lambda = -\dfrac{d(d-1)}{l^2} \quad (l: \text{“}AdS_{d+1}\text{ の半径”}) \\ L_{ij}(\phi) = \delta_{ij} \end{cases}$$

$$\left(S[g,\phi] = \underbrace{S_{loc}[g,\phi]}_{= \int d^dx \sqrt{g} [W(\phi) - R\,\Phi(\phi) + \cdots]} + \Gamma[g,\phi] \right)$$

一方、$W(\phi)$ を次の形に展開する:

$$W(\phi) = -\frac{2(d-1)}{l} + \frac{1}{2} \lambda_i \phi_i^2 + \lambda_{ijk} \phi_i \phi_j \phi_k + \cdots$$

これを関係式

$$V(\phi) = \frac{1}{2} \left(\partial_i W(\phi) \right)^2 - \frac{d}{4(d-1)} \left(W(\phi) \right)^2$$

に代入して、(*) と比較する。

⇓

結果

$$
\begin{cases}
m_i^2 = \lambda_i^2 + d\dfrac{\lambda_i}{\ell} & \longrightarrow \text{まず } \lambda_i \text{ が求まる} \\[2ex]
g_{ijk} = \left(\lambda_i + \lambda_j + \lambda_k + \dfrac{d}{\ell}\right)\lambda_{ijk} & \longrightarrow \lambda_{ijk} \text{ が求まる.}
\end{cases}
$$

実際にとくと

$$\lambda_i = -\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m_i^2\ell^2}$$

一方

$$\beta_i(\phi) = \frac{2(d-1)}{W(\phi)}\partial_i W(\phi)$$

$$= -\lambda_i\phi_i - 3\lambda_{ijk}\phi_j\phi_k + \cdots$$

$$\equiv (d-\Delta_i)\phi_i + \cdots$$

と

scaling dimension:

$$\boxed{\begin{aligned}\Delta_i &= d-\ell\lambda_i \\ &= \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m_i^2\ell^2}\end{aligned}}$$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle \propto \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

一方 λ_{ijk} は 3点関数を与える.

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle \propto \frac{\lambda_{ijk}}{|x-y|^{2\Delta_i+2\Delta_j-2\Delta_k}|y-z|^{\cdots}|z-x|^{\cdots}}$$

§5. Weyl anomaly と continuum limit

[M.F. - Matsuura - Sakai]

• Weyl anomaly の一般的構造

Ⓒ (wt = d の式):

$$-2\, G_{\mu\nu} \frac{\delta P}{\delta G_{\mu\nu}} + \beta^i \frac{\delta P}{\delta \phi^i} = -\frac{1}{[\gamma]_0} \left[\{ S_{loc}, S_{loc} \} \right]_d$$

$$-2\, G_{\mu\nu} \frac{\delta P}{\delta G_{\mu\nu}} \equiv \sqrt{G} \langle T^\mu_\mu \rangle$$

∴

$$-\frac{1}{[\gamma]_0} \left[\{ S_{loc}, S_{loc} \} \right]_d \equiv 2\sqrt{G} \left[\mathcal{W}_d + \nabla_\mu \mathcal{J}^\mu_d \right]$$

$\mathcal{W}_d$: $[\mathcal{L}_{loc}]_0, \cdots, [\mathcal{L}_{loc}]_{d-2}$ だけで書かれる部分

$\nabla_\mu \mathcal{J}^\mu_d$: $[\mathcal{L}_{loc}]_d$ を含む部分.

∴ Weyl anomaly

$$\boxed{\begin{aligned} \mathcal{W}_d + \nabla_\mu \mathcal{J}^\mu_d &= \frac{1}{2[\gamma]_0 \sqrt{G}} \left[\{ S_{loc}, S_{loc} \} \right]_d \\ &= \frac{d-1}{W(\phi) \cdot \sqrt{G}} \left[\{ S_{loc}, S_{loc} \} \right]_d \end{aligned}}$$

• Examples

以下 簡単のため、pure gravity を考える.

$$S_{d+1}[G] = \int d^d x\, dr\, \sqrt{G} \left[2\Lambda - R \right]$$

$$\left(\Lambda = -d(d-1)/2 : \text{cosmol. const} \right)$$

↓

$$\begin{aligned} \mathcal{L}_d &= 2\Lambda - R \\ &\equiv V - R \end{aligned}$$

$$\therefore V = 2\Lambda = -d(d-1)$$

一方

$$S_{loc}[G_{\mu\nu}(x)] = \int d^dx \sqrt{G}\, \mathcal{L}_{loc}$$

$$\mathcal{L}_{loc} = [\mathcal{L}_{loc}]_0 + [\mathcal{L}_{loc}]_2 + [\mathcal{L}_{loc}]_4 + \cdots$$

$$= \underbrace{W}_{0} \underbrace{- \Phi R}_{2} + \underbrace{X R^2 + Y R^{\mu\nu} R_{\mu\nu} + Z R^{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa}}_{4} + \cdots$$

について

$$[\{S_{loc}, S_{loc}\}]_0 = \sqrt{G}\left(-\frac{d}{4(d-1)} W^2\right)$$

$$[\{S_{loc}, S_{loc}\}]_2 = \sqrt{G}\left(\frac{d-2}{2(d-1)} W \cdot \Phi\right)$$

$$\begin{aligned}
[\{S_{loc}, S_{loc}\}]_4 = \sqrt{G}\Big[& -\frac{W}{2(d-1)}\left((d-4)X - \frac{d}{4(d-1)(d-2)^2}\right) R^2 \\
& -\frac{W}{2(d-1)}\left((d-4)Y + \frac{1}{(d-2)^2}\right) R^{\mu\nu} R_{\mu\nu} \\
& -\frac{(d-4)}{2(d-1)} W Z R^{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa} \\
& + \left(2X + \frac{d}{2(d-1)} Y + \frac{2}{d-1} Z\right) \nabla^2 R \Big]
\end{aligned}$$

$$\begin{aligned}
[\{S_{loc}, S_{loc}\}]_6 = \sqrt{G}\, \Phi \Big[& \frac{d+2}{2(d-1)} X R^3 \\
& + \left(4X + \frac{d+2}{2(d-1)} Y\right) R R^{\mu\nu} R_{\mu\nu} \\
& + \left(2(d-3)X + \frac{d-2}{2} Y\right) R \nabla^2 R \\
& + (4X + 2Y) R_{\mu\nu} \nabla^\mu \nabla^\nu R \\
& - 4Y R^{\mu\nu} R^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \\
& - 2Y R^{\mu\nu} \nabla^2 R_{\mu\nu} \Big]
\end{aligned}$$

wt = 0, 2

$$[\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2 = \sqrt{G}\,\mathcal{L}_d$$
$$= \sqrt{G}\left(-d(d-1) - R\right)$$

$$\Rightarrow \quad W = -2(d-1), \quad \Phi = \frac{1}{d-2}$$

d=4

$$\mathcal{W}_4 + \nabla_\mu \mathcal{J}_4^\mu = \frac{3}{W\cdot\sqrt{G}}\,[\{S_{loc}, S_{loc}\}]_4$$
$$= \frac{1}{24}R^2 - \frac{1}{8}R_{\mu\nu}^{\;2} - \underbrace{\left(X + \tfrac{1}{3}Y + \tfrac{1}{3}Z\right)\nabla^2 R}$$

$[\mathcal{L}_{loc}]_4$ から来る
total derivative

d=6

$$\begin{cases} 0 = [\{S_{loc}, S_{loc}\}]_4 & \text{— ①} \\ \mathcal{W}_6 + \nabla_\mu \mathcal{J}_6^\mu = \dfrac{5}{W\sqrt{G}}\,[\{S_{loc}, S_{loc}\}]_6 & \text{— ②} \end{cases}$$

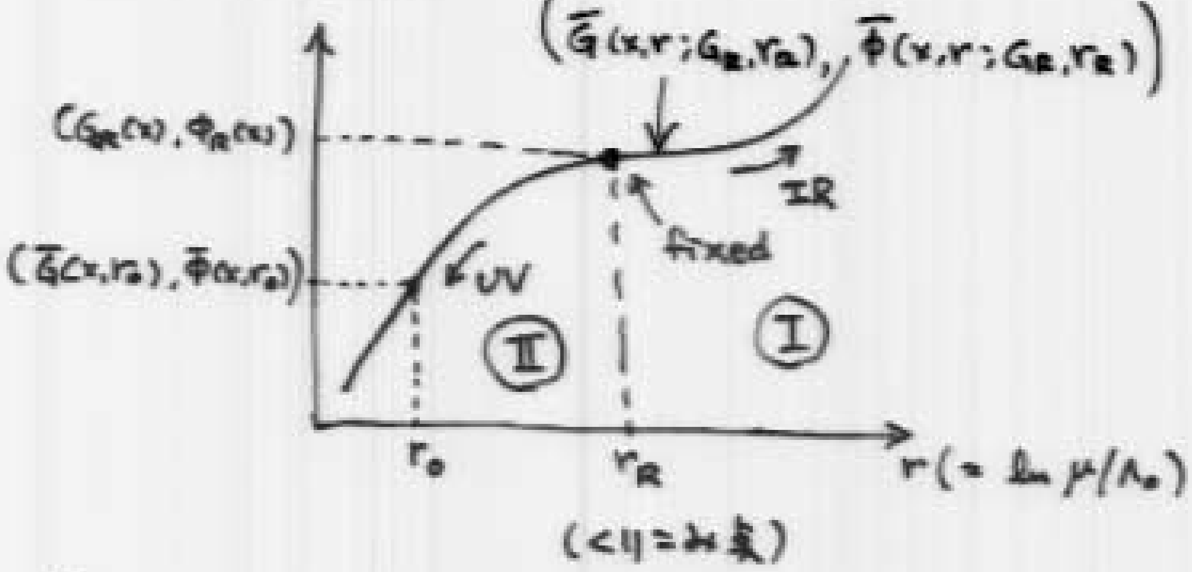
① より $\quad X = \frac{3}{320}, \quad Y = -\frac{1}{32}, \quad Z = 0$

② へ

$$\mathcal{W}_6 = \frac{1}{128}RR_{\mu\nu}R^{\mu\nu} - \frac{3}{3200}R^3 + \frac{1}{64}R^{\mu\nu}R^{\lambda\kappa}R_{\mu\lambda\nu\kappa}$$
$$+ \frac{1}{320}R_{\mu\nu}\nabla^\mu\nabla^\nu R - \frac{1}{128}R^{\mu\nu}\nabla^2 R_{\mu\nu} + \frac{1}{1280}R\nabla^2 R$$

正しく再現

• <u>Continuum limit</u>

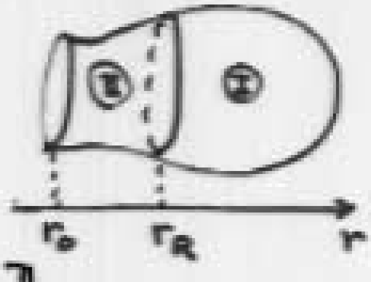


「RG-flow ⇔ classical trajectory」より.

<u>Continuum limit</u> は,

> $S[G_{\mu\nu}(x),\phi^i(x)]$ に $\bar G_{\mu\nu}(x,r_0;G_R,r_R)$, $\bar\Phi^i(x,r_0;\phi_R,r_R)$
> を代入して, $r_0\to-\infty$ の limit をとればよい.

• <u>counter term の評価</u>



$$S[\bar G(x,r_0:G_R,r_R),\ \bar\Phi(x,r_0;\phi_R,r_R)]$$

$$= S_{d+1}[\![\bar G(x,r:G_R,r_R),\ \bar\Phi(x,r;\phi_R,r_R)\ ;\ r_0]\!]$$

$$= S^{Ⅰ}_{d+1}[\![\bar G(x,r;G_R,r_R),\bar\Phi(x,r;\phi_R,r_R);r_R]\!]$$

$$+\int_{r_0}^{r_R}dr\overset{Ⅱ}{}\int d^dx\sqrt{G}\,\mathcal{L}_{d+1}\quad(+\text{ boundary term})$$

$$=\Gamma_R[G_R(x),\phi_R(x)]+S_{CT}[G_R(x),\phi_R(x);r_R,r_0]$$

↑ r_0-dependence は ここのみに現れる.

具体的には.

$$S_{CT}[G_R(x), \phi_R(x) : r_R, r_0]$$

$$= \int_{r_0}^{r_R} dr \int d^d x \sqrt{\bar{G}} \Big[V(\bar{\Phi}) - \bar{R} + \frac{1}{2} L_{ij}(\bar{\Phi}) \bar{G}^{\mu\nu} \partial_\mu \bar{\Phi}^i \partial_\nu \bar{\Phi}^j$$

$$+ \frac{1}{4} (\dot{\bar{G}}_{\mu\nu})^2 - \frac{1}{4} (\bar{G}^{\mu\nu} \dot{\bar{G}}_{\mu\nu})^2$$

$$+ \frac{1}{2} L_{ij}(\bar{\Phi}) \dot{\bar{\Phi}}^i \dot{\bar{\Phi}}^j \Big]$$

ここで $\bar{G}_{\mu\nu}(x, r)$, $\bar{\Phi}^i(x, r)$ は

$$\bar{G}_{\mu\nu}(x, r_R) = G^R_{\mu\nu}(x), \quad \bar{\Phi}^i(x, r_R) = \phi^i_R(x)$$

をみたすもの.

<u>注1</u>

$G_{0\mu\nu}(x) = \bar{G}_{\mu\nu}(x, r_0)$, $\phi^i_0(x) = \bar{\Phi}^i(x, r_0)$ を用いて

書き直すと、r_R, r_0 は 表式から消える:

$$S_{CT}[G_R(x), \phi_R(x) ; \underset{\bar{G}(x, r_0; G_R, \phi_R)}{G_0(x)}, \underset{\bar{\Phi}(x, r_0; \phi_R, r_R)}{\phi_0(x)}]$$

(∵ Hamilton-Jacobi constraint)

<u>注2</u>

$P_R[G_R(x), \phi_R(x)]$ は、再び flow eqn をみたす.

→ β と β_R は 同じ関数形.

anomaly も 〃

• Henningson–Skenderis (hep-th/9806087) との対応

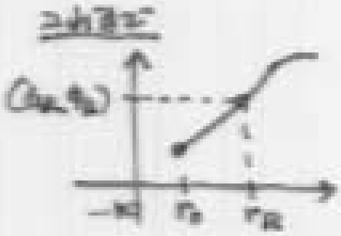
“くりこみの処方の違い”

← [illegible]を含んだ場合の拡張:
[Nojiri–Odintsov (th/9810008)
Nojiri–Odintsov–Ogushi–Sugamoto–Yamamoto (9906066)
Nojiri–Odintsov–Ogushi (0001122)

H-S:

$r_0 \to -\infty$ ($z_0 = e^{r_0} \to 0$) での漸近形を固定する:

$$\begin{cases} \bar{G}_{\mu\nu}(x, r_0) \sim e^{-2r_0}\, g^{(0)}_{\mu\nu}(x) \\ \bar{\Phi}^i(x, r_0) \sim e^{(d-\Delta_i) r}\, \phi^{(0)i}(x) \end{cases}$$



$\bar{G}_{\mu\nu}(x, r_0; G_R, r_R) = e^{-2r_0}(g^{(0)}_{\mu\nu} + \cdots)$

$\bar{\Phi}^i(x, r_0; \phi_R, r_R) = e^{(d-\Delta_i) r_0}(\phi^{(0)i}(x) + \cdots)$

$$\left(\because\ \dot{\bar{\Phi}}^i(x, r) = -L^{ij}(\phi)\, \partial_j W \Big|_{\phi = \bar{\Phi}(x,r)} = -\lambda_i \bar{\Phi}^i + \cdots = (d-\Delta_i)\bar{\Phi}^i + \cdots \quad \therefore\ \bar{\Phi}^i(x, r) \sim e^{(d-\Delta_i) r}\, \phi^{(0)i}(x) \right)$$

すると、cl. soln で要る事から、$r \to -\infty$ の形が逐次的に求まる:

$$\bar{G}_{\mu\nu}(x, r_0) = e^{-2r_0}\left[\underline{g^{(0)}_{\mu\nu}(x)} + e^{2r_0} g^{(2)}_{\mu\nu}(x) + \cdots + e^{d r_0}\left(g^{(d)}_{\mu\nu}(x) + r_0\, h^{(d)}_{\mu\nu}(x)\right) + \cdots \right]$$

$$\bar{\Phi}^i(x, r_0) = e^{(d-\Delta_i) r_0}\left[\underline{\phi^{(0)i}(x)} + e^{2r_0}\, \phi^{(2)i}(x) + \cdots \right]$$

($g^{(2)}, g^{(4)}, \cdots, \phi^{(2)}, \phi^{(4)}, \cdots$ は $\underline{g^{(0)}, \phi^{(0)}}$ で書ける)
← これを「くりこんだ coupling」と解釈する.

S_{CT} に代入すると.　$\epsilon \equiv e^{r_0}$

$$\Rightarrow S_{CT}[g^{(0)}, \phi^{(0)}; \epsilon] = \int d^dx \sqrt{g^{(0)}} \left[\frac{a^{(0)}}{\epsilon^d} + \frac{a^{(2)}}{\epsilon^{d-2}} + \cdots + \frac{a^{(d-2)}}{\epsilon^2} - \ln \epsilon^2 \cdot \underline{W_d} + O(\epsilon^2) \right]$$

↑ Weyl anomaly

注 $(G^R_{\mu\nu}, \phi^{Ri})$ と $(g^{(0)}_{\mu\nu}, \phi^{(0)i})$ の関係は具体的に計算可能.

§6. Conclusion

- Holographic RG

  [d 次元 quantum field theory] + [RG-scale]

  = (d+1) 次元 classical (super) gravity

ESSENCE :

(1) (d+1) 次元一般共変性
(2) asymptotically AdS_{d+1}

- Hamilton-Jacobi constraint と flow eqn

  — Holographic RG を解析する上で有用な枠組

  — BY-PRODUCT :
  任意の次元で Weyl anomaly を計算するのに
  (恐らく最も) 簡単な手続きを与える.

- Holographic RG で記述できる場の理論とは ?

- 未解決

  must be a gauge theory ? (← ZIGZAG SYM)

  " SUSY ? (← $V = (W'(\phi))^2 - (W(\phi))^2$)

- 具体例は存在 (Freedman-Gubser-Pilch-Warner (hep-th/9904017))

  ($N=4$ SYM$_{UV}$ → $N=1$ SYM IR fixed pt) ⟺ ∃ SUGRA dual