

# Noncommutative Quantum Field Theory and its Applications

早川 雅司  
(KEK)

Part I: Overview of Noncommutative (NC)  
Field Theory

Part II: Reparametrization Invariant  
dynamics of NC membrane  
(Ishibashi, Okuyama, M.H.)

Part III: Anomaly in NC YM theory  
and Green-Schwarz mechanism  
(Iso, Kawai, M.H.)

Part IV: Universe filled with impurities  
and curved space-time (M.H.)

---

I, II のみ = 1/2 (7/2)

# Overview of Noncommutative (NC) Field Theory

## § 1.1. 背景

### NC Yang-Mills (NCYM) 理論

- ① 摂動的な弦理論で
- ② string の非摂動的な構築を組む  
行列理論の一部として  
現われた。

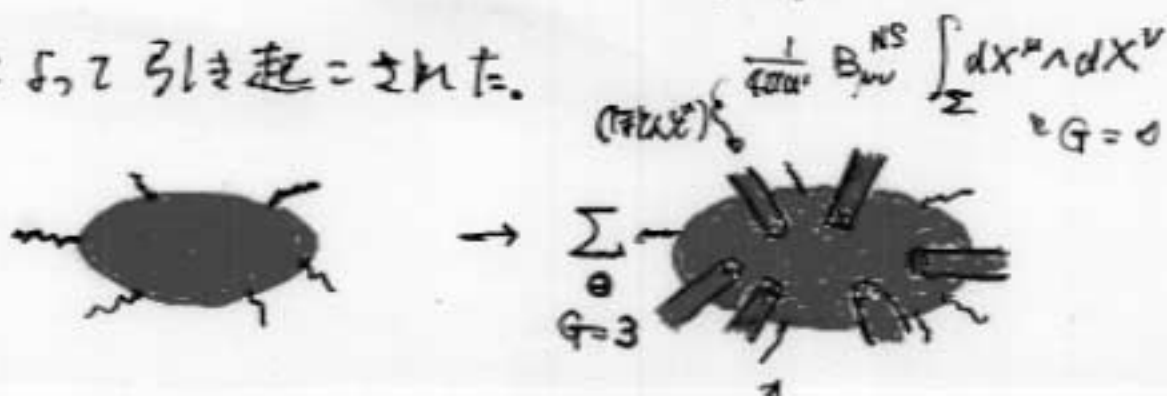
①の立場からは


open string sector の代数構造の変形

$$(a) \quad f \circledast g \rightarrow f \star g$$

は、closed string mode の condensation  
( $B_{\mu\nu}^{NS} \neq 0$ )

による引き起こされた。

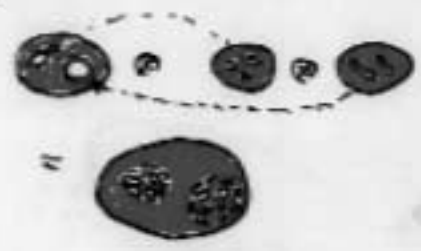


{ , its composition law }

小さい disk の operad

⋮

✦ 与えられた小道具の1つ



変形 (A) の操作 加える代数構造

⇕ 共通 -- (Gerstenhaber) algebra

closed string field theory の vertex の building block : string-積 の代数構造

(A, B<sub>1</sub>, ..., B<sub>n</sub>)  
string field  
[B<sub>1</sub>, ..., B<sub>n</sub>]  
↓  
(A, [B<sub>1</sub>, ..., B<sub>n</sub>])  
→ action ε 描く

(この link の定式化: (Hofman, Ma))

② の立場からすると、これは模型の中に

見い出せるとうれしい (かも知らない)。

行列模型の立場から5は

量子論 (NC quantum field theory)

であるべき。

それはどんなものか?

それは string 的, or 重力の要素

を垣間見ることかできるか?

- Nonperturbative analysis

(Ishibashi, Iso, Kawai, Kitazawa)

↑                    ↑                    (00mcs) の ↑  
 ⇒ 11d の ⇒                    2d ⇒

- Perturbative analysis

今回は、"NC quantum field theory" の

- UV-極限

= large N field theory の UV-極限

- IR-極限

UV-極限との相関

## 2. 非可換空間上の量子場の理論 (2.1) (NC field theory)

例) と (2) 4次元 Euclidean  $\phi^4$ -theory:

$$\int_{\text{NC}} \phi_4^4 = \int d^4x \left( \frac{1}{2} \partial_\mu \phi * \partial_\mu \phi + \frac{1}{2} m^2 \phi * \phi + \frac{\lambda}{4} \phi * \phi * \phi * \phi \right)$$

$c^{\nu\mu} = -c^{\mu\nu}$   
 $[長文]^2$   
 $(f * g)(x) = f(x) e^{\frac{1}{2} \overleftarrow{\partial}_\mu c^{\mu\nu} \overrightarrow{\partial}_\nu} g(x)$   
 $= \overleftarrow{\partial} \wedge \overrightarrow{\partial}$

Feynman rule :=  $\int \int$  振動的 := 定義

momentum space :=  $\rightarrow \rightarrow$  と便利:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot x} \tilde{\phi}(p)$$

2次の項:

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \tilde{\phi}(-p) \left( e^{\frac{i}{2} \frac{p \wedge p}{0}} \right) (p^2 + m^2) \tilde{\phi}(p)$$

" /

$$\Rightarrow \overline{\underset{\vec{p}}{\quad}} = \frac{1}{p^2 + m^2}$$

$\phi^4$  - 結合:

$$p_1 \quad p_2 \quad p_3 \quad p_4$$

$$\phi * (\phi * (\phi * \phi))$$

⋮

⋮

⋮

$$e^{ip_3 \cdot x} e^{\frac{i}{2\omega} \vec{\sigma} \wedge \vec{\sigma}} e^{ip_4 \cdot x}$$

位相  $\propto 12 \quad \frac{1}{2} p_3 \wedge p_4$

$$\frac{1}{2} p_2 \wedge (p_3 + p_4)$$

$$\frac{1}{2} p_1 \wedge (p_2 + p_3 + p_4)$$

$$X \Rightarrow \int \prod_{j=1}^4 \frac{d^4 p_j}{(2\pi)^4} \boxed{e^{\frac{i}{2} \sum_{j < k} p_j \wedge p_k}}$$

$$\times \frac{\pi}{4} \tilde{\Phi}(p_1) \tilde{\Phi}(p_2) \tilde{\Phi}(p_3) \tilde{\Phi}(p_4)$$

① 普通の  $\phi^4$  相互作用:

(2.3.)

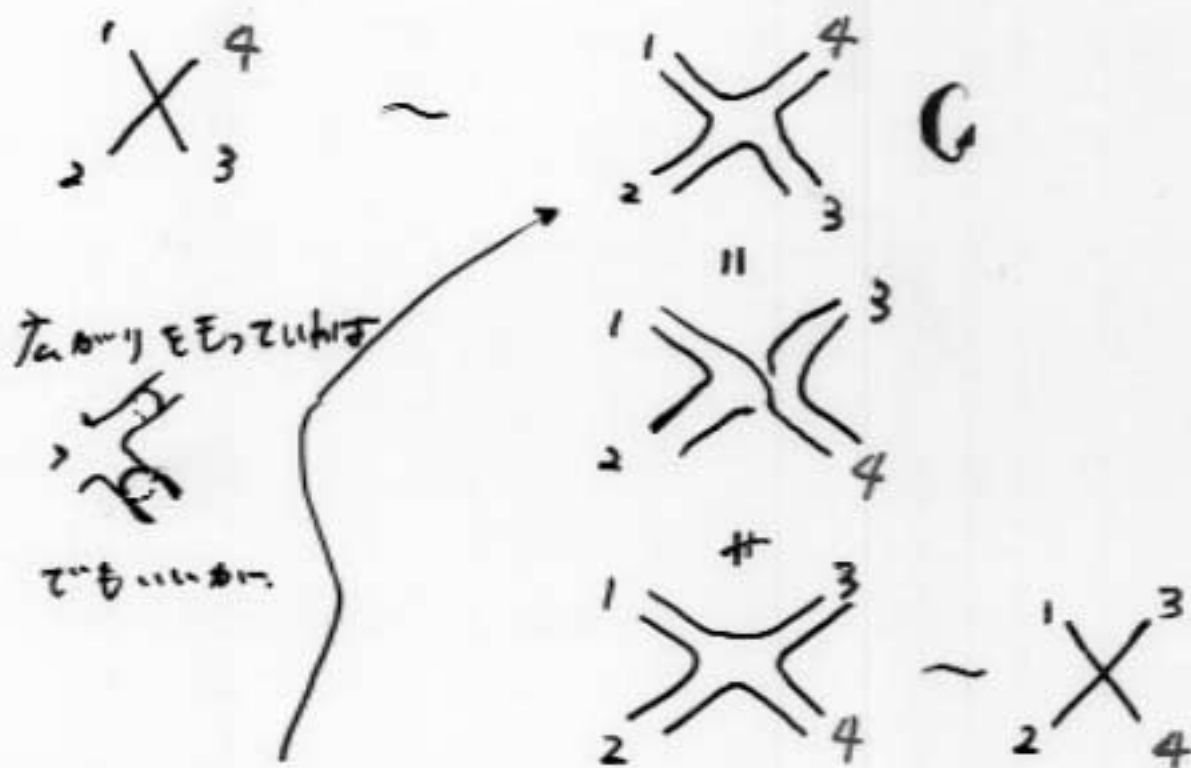
$$\begin{array}{c} 1 \\ \diagdown \\ 2 \end{array} \times \begin{array}{c} 4 \\ \diagup \\ 3 \end{array} \text{G} = \begin{array}{c} 1 \\ \diagdown \\ 2 \end{array} \times \begin{array}{c} 4 \\ \diagup \\ 3 \end{array}$$

総計  $4!$  の置換する  $\sim$  のものが等価

(vertex は本当に「点」)

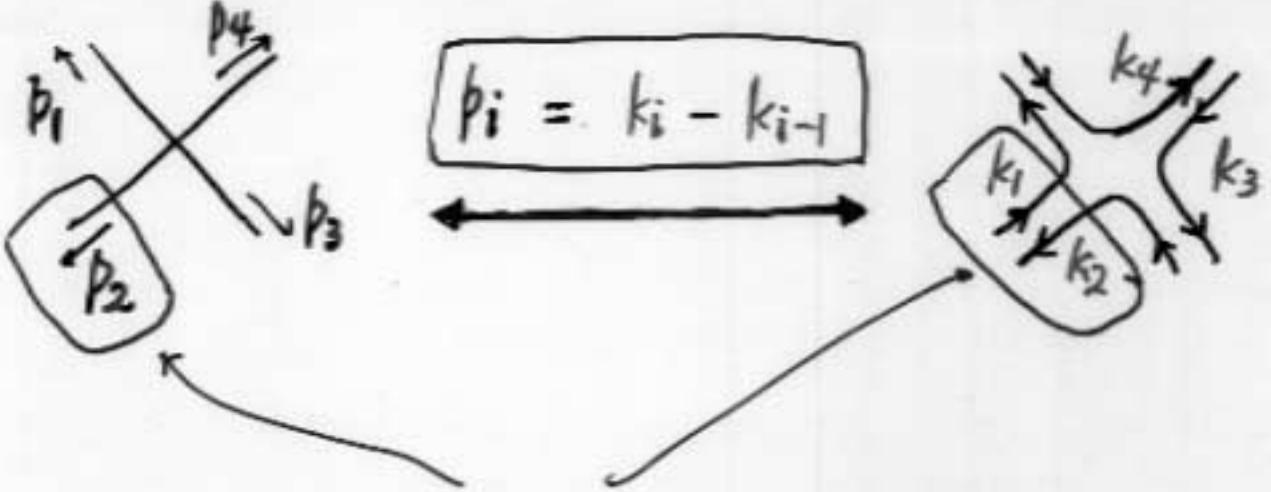
② NC の  $\phi^4$  相互作用:

phase factor のため、cyclic symmetry のよ  
しからい。



2本線 で書き表せる点に7112.

( $\Rightarrow$  large-N field theory との接点を与える)



$k_2$  分のみ出て  $k_1$  分のみ入る。

正味、 $k_2 - k_1 = p_2$  の運動量が出てくる。

$$\tilde{\Phi}(p_i) = \tilde{\Phi}(k_i - k_{i-1})$$

$$\begin{aligned} \sum_{i < j} p_i \wedge p_j &= \sum_i k_i \wedge k_{i+1} \\ &= k_1 \wedge k_2 + k_2 \wedge k_3 \\ &\quad k_3 \wedge k_4 + k_4 \wedge k_1 \end{aligned}$$

$$\Rightarrow e^{\frac{i}{2} k_1 \wedge k_2} \tilde{\Phi}(k_2 - k_1)$$

このときだけ目撃してやる。

$\Rightarrow$



$$X \Leftrightarrow \int 4^4 \frac{4}{j=1} \frac{d^4 k_j}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \dots + k_4) \quad (2.2.)$$

$$\times \frac{\lambda}{4} \left[ e^{\frac{i}{2} k_4 \wedge k_1} \tilde{\phi}(k_1 - k_4) \right]$$

→ ... →

$$\left[ e^{\frac{i}{2} k_3 \wedge k_4} \tilde{\phi}(k_4 - k_3) \right]$$

∴

$$\tilde{\phi}[k_1, k_4] \equiv e^{-\frac{i}{2} k_1 \wedge k_4} \tilde{\phi}(k_1 - k_4)$$

という量  $\varepsilon$  を  $\tau$  と  $\tau \neq \tau$  としておく。

$$(\tilde{\phi}[k_1, k_4])^* = e^{\frac{i}{2} k_1 \wedge k_4} (\tilde{\phi}(k_1 - k_4))^*$$

$$\phi(x) : \text{real} \Rightarrow \tilde{\phi}(p)^* = \tilde{\phi}(-p) \Rightarrow \tilde{\phi}(k_4 - k_1)$$

$$= e^{-\frac{i}{2} k_4 \wedge k_1} \tilde{\phi}(k_4 - k_1)$$

$$= \tilde{\phi}[k_4, k_1]$$

∴  $\tilde{\phi}^\dagger[k_1, k_4] \equiv (\tilde{\phi}[k_4, k_1])^*$  (エルミート共役)

と訂正

$$\tilde{\phi}^\dagger[k_1, k_4] = \tilde{\phi}[k_1, k_4]$$

∞次元  
エルミート行列

今考えている real  $\phi^4$  理論 の action:

$$\int_{\phi_0}^{NC} = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

で記述出来るように、hermitian  $N \times N$  行列場  $\Phi_i^j(x)$

で action

$$S[\Phi^4]_N = \int d^4x \text{tr} \left[ \frac{1}{2} \partial_\mu \Phi \partial_\mu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda_H}{4N} \Phi^4 \right]$$

$N \times N$  行列  $\Rightarrow$

ここで large- $N$  極限を考慮する

で記述出来るように比較して考える。

後者の場の Feynman rule:

$$\begin{matrix} i \rightarrow k \\ j \leftarrow l \\ \vec{p} \end{matrix} = \delta^{ik} \delta_{jl} \frac{1}{p^2 + m^2}$$

$$\begin{matrix} (k_4, i_4) \\ (k_1, i_1) \\ (k_2, i_2) \\ (k_3, i_3) \end{matrix} \Leftrightarrow \int d^4 \frac{d^4 k_j}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \dots + k_4) \cdot \frac{\lambda_H}{4N} \tilde{\Phi}_{i_4}^{i_1}(k_1 - k_4) \dots \tilde{\Phi}_{i_3}^{i_2}(k_4 - k_3)$$

$$\Phi_i^j(x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{\Phi}_i^{j_2}(p) e^{ip \cdot x}$$

NC field theory :

$$\frac{\lambda}{4} [ e^{\frac{i}{2} k_4 \wedge k_1} \Phi(k_1 - k_4) ]$$

$$\times \dots \times [ e^{\frac{i}{2} k_3 \wedge k_4} \Phi(k_4 - k_3) ]$$

large - N field theory :

$$\frac{1}{N} \frac{\lambda_H}{4} \Phi_{i_4}^{i_1} (k_1 - k_4) \times \dots \times \Phi_{i_3}^{i_4} (k_4 - k_3)$$

double - line 表示を 通して分かること :

- 両者の Feynman rule は 同一。
- 描かれる Feynman diagram は 同一。
- それに伴う combinatoric factor は 同一。

以下で議論したいこと :

large - N 側で color index が果たしている役割  
(この contraction のパターン)

と  
NC-field theory 側で phase factor が果たす役割  
の比較

e.g. 2-loop effect on free energy

(2.8.)



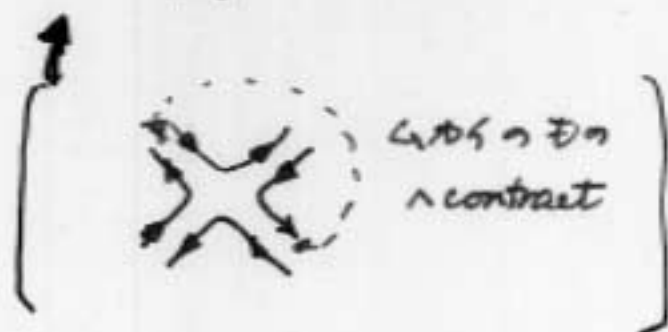
(a)



(b)



$\times \underline{2}$



$\times \underline{1}$

どちらの立場からしても同じ topology の diagram を書け. 伴う combinatoric factor は共通.

large N  $\phi^4$  - theory の立場から

(a)  $\sim N^3 \cdot \frac{\lambda_H}{N} = N^2 \lambda_H$  planar



(b)  $\sim N \cdot \frac{\lambda_H}{N} = \lambda_H$   
nonplanar



$N \rightarrow \infty$  (a)  $\gg$  (b) (planar type が dominate)

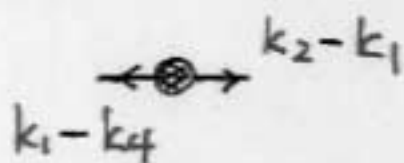
# NC field theory 例の事情

(2.10.9)

(a) planar contribution

$$\int 4^4 \prod_{i=1}^4 \frac{d^4 k_i}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2 + k_3 + k_4)$$

$$\begin{aligned} & \cdot 2 \times \frac{\lambda}{4} \times \left. \begin{aligned} & e^{\frac{i}{2} k_4 \wedge k_1} \tilde{\Phi}(k_1 - k_4) \\ & e^{\frac{i}{2} k_1 \wedge k_2} \tilde{\Phi}(k_2 - k_1) \end{aligned} \right\} \\ & \left. \begin{aligned} & e^{\frac{i}{2} k_2 \wedge k_3} \tilde{\Phi}(k_3 - k_2) \\ & e^{\frac{i}{2} k_3 \wedge k_4} \tilde{\Phi}(k_4 - k_3) \end{aligned} \right\} \end{aligned}$$



$$(k_1 - k_4) + (k_2 - k_1) = 0$$

$$\therefore k_2 = k_4$$

$$\begin{aligned} (\text{phase factor}) &= e^{\frac{i}{2} k_2 \wedge k_1} e^{\frac{i}{2} k_1 \wedge k_2} (\Leftrightarrow \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} = N) \\ & e^{\frac{i}{2} k_3 \wedge k_3} e^{\frac{i}{2} k_3 \wedge k_2} \\ & = 1 \end{aligned}$$

$$(a) = 2 \times \frac{\lambda}{4} \int \frac{d^4 p_1}{(2\pi)^4} \left( \int \frac{d^4 p_3}{(2\pi)^4} \frac{1}{p_3^2 + m^2} \right) \frac{1}{p_1^2 + m^2}$$

$$= a_1 \lambda \Lambda^4$$



$$\sim \Lambda^2 \quad (\Lambda: \text{cut-off})$$

$(a) \text{ in large } N = N^2 a_1 \lambda \Lambda^4$

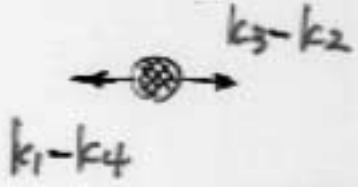
NC field theory 例の事情

(2.12.9.)

(b) nonplanar contribution

$$\int 4^4 \prod_{j=1}^4 \frac{d^4 k_j}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + k_2 + k_3 + k_4)$$

$$\begin{aligned} & \times I \times \frac{\lambda}{4} \times \begin{aligned} & e^{\frac{i}{2} k_4 \wedge k_1} \tilde{\Phi}(k_1 - k_4) \\ & e^{\frac{i}{2} k_1 \wedge k_2} \tilde{\Phi}(k_2 - k_1) \\ & e^{\frac{i}{2} k_2 \wedge k_3} \tilde{\Phi}(k_3 - k_2) \\ & e^{\frac{i}{2} k_3 \wedge k_4} \tilde{\Phi}(k_4 - k_3) \end{aligned} \end{aligned}$$



$$\rightarrow (k_1 - k_4) + (k_3 - k_2) = 0$$

$$\therefore k_1 + k_3 = k_2 + k_4$$

$$\Rightarrow k_1 + k_2 + k_3 + k_4 = 0 \text{ a.s.}$$

$$k_1 + k_3 = 0 = k_2 + k_4$$

$\delta_i^k \delta_k^j = \delta_i^j$

$\rightarrow$

$$\text{(phase factor)} = e^{-\frac{i}{2} (-k_2) \wedge k_1} e^{\frac{i}{2} k_1 \wedge k_2}$$

$$\times e^{\frac{i}{2} k_2 \wedge (-k_1)} e^{\frac{i}{2} (-k_1) \wedge (-k_2)}$$

$$= e^{2i k_1 \wedge k_2}$$

$$(b) = \frac{\lambda}{4} \int \frac{d^4 p}{(2\pi)^4} \left( \int \frac{d^4 q}{(2\pi)^4} e^{i p \wedge q} \frac{1}{q^2 + m^2} \right) \frac{1}{p^2 + m^2}$$



$$\int \frac{d^4 q}{(2\pi)^4} e^{i p \wedge q} \frac{1}{q^2 + m^2}$$

$$p \wedge p = \underbrace{g_{\mu\nu} (c^{\mu\nu} p_\nu)}_{\equiv \tilde{p}^\mu}$$

$$= \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot \tilde{p}} \frac{1}{q^2 + m^2}$$

(長さの次元)

scalar field の tree propagator の "座標"  $\tilde{p}^\mu$  表示

その赤外極限  $p_\mu \rightarrow 0 \iff$  "座標"  $\tilde{p}^\mu \rightarrow 0$  の short distance limit

$$\rightarrow \text{singular} \rightarrow \frac{1}{4\pi^2} \frac{1}{p^2}$$

$$\tilde{p}^\mu = a(1, 1, 1, 1) \equiv n^\mu \quad (a > 0) \text{ とし } q \rightarrow \frac{1}{a} q$$

と rescale すると.

$$= \frac{1}{a^4} \int \frac{d^4 q}{(2\pi)^4} e^{-i n \cdot q} \frac{1}{\frac{1}{a^2} q^2 + m^2} \xrightarrow{a \rightarrow 0} \frac{a^2}{q^2}$$

$$\xrightarrow{a \rightarrow 0} \frac{1}{a^2} \int \frac{d^4 q}{(2\pi)^4} e^{-i n \cdot q} \frac{1}{q^2}$$

- (1.3.15)  $p^0 = 0$  としたとき, phase factor は  
 「(1.3.1)」、planar diagram とする積分は、

これは 2次発散している。

これを反映し、 $p^0 \rightarrow 0$  で 2次的に発散  
 ( $\Rightarrow$  後で詳しく)  $\sim \frac{1}{p^2}$

- $\frac{1}{p^2}$  は  $p^0 = 0$  の massless pole

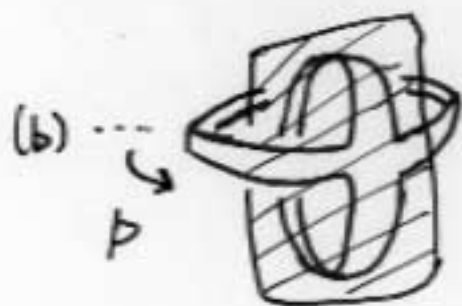
にも見える。 ( $\Rightarrow$  後で)



$|p| \rightarrow \infty$ 

$$\int \frac{d^4 q}{(2\pi)^4} e^{-i\tilde{p}\cdot q} \frac{1}{q^2 + m^2}$$

$$\rightarrow \frac{1}{4\pi^2} \sqrt{\frac{\pi}{2}} \frac{(\tilde{p}^2 m^2)^{1/2}}{\tilde{p}^2} e^{-\sqrt{\tilde{p}^2 m^2}}$$

 $p \rightarrow \infty$  のときは有限

他方

$$\sim \int_{p: \text{small}} d^4 p \frac{1}{\tilde{p}^2} \frac{1}{\underline{p^2 + m^2}} : \text{有限}$$

 $\therefore$  (b) は有限

NC-field theory 側

(a)  $\gg$  (b) for  $\lambda \rightarrow \infty$

Planar は 運動量に依らず phase E 含む

Nonplanar は、それら non-trivial の phase E 含む

NC - field theory の 紫外極限

||

large N .field theory の planar limit の  
紫外極限

(e.g.)

free energy  $d=4$

$$F_{NC} = * \Lambda^4 ( a_0 + a_1 \lambda + a_2 \lambda^2 + \dots ) + \dots$$

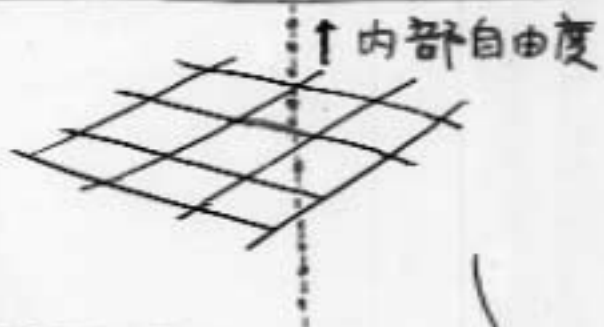
$$F_{large N} = * N^2 \Lambda_H^4 ( a_0 + a_1 \lambda_H + a_2 \lambda_H^2 + \dots )$$

$$+ * N^0 \Lambda_H^4 ( b_1 \lambda_H + b_2 \lambda_H^2 + \dots )$$

+ ...

large N Yang Mills theory  
in planar limit

reduced model  
(Eguchi-Kawai model)

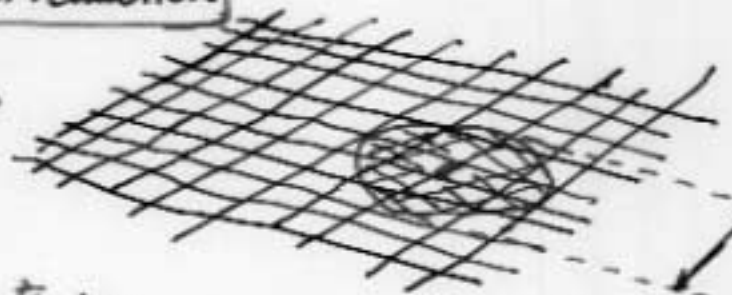


時空間の reduction

UV side is  
open

U(1) Noncommutative Yang-Mills theory

内部空間の reduction

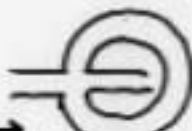


内部自由度が

$$l_{NC} = 2\pi / C^{UV} / 2$$

\*-種の構造にfy. 時空間に埋め込めた。

3. IR 側の振る舞いと UV 側の振る舞いの (3.1.)  
 関連

$\vec{p}$   =  $\epsilon$  決めた積分:

$$\int \frac{d^4 q}{(2\pi)^4} e^{i q \cdot \tilde{p}} \boxed{\frac{1}{q^2 + m^2}} = \int_0^\infty d\alpha e^{-\alpha(q^2 + m^2)}$$

$$= \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-(\alpha m^2 + \frac{1}{4\alpha} \tilde{p}^2)}$$

$q \rightarrow$  大か  
 小かは  
 $\alpha \rightarrow$  小

$\alpha \rightarrow 0$  の紫外発散  $\epsilon$  導き得る

$$\sim \int_0 \frac{d\alpha}{\alpha^2} e^{-\frac{1}{4\alpha} \tilde{p}^2} \quad \text{と } p^\mu \neq 0 \text{ の有限に}$$

$\approx 2\pi$

(planar type) :  $\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2}$

$$= \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2}$$

$$\propto \int_0 \frac{d\alpha}{\alpha^2} : \text{発散}$$

$\Rightarrow$  regularization :

$$\frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-(\alpha m^2 + \frac{1}{4\alpha} \frac{1}{\Lambda^2})}$$

Nonplanar diagram の支配する  $p^\mu \rightarrow 0$  の singular の振る舞い  
 (IR limit)  $(\sim 1/\tilde{p}^2)$

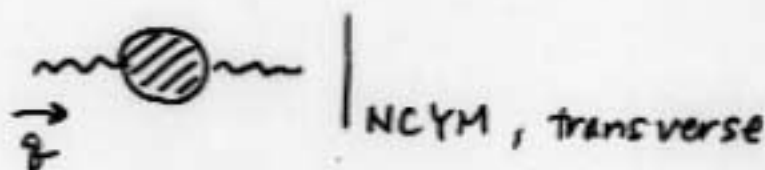
**関連**

$\equiv$  Planar diagram の支配する  $\Lambda \rightarrow \infty$  (UV 極限) の singular の振る舞い

U(1) 非可換 Yang-Mills 理論 で

(3.3)

NCYM



UV  $\rightarrow$   $-\frac{g^2}{16\pi^2} \frac{10}{3} \ln \tilde{q}^2 (\delta_{\mu\nu} \tilde{q}^2 - \tilde{q}_\mu \tilde{q}_\nu)$   
 $|B| \gg \sqrt{|C|}$



$Z_A = 1 - \left(-\frac{g^2}{16\pi^2} \frac{10}{3}\right) \ln \Lambda^2$

IR

$\rightarrow -\frac{g^2}{16\pi^2} \frac{10}{3} \frac{[-\ln(\tilde{q}^2)]}{> 0 \text{ for } |B| \ll |C|} (\delta_{\mu\nu} \tilde{q}^2 - \tilde{q}_\mu \tilde{q}_\nu)$   
 $|B| \ll \sqrt{|C|}$



① IR 側と UV 側の振る舞いの singularity は一致 (log 的)

(理由は  $\phi^4$  のときと同じ)

② さらに係数も込めて - 教

$\hat{x}^\mu$  の生成する

operator の代数  $[\hat{x}^\mu, \hat{x}^\nu] = -i C^{\mu\nu} \hat{1}$

122

\* の関数の代数

$$\Delta x^1 \Delta x^2 \geq 2\pi |C^{12}|$$

この面積

$\Delta x^2 \downarrow$



$\Delta x^1$

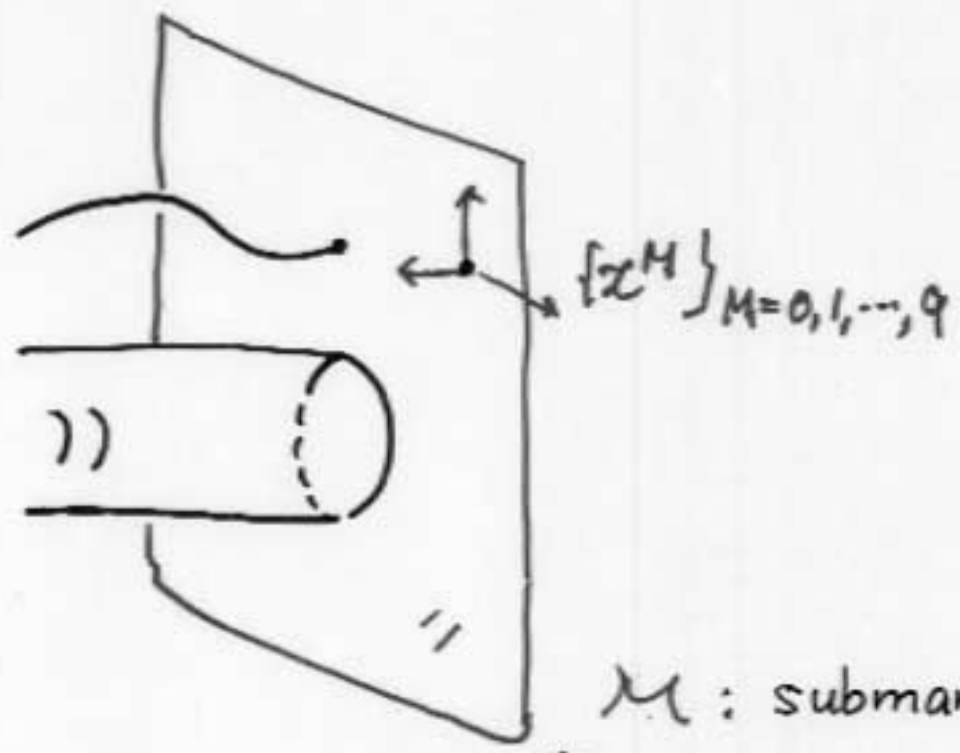
$\Delta x^1$  が小さい

とすると

より長距離的側面も含まれるようになる。

# §2.1. Reparametrization invariance of D-brane

one  $D_p$ -brane in flat space-time  
(BPS object)

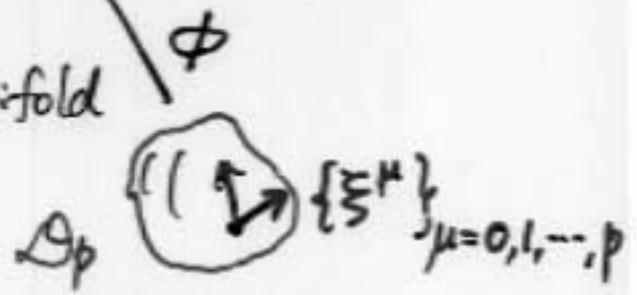


$\mathcal{M}$  : submanifold  
deficit  
of space-time

$$\mathcal{M} = \phi(\underline{\mathcal{D}}_p) \subseteq \text{space-time}$$

$(p+1)$ -dimensional manifold

embedding map  
 $\delta(?)$



$\phi$

$$\left\{ \phi^M(\xi^0, \xi^1, \dots, \xi^p) \right\}_{M=0,1,\dots,9}$$

in respective local coordinate system

For instance, choice of  $(\xi^0, \xi^1, \dots, \xi^p)$

is irrelevant to describe  $D_p$ -brane

reparametrization (diffeomorphism)

invariance of world-volume

$$\left( \text{Diffeo} \dots \frac{\partial(\xi^{0'}, \xi^{1'}, \dots, \xi^{p'})}{\partial(\xi^0, \xi^1, \dots, \xi^p)} \neq 0 \right)$$



reparametrization - invariant description

of D-brane



response to local fluctuation

⇒ world-volume action

(Aganagic, Popescu, Schwarz)

variables:

→  $\phi^M(\xi)$  ( $M=0, 1, \dots, 9$ )

$A_\mu(\xi)$  ( $\mu=0, 1, \dots, p$ )

for data of gauge bundle



⇒ A D-brane has

$\mathcal{N}=2$  space-time SUSY

$U(1)$ -local symmetry

→  $\Theta_a^{\dot{\alpha}}(\xi)$  ( $\dot{\alpha}=1, 2$ , Majorana<sub>2</sub>Weyl)

# Redundant variables

D-4

( physically relevant mode  
↔ generator of broken space-time symmetry )

①  $(p+1)$ -number of  $\left\{ \begin{array}{l} \phi^M \\ \Delta_\mu \end{array} \right\}$

② half of  $\Theta_\alpha^{\dot{\alpha}}$

• ① is eliminated

by using reparametrization invariance

• ② is eliminated

by using local  $\kappa$ -symmetry  
of world-volume

( D-brane has local  $\kappa$ -symmetry )

(  $\downarrow$  T&L  
non BPS )

conventional gauge

$\Rightarrow$  static gauge  $\xi^\mu = x^\mu$  ( $\mu = 0, 1, \dots, p-1$ )

$$\phi^\mu = 0$$

half of (4) is eliminated

respecting the above

$$\Rightarrow \{A_\mu, \phi^I, \lambda\}$$

$$(I = p, \dots, 9)$$

... content of



$\mathcal{N} = 1$  (SYM) $_{1+9}$  multiplet

reduced to  $(1+p)$ -dimension

## §2.2. One application of NC-field theory

(in progress with Ishibashi & Okuyama)

$$\text{constant } F_{\mu\nu} = B_{\mu\nu}^{\text{NS}} + 2\pi\alpha' \tilde{F}_{\mu\nu}$$
$$\tilde{F} \equiv dA$$

とも  $D$ -brane を考える。

便宜上、 $\Lambda$ -変換: 
$$\begin{cases} B^{\text{NS}} \mapsto B^{\text{NS}} + d\Lambda \\ A \mapsto A - \frac{1}{2\pi\alpha'} \Lambda \end{cases}$$

$\therefore B^{\text{NS}} = 0$  とする。

constant  $\tilde{F}_{\mu\nu}$  の時、 $T = D$ -brane  
を考える。

### 目的 1.

gauge 固定  $\varepsilon$  (static gauge  $\varepsilon = \tau = 1$ )

うまく選んで

fixed  $\varepsilon$  かつ  $T =$  reparametrization  $\varepsilon$

系統的に追跡しよう。

# gauge 固定

... world-volume 上には Noncommutative Yang-Mills theory  $\varepsilon$  induce するものを  $\varepsilon$  選ぶ。

reparametrization の生成子 (current)

の同-視を可能にする点、

... 変形量子化における

• Kishimoto (and Asakawa) 's talk

formality theorem

(c.f.) Kontsevich

Jurco, Schupp & Wess

Manchon

Cattaneo, Felder

## 目的 2.

主に D2-brane に注目したとき、

reparametrization の量子論的な

consistency を NC quantum field theory

を用いて調べる。

$\Rightarrow$  Critical dimension

DLCQ-Matrix theory における membrane sector の量子論的整合性

$N = \underset{10}{4}$  (NCYM)<sub>(1+2)</sub> is "normal"

$N = \overset{6}{2}, \overset{4}{1}, \overset{3}{0}$  (NCYM)<sub>(1+2)</sub>

is "abnormal"

(...と主張したは結果では正しくないです。)

以下

§ 2.3. Boundary state description  
of D-brane with constant  $\tilde{F}_{uv}$

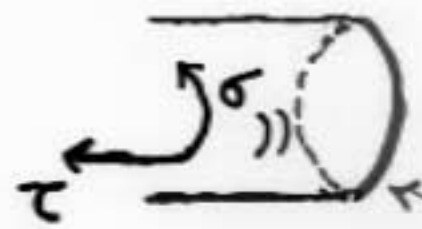
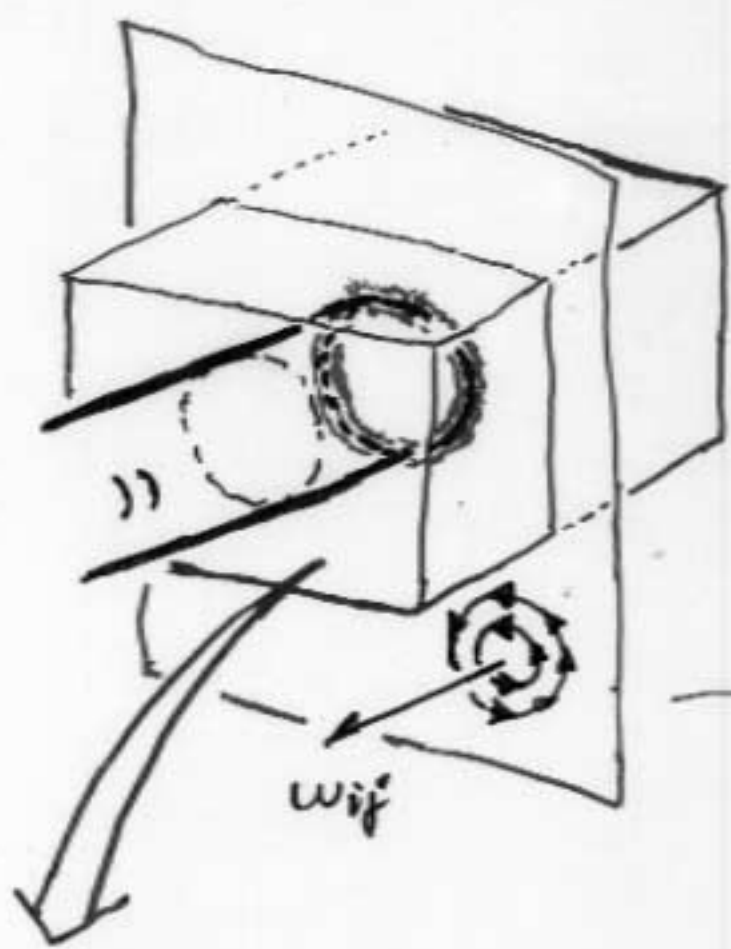
§ 2.4. Current for reparametrization

§ 2.5. Discussion and remained & related  
issues

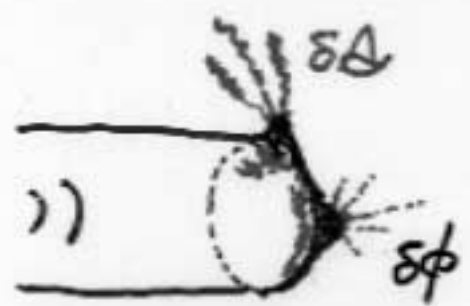
§2.3. Boundary state description of D-brane with constant  $\tilde{F}_{\mu\nu}$

$$\omega_{\mu\nu}$$

$$\left( \begin{matrix} \omega_{0j} = 0 \\ \tau_{23} \end{matrix} \right)$$



: closed string world-sheet beginning with boundary state



local fluctuation

$|\Delta, \phi, (\ominus)\rangle :=$  boundary state

BS-2

coherent state of closed string Hilbert space

which respects

world-sheet conformal symmetry:

$$Q_{\text{BRST}} |\Delta, \phi\rangle = 0$$

and is specified by

the boundary conditions ( $\leftarrow \Delta, \phi$ )



First we consider

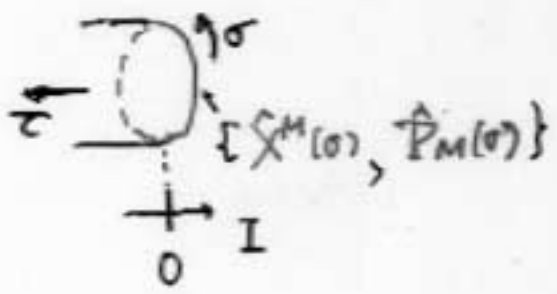
rigid D-brane with  $w_{ij}$

and the boundary state for it ;  $|B\rangle_w$

$w_{ij}=0$  ... boundary condition :

$$\hat{P}_\mu(\sigma) |B\rangle_{w=0} = 0 = \hat{X}^I(\sigma) |B\rangle_{w=0}$$

( $I = p, \dots, 9$ )



open string channel

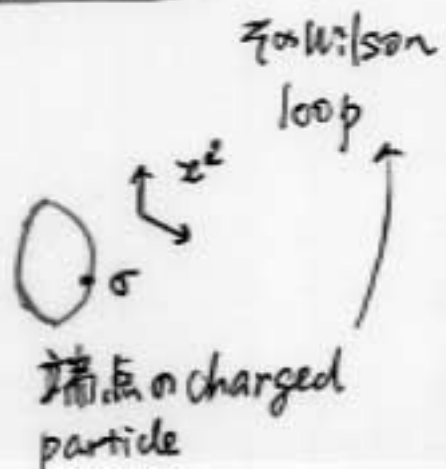
Neumann B.C.  
 $\partial_\sigma x^i = 0$   
 $\downarrow$   
 $g_{\mu\nu} \partial_\sigma x^\nu + \tilde{g}_{\mu\nu} \partial_\tau x^\nu = 0$

$w_{ij}$  = switch on

$$\Rightarrow \begin{cases} (\hat{P}_j(\sigma) - w_{ji} \partial_\sigma \hat{X}^i(\sigma)) |B\rangle_w = 0 = \hat{P}_0(\sigma) |B\rangle_w \\ \hat{X}^I(\sigma) |B\rangle_w = 0 \end{cases}$$

$$|B\rangle_w = \int Dx \left[ e^{i \int d\sigma \Delta_j(x) \partial_\sigma x^j} \right] |x\rangle$$

$$\begin{cases} \hat{X}^i(\sigma) |x\rangle = \underline{x^i(\sigma)} |x\rangle \\ \hat{P}_0(\sigma) |x\rangle = 0 \\ \hat{X}^I(\sigma) |x\rangle = 0 \end{cases} \begin{pmatrix} \text{c-number} \\ \text{profile of} \\ \text{boundary} \end{pmatrix}$$



$$0 = \int \Delta x \frac{\delta}{\delta x^i(\sigma)} e^{i \int d\sigma' A_j(x) \partial \sigma' x^j} |x\rangle \quad \text{BS-4}$$

$$= \int \Delta x e^{i \int d\sigma' A_j(x) \partial \sigma' x^j} \cdot \left[ \begin{array}{l} \textcircled{1} \quad \textcircled{2} \\ i(\partial_{\sigma} x^j \partial_i A_j(x) - \partial_{\sigma} x^j \partial_j A_i(x)) |x\rangle \\ \textcircled{3} \\ = i \tilde{F}_{ij}(x) \partial_{\sigma} x^j = i \tilde{F}_{ij}(\hat{X}) \partial_{\sigma} \hat{X}^j \\ + \frac{\delta}{\delta x^i(\sigma)} |x\rangle \end{array} \right]$$

If one can show that

$$-\frac{\delta}{\delta x^i(\sigma)} |x\rangle = i \tilde{P}_i(\sigma) |x\rangle \quad (*)$$

then

$$0 = i \left( \underbrace{\tilde{F}_{ij}(\hat{X}) \partial_{\sigma} \hat{X}^j}_{\text{" } \omega_{ji} \partial_{\sigma} \hat{X}^j} - \tilde{P}_i(\sigma) \right) \underbrace{\int \Delta x e^{i \int d\sigma' A_j(x) \partial \sigma' x^j} |x\rangle}_{\text{" } |B\rangle_{\omega}}$$

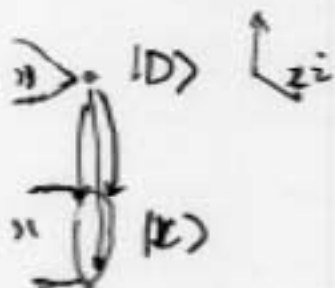
as required

$$\hat{X}^i(\sigma) |x\rangle = x^i(\sigma) |x\rangle$$

BS-5

$$|x\rangle = \exp\left[-i \int d\sigma \hat{P}_j^i(\sigma) x^j(\sigma)\right] |D\rangle$$

$|x=0\rangle$



$$[X^i(\sigma), \hat{P}_j^k(\sigma')] = i \delta^i_j \delta(\sigma - \sigma')$$

$$\hat{X}^i(\sigma) |D\rangle = 0$$

$$\frac{\delta}{\delta x^j(\sigma)} |x\rangle = -i \hat{P}_j^i(\sigma) |x\rangle$$

as required

We get

$$|B\rangle_\omega = \int Dx \exp\left[i \int d\sigma \left( \frac{1}{2} x^i \omega_{ij} \partial_\sigma x^j - \hat{P}_j^i(\sigma) x^j(\sigma) \right)\right] |D\rangle$$

$$\frac{1}{2} x^i \omega_{ij}$$

D-brane fluctuation

$$\frac{1}{2} x^i \omega_{ij} \partial_\sigma x^j + \delta \Delta_j(x)$$

$$x^j + \delta \phi^j(x)$$

new  $\Delta_j(x)$

$\phi^j(x)$

Further add  $\Delta_0(x) \partial_\sigma x^0$  with  $x^0$ -coordinate dependence,

$$- \hat{P}_I^0(\sigma) \phi^I(x) \Rightarrow - \hat{P}_0(\sigma) \phi^0(x) \Rightarrow$$

$$|A, \phi\rangle$$

$$= \int \underline{[Dx]_{\phi}} e^{i \int d\sigma [A_{\mu}(x(\sigma)) \dot{x}^{\mu}(\sigma) - \hat{P}_M(\sigma) \dot{\phi}^M(x(\sigma))]} |0\rangle$$

measure invariant under reparametrization

$$h_{\mu\nu}(\xi) \equiv \partial_{\mu} \phi^M(\xi) \partial_{\nu} \phi^N(\xi) \eta_{MN}$$

$\xi$  on



$\Rightarrow$  norm on  $\{x(\sigma)\}$ -space :

$$\|\delta x\|_{\phi}^2 = \int d\sigma \sqrt{h(x(\sigma))} h_{\mu\nu}(x(\sigma)) \delta x^{\mu}(\sigma) \delta x^{\nu}(\sigma)$$

$\Rightarrow [Dx]_{\phi}$

• reparametrization invariance of

$$|\mathcal{A}, \phi\rangle = \int [Dx]_{\phi} e^{i \int d\sigma [\mathcal{L}_{\mu}(x) \partial_{\sigma} x^{\mu} - \hat{P}_M \phi^M(x)]} |D\rangle$$

Under

$$\left\{ \begin{aligned} \delta_{\nu} \mathcal{A}_{\mu} &= \partial_{\mu} v^{\nu} \mathcal{A}_{\nu} + v^{\nu} \partial_{\nu} \mathcal{A}_{\mu} \\ &= \partial_{\mu} (v \cdot \mathcal{A}) + v^{\nu} \tilde{F}_{\nu\mu} \\ &= \left[ (\underline{d} i_{\nu} + i \underline{v} d) \mathcal{A} \right]_{\mu}, \\ &\quad \text{interior product} \end{aligned} \right.$$

$$\delta_{\nu} \phi^M = v^{\mu} \partial_{\mu} \phi^M,$$

$$\delta_{\nu} |\mathcal{A}, \phi\rangle$$

$$= \int [Dx]_{\phi} e^{i \int d\sigma [\mathcal{L}_{\mu}(x) \partial_{\sigma} x^{\mu} - \hat{P}_M \phi^M(x)]} \cdot i \int d\sigma \left[ \underbrace{(\partial_{\mu} (v \cdot \mathcal{A}) + v^{\nu} \tilde{F}_{\nu\mu})}_{=0} \partial_{\sigma} x^{\mu} - \hat{P}_M v^{\mu} \partial_{\mu} \phi^M \right] |D\rangle$$

(partial integration)

$$= -i \int [Dx]_{\phi} \int d\sigma \partial_{\sigma} v^{\mu}(x) e^{i \int d\sigma [\mathcal{L}_{\mu}(x) \partial_{\sigma} x^{\mu} - \hat{P}_M \phi^M(x)]} = 0$$

gauge fix

BS-8

① static gauge

$$A_j(x) = \frac{1}{2} x^i \omega_{ij} + A_j(x)$$

$$A_0(x) = A_0(x) \rightarrow \text{world-volume gauge field}$$

$$\phi^\mu(x) = x^\mu \quad \text{fixed}$$

$$\phi^I(x) \rightarrow \text{transverse fluctuation}$$

$$|A, \phi\rangle_{st} = \int Dx e^{i \int d\sigma \left[ \left( \frac{1}{2} x^i \omega_{ij} + A_j \right) \partial_\sigma x^j + A_0 \partial_\sigma x^0 - \hat{P}_\mu x^\mu - \hat{P}_I \phi^I \right]} |D\rangle$$

② NC-gauge

$$|A, \phi\rangle = \int [Dx]_\phi e^{i \int d\sigma [A_\mu \partial_\sigma x^\mu - \hat{P}_M \phi^M]} |D\rangle$$

$$\text{diffeo} \Rightarrow \int \underline{[Dx']} e^{i \int d\sigma [A'_\mu \partial_\sigma z'^\mu - \hat{P}_M \frac{\phi^M}{f_\phi(x')}] } |D\rangle$$

flat measure in  $\{x'\}$

call new  $\phi$

(Okuyama)

and,

$$A_j(x) = \frac{1}{2} x^i \omega_{ij} \quad (\dots \text{no fluctuation})$$

$$A_0(x) \Rightarrow \hat{A}_0(x) \quad (\dots \text{not operator})$$

$$\phi^i(x) = x^i + \theta^{ij} \hat{A}_j(x) \quad (\theta \equiv \omega^{-1})$$

$\phi^0(x) = x^0$   
 $\phi^I(x)$

... naturally obtained from IIB matrix model

$$\begin{cases} [\hat{z}^i, \hat{z}^j] = i\theta^{ij} \\ \hat{\varphi}^i = \hat{z}^i + \theta^{ij} \hat{A}_j \end{cases} \quad \begin{matrix} i \\ \text{no } A_\mu \end{matrix}$$

(  
 • Aoki, Ishibashi, Iso, Kawai, Kitazawa & Tada  
 • Ishibashi

$|A, \phi\rangle_{\text{NC}}$

$$= \int \underline{Dx} \ e^{i \int dt \left[ \frac{1}{2} x^i \omega_{ij} \partial_t x^j + \hat{A}_0(x) \partial_t x^0 \right]}$$

$$- \hat{P}_0 x^0 - \hat{P}_i (x^i + \theta^{ij} \hat{A}_j(x)) - \hat{P}_I \phi^I \Big| 0 \rangle$$

## §2.2. Current for reparametrization

$$\text{action: } \hat{S}_W[\Phi] = \text{Tr } L^*[\Phi]$$

$\omega_{\mu\nu}$  : 一般の symplectic form

$*$  : 対応する star 積

given by Kontsevich deformation  
quantization procedure

(or by Fedosov procedure  
⇒ Asakawa's talk)

$$\Phi = (\hat{A}_\mu, \phi^I, \lambda)$$

加書けていくとする。

original  $(A, \phi)$  -- reparametrization 子変

$\omega_{\mu\nu}$  を力学変数の一部  $\phi^I$  とする。

適切に 変換していい限り。

action  $\hat{S}_W[\Phi]$  (=  $\hat{S}[W, \Phi]$  と書くべき)

は不変になる。

↓  
(もう少し詳しく。)



## Reparametrization

$$x^\mu \mapsto x'^\mu$$

のとき

$$\Phi(x) \mapsto \Phi'(x')$$

$$\omega_{\mu\nu}(x) \mapsto \omega'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \omega_{\alpha\beta}(x)$$

↓ 対応して

$$* \mapsto *'$$

変形量子化

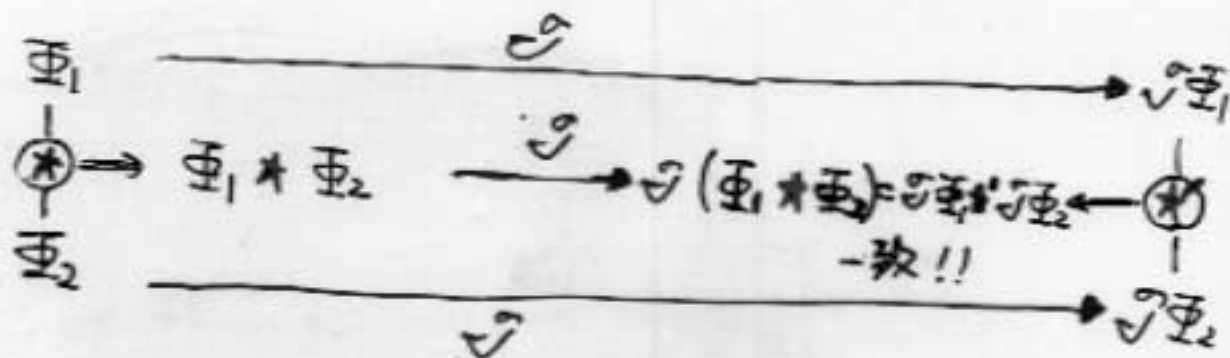
...  $*$  と  $*'$  は "gauge" equivalent:

differential operator  $\checkmark$

$$\left\{ \begin{array}{l} \Phi'(x') = \checkmark \Phi(x) \\ \checkmark \Phi_1 *' \checkmark \Phi_2 = \checkmark (\Phi_1 * \Phi_2) \end{array} \right.$$

$$\checkmark \Phi_1 *' \checkmark \Phi_2 = \checkmark (\Phi_1 * \Phi_2)$$

好むのかもある。



今

G-3

$$\text{Tr} \mapsto \text{Tr}' = \text{Tr} \hat{V}^{-1}$$

加 定義できたとすると.

$$\begin{aligned} \text{Tr} L^*(\Phi(x)) &\mapsto \text{Tr}' L'^*(\Phi'(x)) \\ &= \text{Tr} \hat{V}^{-1} \hat{V} L^*(\Phi(x)) \\ &= \text{Tr} L^*(\Phi(x)) \end{aligned}$$

$\therefore$  action は 不変:

$$S_{w'}[\Phi] = S_w[\Phi]$$

reparametrization の current は ?

$$\bullet \int w = \text{const} [\Phi] = \int \underbrace{N_{\text{CYM}}}_w [\Phi]$$

↑  
不変

$w$ : "metric"

$$\text{(c.f.) } \int dx \delta g^{\mu\nu}(x) \underline{T}_{\mu\nu}(x) \quad \text{energy-momentum tensor}$$

$$= S[g_{\mu\nu} + \delta g_{\mu\nu}, \Phi] - S[g_{\mu\nu}, \Phi]$$

$\omega \xrightarrow{\text{constant}} \omega + \delta\omega$  と (左と右の状態):

Gr-4

$$S_{\omega+\delta\omega}[\Phi] - S_{\omega}[\Phi]$$

$$= \text{Tr} \frac{\delta\theta^{\mu\nu}(x)}{f} \underbrace{J_{\mu\nu}(x)}_{\text{current !!}}$$

$$(\theta + \delta\theta) \cdot (\omega + \delta\omega) = 1$$

$$\Rightarrow \delta\theta = -\omega \cdot \theta \cdot \omega$$

$$\text{左辺の量} = S_{\omega+\delta\omega}[(\Phi + \delta\Phi) - \delta\Phi] - S_{\omega}[\Phi]$$

$$\cong (S_{\omega+\delta\omega}[\Phi + \delta\Phi] - S_{\omega}[\Phi]) = 0$$

$$- \text{Tr} \delta\Phi(x) \frac{\delta S_{\omega}[\Phi]}{\delta\Phi(x)}$$

$$\cong - (S_{\omega}[\Phi + \delta\Phi] - S_{\omega}[\Phi])$$

$$\Rightarrow \boxed{\begin{aligned} & - \text{Tr} (\delta\theta^{\mu\nu}(x) J_{\mu\nu}(x)) \\ & = S_{\omega}[\Phi + \delta\Phi] - S_{\omega}[\Phi] \end{aligned}}$$

$\delta\Phi$  associated with  $\delta\omega$  1? ?

T-dual coordinate description

$$|D\rangle \equiv |x=0\rangle = \int D\varphi \check{|\varphi\rangle}$$
$$\hat{P}_I(\sigma) \check{|\varphi\rangle} = \partial_\sigma \varphi_I(\sigma) \check{|\varphi\rangle}$$

(E.f.) SW-map ( $A_\mu \leftrightarrow \hat{A}_\mu$ )  
obtained by Okuyama

$$|A', \phi'\rangle_{NC} = |A, \phi\rangle_{NC}$$

$$\Delta_\mu = \frac{1}{2} x^\nu w_{\nu\mu} \delta S_A \text{ (not)}$$

変位LEの ;

$$\delta W = dS_A$$

$$A' = A + \delta A \leftarrow \text{input}$$

$$\phi'^i(x) = x^i + \theta^{ij} \hat{A}_j(x)$$

$$= \hat{A}_j(x) + \delta \hat{A}_j(x)$$

変位LE

$\delta \hat{A}_j$  に関する (余計な項は略)

$$|A', \phi'\rangle_{NC}$$

$$= \int Dx \exp \left[ i \int d\sigma \left( \left( \frac{1}{2} x^i w_{ij} + \delta A_j(x) \right) \partial_\sigma x^j \right. \right.$$

$$\left. \left. - \hat{P}_i(\sigma) (x^i + \theta^{ij} \hat{A}_j(x)) \right) \right] |D\rangle$$

$$\int Dy |\tilde{y}\rangle$$

$$= \int Dy Dx \exp \left[ i \int d\sigma \left( \left( \frac{1}{2} x^i w_{ij} + \delta A_j(x) \right) \partial_\sigma x^j \right. \right.$$

$$\left. \left. - \hat{P}_i(\sigma) (x^i + \theta^{ij} \hat{A}_j(x)) \right) \right] |\tilde{y}\rangle$$

$$y^i \equiv \theta^{ij} y_j \rightarrow = \int d\sigma \left( \frac{1}{2} (x^i - y^i) w_{ij} \partial_\sigma (x^j - y^j) \right.$$

$$\left. - \frac{1}{2} y^i w_{ij} y^j \right)$$

$$|\Delta', \phi'\rangle_{NC}$$

$$= \int D\psi e^{-i \int d\sigma \frac{1}{2} \psi^i \omega_{ij} \dot{\psi}^j}$$

$$\times \int D\xi \exp \left[ i \int d\sigma \left( \frac{1}{2} \xi^i \omega_{ij} \partial_\sigma \xi^j + \delta A_j (\psi + \xi) \partial_\sigma (\psi^j + \xi^j) - \partial_\sigma \psi^j \hat{A}_j' (\psi + \xi) \right) \right] |\psi\rangle$$

①

show

$$|\Delta, \phi\rangle_{NC}$$

$$= \int D\psi e^{-i \int d\sigma \frac{1}{2} \psi^i \omega_{ij} \dot{\psi}^j}$$

$$\times \int D\xi \exp \left[ i \int d\sigma \left( \frac{1}{2} \xi^i \omega_{ij} \partial_\sigma \xi^j - \partial_\sigma \psi^j \hat{A}_j (\psi + \xi) \right) \right] |\psi\rangle$$

②

と等しくなることを示す。

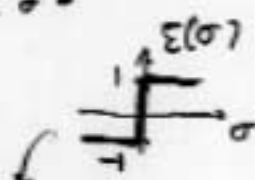
$$\Rightarrow \textcircled{1} = \textcircled{2} \text{ for } \psi$$

$$\Rightarrow \delta \hat{A}_\mu \equiv \hat{A}'_\mu - \hat{A}_\mu \text{ に対する}$$

perturbation theory with

$$\langle \xi^i(\sigma_1) \xi^j(\sigma_2) \rangle = i \frac{\theta^{ij}}{2} \epsilon(\sigma_1 - \sigma_2)$$

after Pauli-Villars regularization



⇒

$$\delta \hat{A}_\mu = -\delta A_\mu$$

$$+ 2i \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left( \partial_{\mu_1} \dots \partial_{\mu_{2k+1}} \hat{A}_\mu \right) \cdot \left( \frac{i}{2} \theta^{\mu_1 \nu_1} \right) \dots \cdot \left( \frac{i}{2} \theta^{\mu_{2k+1} \nu_{2k+1}} \right) \cdot \left( \partial_{\nu_1} \dots \partial_{\nu_{2k}} \delta A_{\nu_{2k+1}} \right)$$

⇒

$$-\text{Tr}(\delta \theta^{ij}(x) J_{ij}(x))$$

$$= \int_{\Lambda} \omega[\Phi + \delta \Phi] - \int_{\Lambda} \omega[\Phi]$$

$$= \Lambda^3$$

1-loop renormalization of current  $J_{ij}$

$$\langle [J_{ij}](x) \Phi_i(x_1) \dots \Phi_n(x_n) \rangle$$

如有限 (= 03 53) = ( )  $\in \mathbb{R}^4$

... OK

おの量  $\in$  評価中

## §2.5. Discussion and remained & related issues

▷ reparametrization (of world volume)

の生成子としての current  $J_m$

noncommutative field theory

( & deformation quantization )

の枠組で出せる。

▷ Membrane の reparametrization

の量子論的側面を調べるために

NC quantum field theory を用いることが

できる。

▷ やらねばならない :

・ 正確な fermion 場  $\lambda$  の変換性  $\delta\lambda$

を知らねばならない。

・ string field theory の変換性

の同一視と reparametrization symmetry

の代数 (群) 構造の把握。

( homotopic associative algebra ? )