BPS State Counting and Related Physics

YITP Workshop 2005

Kazutoshi Ohta

Theoretical Physics Laboratory
RIKEN

Introduction

- BPS objects are very important to understand the non-perturbative dynamics in gauge/string theory
  (Instantons, monopoles, D-branes, etc.)
- Statistical counting of BPS states plays essential roles
- BPS states counting may give the non-perturbative formulation of gauge / string theory

Introduction

Recently,

- Topological string amplitude from BPS state counting [Gopakumar-Vafa 1998]
- Exact instanton contribution to the prepotential for 4d N=2 theory [Nekrasov 2002]
- Exact effective superpotential for 4d N=1 theory by using the matrix model technique [Dijkgraaf-Vafa 2002]

But, SUSY and holomorphy are required

Introduction

In these analysis, we encounter

- Extended Young diagram, plane partition...
- Lower dimensional *bosonic* gauge theory
  - 3d Chern-Simons [Gopakumar-Vafa 1998]
  - 2d Yang-Mills [Matsuo-Matsuura-KO 2004]
- Free fermions, CFT... [Losev-Marshakov-Nekrasov 2003]
- Interesting statistical models (melting crystal, random walks...)

Introduction

But, why?? Is this accidental?

The answer should be
in string dualities

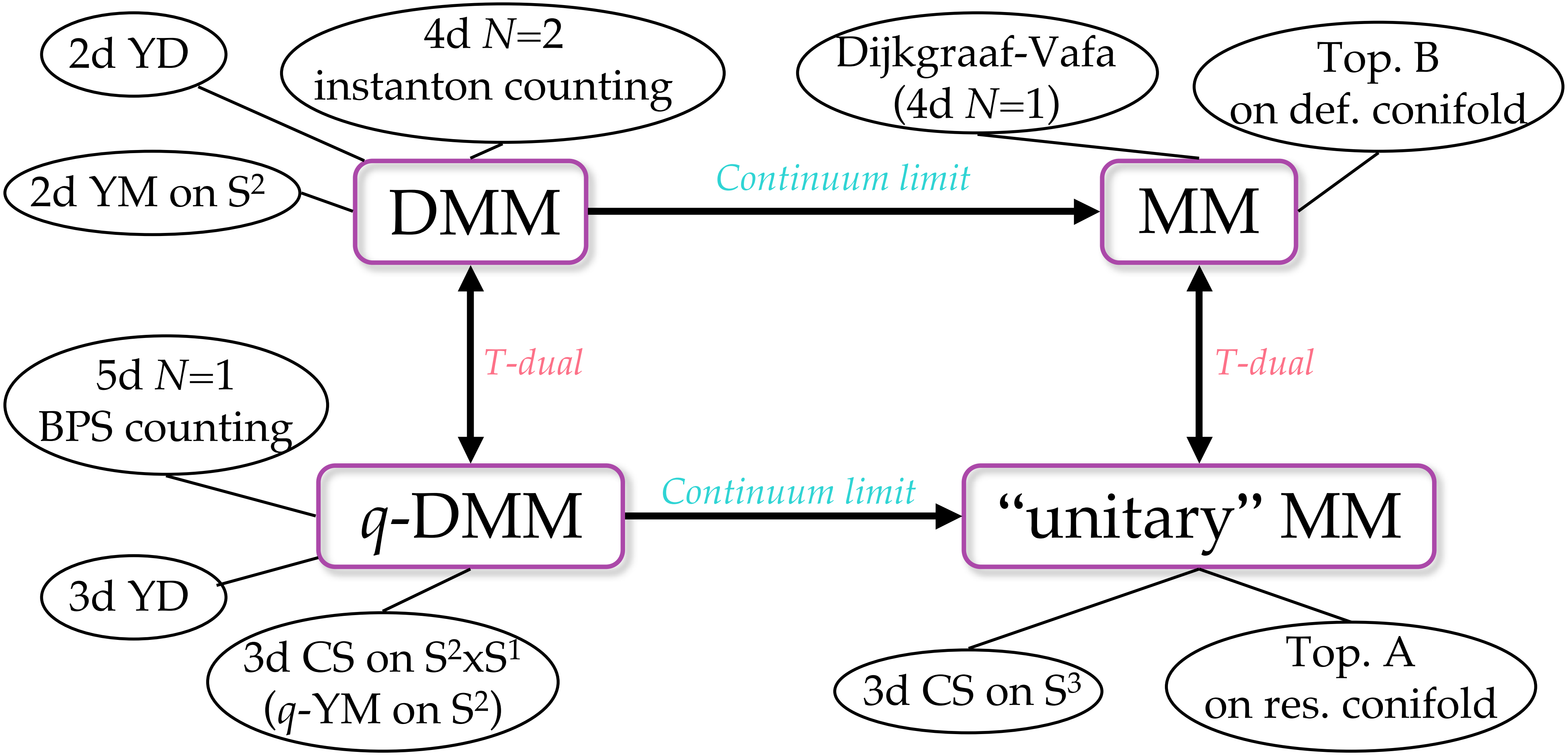
Integrable subset

Topological / Non-critical M-theory

7 dim *3 dim*

Overview of This Talk

Discrete Matrix Model exists behind various theories



Nekrasov's Instanton Counting

Prepotential of 4d N=2 SU(r) gauge theory has the following instanton expansion

$$\mathcal{F}(a_l) = \sum_{k=0}^{\infty} \mathcal{F}_k(a_l) \Lambda^{2rk}$$

Adjoint Higgs vev

k-instanton contribution is given by the "volume" of the instanton moduli space

$$\mathcal{F}_k(a_l) = \int_{\hat{\mathcal{M}}_{r,k}} 1$$

Nekrasov's Instanton Counting

To calculate k-instanton contribution, we utilize the D-instanton effective action

$$(\mu, \nu = 1, \ldots, 6)$$

$$S_{\rm inst} = \frac{1}{g^2} \mathrm{Tr} \left(\frac{1}{4} [X_\mu, X_\nu]^2 + \frac{1}{2} \bar{\Psi} \Gamma^\mu [X_\mu, \Psi] \right)$$

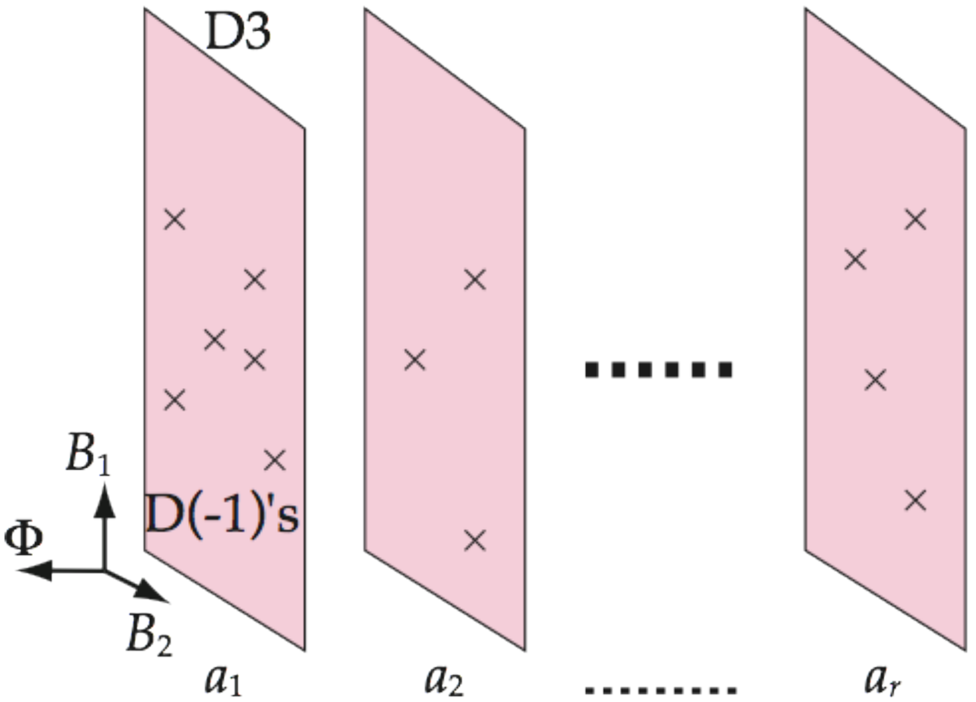
which is a reduced matrix model from 6d N=1 SU(k) Yang-Mills theory

The k-instanton contribution is obtained from the partition function

$$Z_k = \int \mathcal{D}X \mathcal{D}\Psi e^{-S_{\rm inst}}$$

Nekrasov's Instanton Counting

$$B_1 = X^1 + iX^2,\ B_1 = X^3 + iX^4,\ \Phi = X^5 + iX^6,\ \ldots$$



obeys ADHM eqs.

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta$$

$$[B_1, B_2] + IJ = 0$$

Nekrasov's Instanton Counting

$$Z_k = \int \mathcal{D}X\mathcal{D}\Psi e^{-S_{\rm inst}}$$

Topological twist

[Hirano-Kato]

+

Ω-background $\Omega_{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}$

[Moore-Nekrasov-Shatashvili]

$$Z_k(a_l|\epsilon) = \int \mathcal{D}\vec{\mathcal{B}}\mathcal{D}\vec{\mathcal{F}}\mathcal{D}\Phi e^{-Q_\epsilon \mathrm{Tr}\,\Xi[\vec{\mathcal{B}},\vec{\mathcal{F}},\Phi]}$$

$$= \frac{\epsilon^k}{k!(2\pi i\epsilon_1\epsilon_2)^k}\oint\prod_I \frac{d\phi_I}{P(\phi_I)P(\phi_I-\epsilon)}\prod_{I<J}\frac{\phi_{IJ}^2(\phi_{IJ}^2-\epsilon)}{(\phi_{IJ}^2-\epsilon_1^2)(\phi_{IJ}^2-\epsilon_2^2)}$$

$$(\phi_{IJ} = \phi_I - \phi_J,\ \epsilon = \epsilon_1 + \epsilon_2)$$

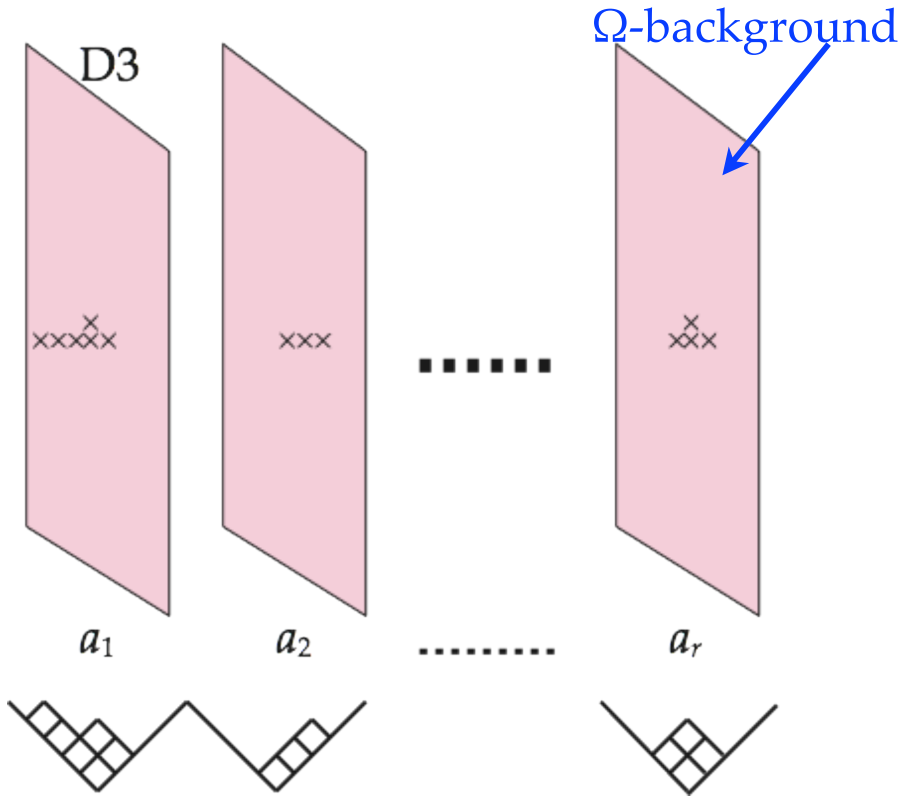
Nekrasov's Instanton Counting

Poles at the fixed points of

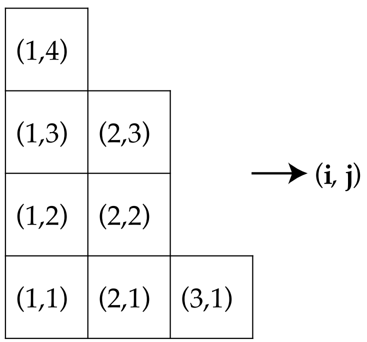
$$[B_l, \Phi] = \epsilon_l B_l$$

$$\Phi I - aI = 0$$

$$J\Phi - aJ = (\epsilon_1 + \epsilon_2)J$$



Young diagrams



$$B_1^{(i-1)} B_2^{(j-1)} \neq 0$$

$$\phi_{(i,j)} = a_l + \epsilon_1(i-1) + \epsilon_2(j-1)$$

Nekrasov's Instanton Counting

Finally, we obtain after setting $\epsilon_1 = -\epsilon_2 = \epsilon$

$$\boxed{Z_k(a_l|\epsilon) = \sum_{\vec{Y}} \prod_{(l,i)\neq(n,j)} \frac{a_{ln} + \epsilon(k_i^{(l)} - k_j^{(n)} - i + j)}{a_{ln} + \epsilon(-i + j)}}$$

where $a_{ln} = a_l - a_n$ and $\vec{Y}$ is a set of YDs with k total boxes

The prepotential of 4d N=2 theory is recovered by taking the limit of $\epsilon \to 0$

$$\mathcal{F}(a_l) = \lim_{\epsilon \to 0} \epsilon^2 \ln \left(\sum_k Z_k(a_l|\epsilon) \Lambda^{2rk} \right)$$

Nekrasov's Instanton Counting

Important point is:

Integration over instanton moduli space → diverge

Localization

Regularized summation over fixed points

||

Summation over sets of Young diagrams

D-brane Counting

D5-brane compactified on S^2 realizes 4d N=2 theory

Area of S^2

k-instanton contribution ~ $e^{-\frac{A}{g_s}k}$

~ k D1′s wrapping on S^2

r D5-branes compactified on S^2

|| *Large N reduction*

r sets of large N D-strings

D-brane Counting

Effective theory on large N D-strings

||

Topologically twisted 2d U(N) gauge theory

[Bershadsky-Sadov-Vafa 1996]

Grand canonical ensemble for D1+D(-1) bound state

$$\left\langle \exp\left[-\frac{1}{g_s}\int_{S^2}\mathrm{Tr}\,(i\Phi F+\lambda\wedge\lambda)-\frac{\mu}{2g_s}\int_{S^2}\mathrm{Tr}\,\Phi^2\right]\right\rangle_{\mathrm{top}}$$

$\wr||$ *Localization*

$$Z_{\mathrm{2dYM}}=\int\mathcal{D}A\mathcal{D}\Phi e^{-\frac{1}{g_s}\int_{S^2}\mathrm{Tr}\left(i\Phi F+\frac{\mu}{2}\Phi^2\right)}$$

Discrete Matrix Model

We can evaluate the 2d YM partition function exactly
[Migdal 1975, Blau-Thompson 1993]

$$Z = \int \mathcal{D}A\mathcal{D}\Phi e^{-\frac{1}{g_s}\int_{S^2} \mathrm{Tr}\left(i\Phi F + \frac{\mu}{2}\Phi^2\right)}$$

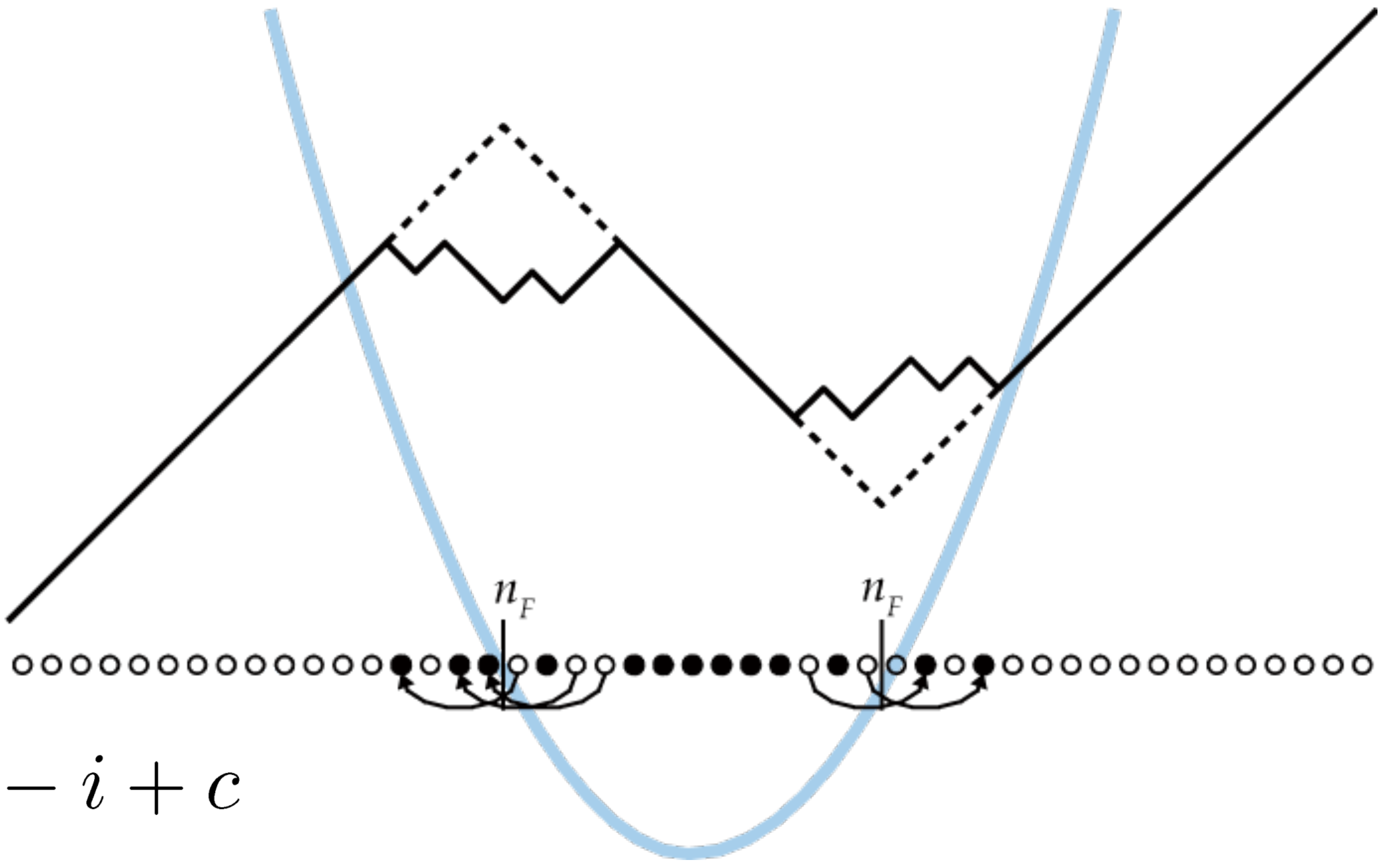
$$\Big\downarrow \quad \int_{S^2} dA_i = 2\pi p_i, \quad p_i \in \mathbb{Z}$$

$$Z = \frac{1}{N!}\prod_k \sum_{p_k\in\mathbb{Z}} \int d\lambda_k \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp\left[-\frac{1}{g_s}\left(2\pi i \lambda_k p_k + \frac{\mu A}{2}\lambda_k^2\right)\right]$$

$$\Big\downarrow$$

$$\boxed{Z = \sum_{n_1>n_2>\cdots>n_N} \prod_{i<j}(g_s n_i - g_s n_j)^2 \exp\left[-\frac{g_s \mu A}{2}\sum_i n_i^2\right]}$$

Eigenvalues



$$n_i = k_i - i + c$$

Eigenvalues ⟷ Free fermion

Instanton contributions ⟷ Fermion excitations ⟷ Young diagram

Difference Equation

Define

$$\Delta_{g_s} f(x) \equiv f(x+g_s) - f(x) \;\; \text{and} \;\; \rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - g_s n_i)$$

Vandermonde determinant (measure) part becomes

$$2N^2 \fint dxdy \, \log(x-y)\rho(x)\rho(y) = -2N^2 \fint dxdy \, \gamma(x-y|g_s)\Delta_{g_s}\rho(x)\Delta_{g_s}\rho(y)$$

where $\gamma(x|g_s)$ satisfies

$$\boxed{\Delta_{g_s}\Delta_{-g_s}\gamma(x|g_s) = 2\gamma(x|g_s) - \gamma(x+g_s|g_s) - \gamma(x-g_s|g_s) = -\log x}$$

Solution to Difference Equation

Recall

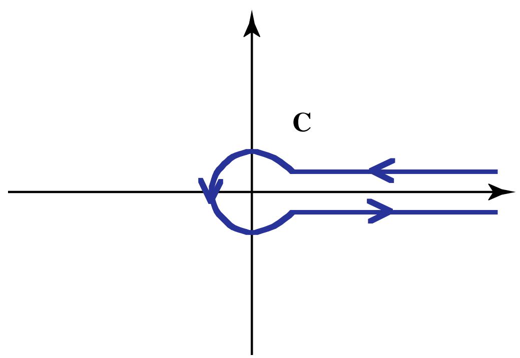
$$\log\Gamma(x+1) - \log\Gamma(x) = \log x$$

Solution to the 2nd order difference eq. is given by the *Barnes Double Gamma Function*

$$\gamma(x|g_s) = -\ln\Gamma_2(x|g_s)$$

or

$$\gamma(x|g_s) = \frac{1}{2\pi i}\int_C \frac{dt}{t}\frac{(\gamma+\log(-t))e^{-xt}}{(2\sinh\frac{g_s t}{2})^2}$$



Hankel contour

Schwinger's one-loop computation

(BPS particle pair creation in graviphoton background)

[Gopakumar-Vafa]

Asymptotic Expansion

The kernel function $\gamma(x|g_s)$ has the following Stirling like asymptotic expansion

$$\gamma(x|g_s) = \frac{1}{g_s^2}\left(\frac{1}{2}x^2\log x - \frac{3}{4}x^2\right) - \frac{1}{12}\log x + \sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\left(\frac{g_s}{x}\right)^{2g-2}$$

$$\Downarrow$$

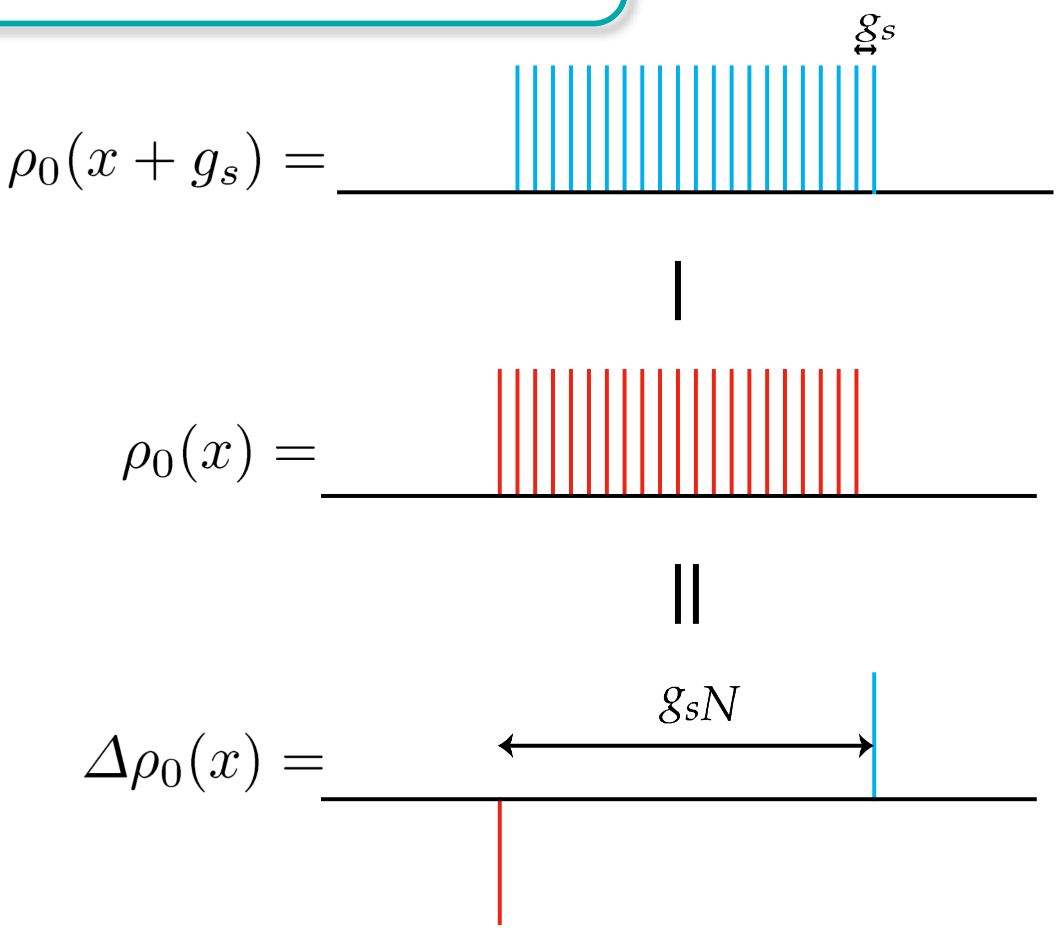
- Perturbative part of the 4d N=2 prepotential
- B-model topological string amplitude on deformed conifold
- c=1 string amplitude at self-dual radius

Ground State

$$-2N^2 \fint dx dy\, \gamma(x-y|g_s) \Delta\rho_0(x) \Delta\rho_0(y)$$

Ground state

0-instanton contribution
(perturbative part)



Perturbative Part

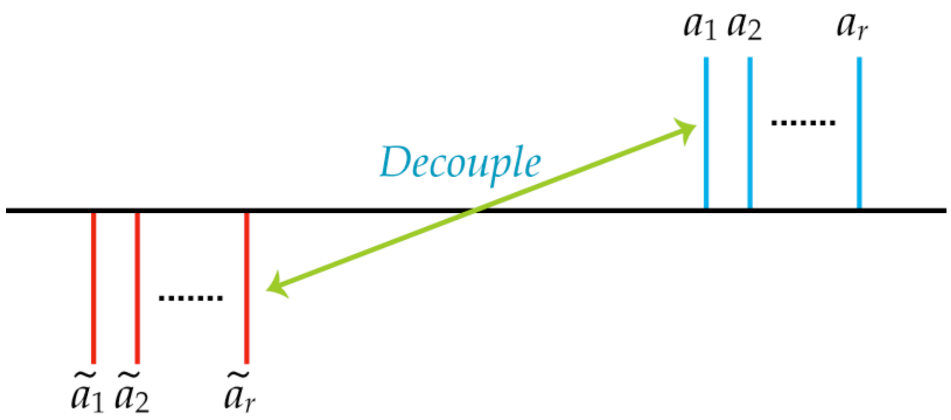
Interesting fact in one-cut solution

$$\int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\frac{1}{g_s}\sum_i \lambda_i^2} \simeq e^{2\gamma(g_s N|g_s)}$$

$$= 1/\mathrm{Vol}(U(N))$$

For multi-cut solution (multiple fermi surfaces)

$$\boxed{\mathcal{F}_{\mathrm{pert}}(a_l|g_s) = 2\sum_{l\neq n}\gamma(a_l - a_n|g_s)}$$



Large N Limit

In the large N limit of the multi-cut solution, two fermi surfaces are completely decoupled

So we get

$$Z_{\mathrm{DMM}} \underset{\text{Large } N}{\simeq} \left|Z_{\mathrm{Nekrasov}}\right|^2$$

Trigonometric Extension

$$[\Phi, \cdot] \longleftrightarrow D_t = \partial_t + [\Phi, \cdot]$$

T-dual

$$\prod_{i<j} (\lambda_i - \lambda_j)^2 \longleftrightarrow \prod_{n, i<j} (i\frac{n}{\beta} + \lambda_i - \lambda_j)^2 \simeq \prod_{i<j} \left(\frac{1}{\beta} \sinh \beta(\lambda_i - \lambda_j) \right)^2$$

Then we have

$$\boxed{Z = \sum_{n_1 > n_2 > \cdots > n_N} \prod_{i<j} \left(\frac{1}{\beta} \sinh \beta(g_s n_i - g_s n_j) \right)^2 \exp \left[-\frac{g_s \mu A}{2} \sum_i n_i^2 \right]}$$

q-deformed 2d YM [Aganagic-Ooguri-Saulina-Vafa]

Trigonometric Extension

This model relates to

- The prepotential of 5d N=1 gauge theory
- 3d Chern-Simons gauge theory on S^2xS^1
- Microscopic BPS blackhole state counting
  [Aganagic-Ooguri-Saulina-Vafa 2004]
- Non-perturbative formulation of topological B-model on conifold / c=1 string at self-dual radius

since in the $\beta\rightarrow 0$ limit, we recover 2d YM / Discrete Matrix Model

q-Difference Equation

Similar to the DMM (2d YM) case, the measure part becomes

$$2N^2 \fint dx dy \, \log\left(\frac{1}{\beta}\sinh\beta(x-y)\right)\rho(x)\rho(y)$$

$$= -2N^2 \fint dx dy \, \tilde{\gamma}(x-y|g_s;\beta)\Delta_{g_s}\rho(x)\Delta_{g_s}\rho(y)$$

where $\tilde{\gamma}(x|g_s;\beta)$ satisfies

$$\boxed{\Delta_{g_s}\Delta_{-g_s}\tilde{\gamma}(x|g_s;\beta) = -\log\left(\frac{1}{\beta}\sinh\beta x\right)}$$

Multiple Gamma Functions

$$\Gamma_2(x|\epsilon_1,\epsilon_2) \simeq \prod_{(m,n)\in\mathbb{N}^2} (x+m\epsilon_1+n\epsilon_2)^{-1}$$

4d case
2d YD

$$\gamma(x|g_s) = -\ln\Gamma_2(x|\epsilon_1,\epsilon_2)|_{\epsilon_1=-\epsilon_2=g_s}$$

$$\Gamma_3(x|\epsilon_1,\epsilon_2,\epsilon_3) \simeq \prod_{(l,m,n)\in\mathbb{N}^3} (x+l\epsilon_1+m\epsilon_2+n\epsilon_3)^{-1}$$

5d case
3d YD

$$\tilde{\gamma}(x|g_s;\beta)$$

$$= -\ln\Gamma_3(x|\epsilon_1,\epsilon_2,\epsilon_3) - \ln\Gamma_3(\epsilon_3 - x|\epsilon_1,\epsilon_2,\epsilon_3)|_{\substack{\epsilon_1=-\epsilon_2=g_s \\ \epsilon_3=\frac{i}{\beta}}}$$

Positive KK modes *Negative KK modes*

Recall $\quad -\ln\Gamma(x) - \ln\Gamma(1-x) \simeq \ln\sin x$

Integral Representation

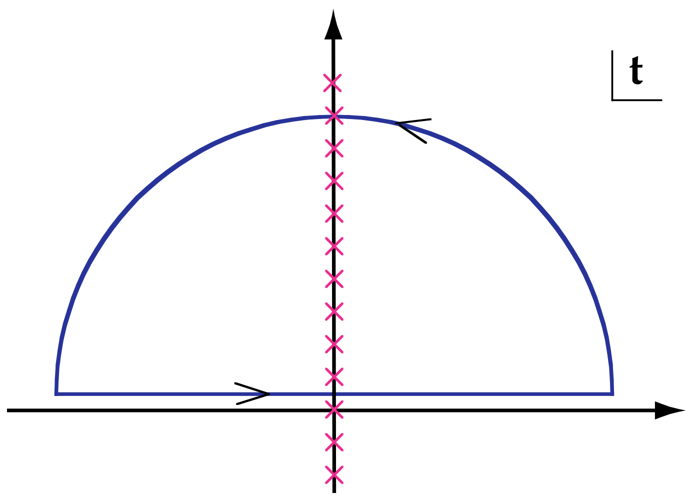
$$\tilde{\gamma}(x|g_s;\beta) = -\frac{i\pi}{6}B_{3,3}(x|\epsilon_1,\epsilon_2,\epsilon_3) + \int_{\mathbb{R}+i\delta}\frac{dt}{t}\frac{e^{-xt}}{\prod_{i=1}^{3}(1-e^{-\epsilon_i t})}\Bigg|_{\substack{\epsilon_1=-\epsilon_2=g_s\\ \epsilon_3=\frac{i}{\beta}}}$$

3rd order polynomial

➠ Perturbative part of 5d gauge theory

Residues of the integral

➠ Non-perturbative corrections of 5d gauge theory



Genus Expansion

More explicitly,

$$\tilde{\gamma}(x|g_s;\beta) = \frac{\beta x^3}{6g_s^2} - \frac{x^2}{2g_s^2}\log\beta - \frac{\beta x}{12} - \sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-2\beta n x}}{(\sinh g_s\beta n)^2}$$
$$= \sum_g \mathcal{F}_g g_s^{2-2g}$$

where

$$\mathcal{F}_0 = \frac{\beta x^3}{6} - \frac{x^2}{2}\log\beta + \frac{1}{\beta^2}\mathrm{Li}_3(e^{-2\beta x})$$

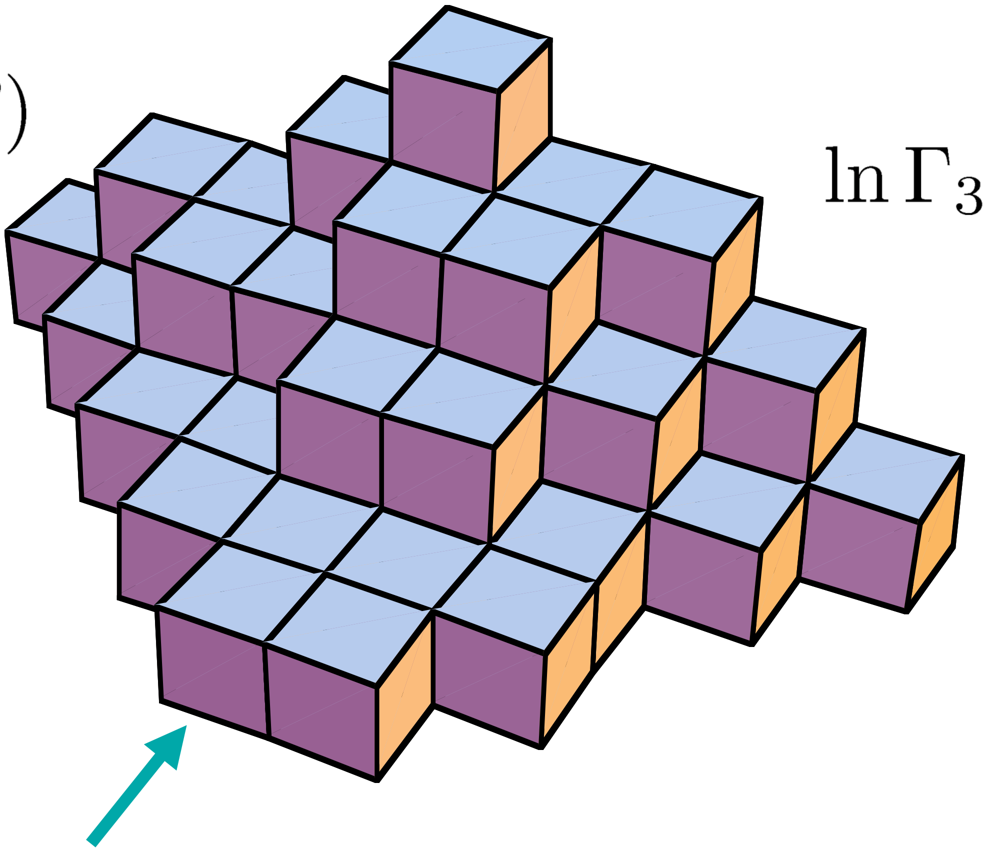
$$\mathcal{F}_1 = -\frac{1}{12}\log\sinh\beta x$$

Topological A-model on resolved conifold

$$\mathcal{F}_g = \frac{B_{2g}(2\beta)^{2g-2}}{2g(2g-2)}\mathrm{Li}_{3-2g}(e^{-2\beta x})$$

Random Plane Partition

$$\ln \Gamma_3(\frac{i}{\beta} - x | g_s; \beta)$$



$$\ln \Gamma_3(x | g_s; \beta)$$

2d partition (2d YD) $\rho(x)$

[Maeda-Nakatsu-Takasaki-Tamakoshi]

Chern-Simons Partition Function

Similar to the 4d case, we find

$$Z_{\mathrm{CS}} = \int \prod_i d\lambda_i \prod_{i<j} \sinh^2(\lambda_i - \lambda_j) e^{-\frac{1}{g_s}\sum_i \lambda_i^2} \simeq e^{2\tilde{\gamma}(g_s N | g_s; \beta=1)}$$

Alternatively, using the Weyl formula

$$Z_{\mathrm{CS}} = \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k+N} \right)^{N-j}$$

where we set

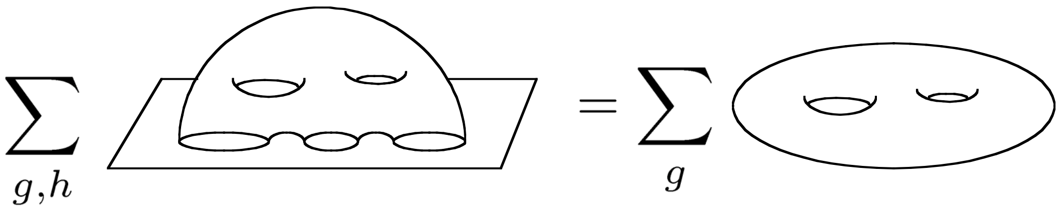
$$g_s = \frac{2\pi}{k+N}$$

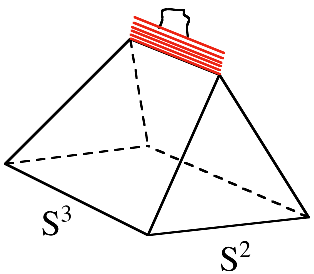
Topological String Amplitude

$$\sum_g \mathcal{F}_g^{\mathrm{NP}} g_s^{2g-2} + \sum_{g,h} \mathcal{F}^{\mathrm{P}}_{g,h} g_s^{2g-g+h} N^h = \sum_g \mathcal{F}_g^{\mathrm{top}}(t) g_s^{2g-2}$$

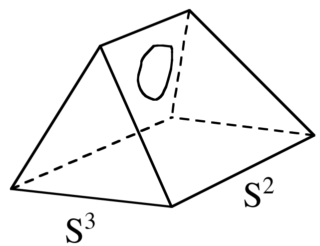
Chern-Simons on S^3
(open string)

Closed top. A-model
on resolved conifold





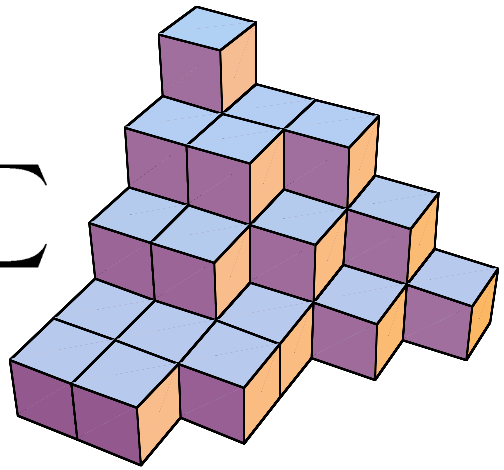
open



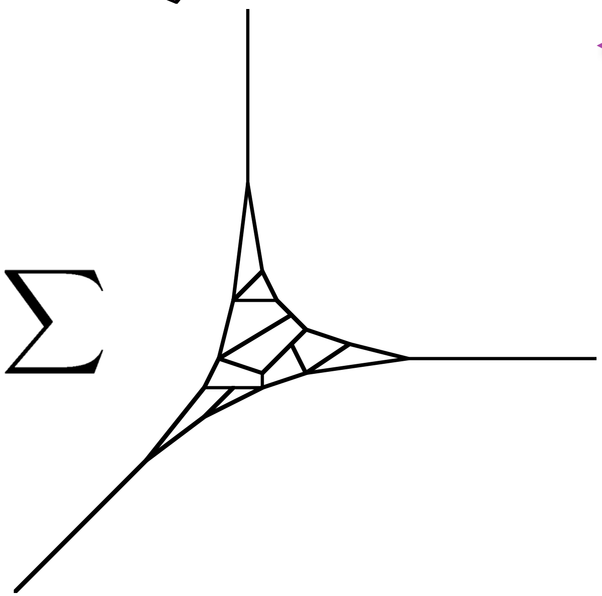
closed

Quantum Foam

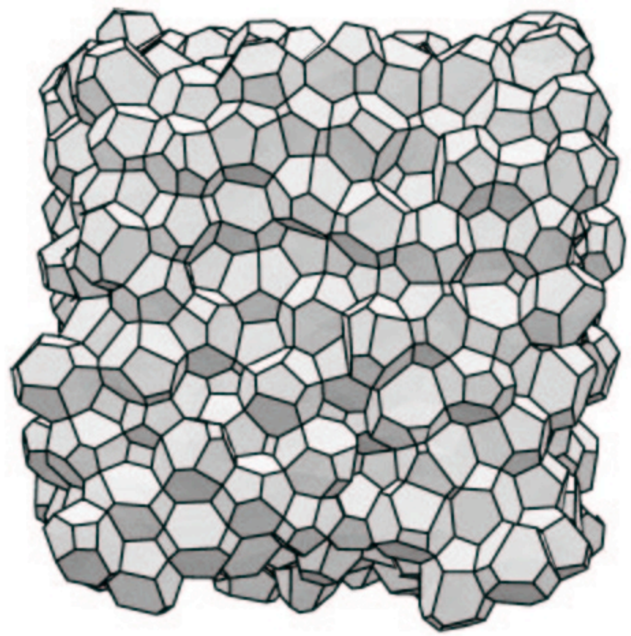
Young diagram $\sum$



Toric diagram $\sum$



$\sum$



Space-time foam

Toplogical M(atrix) Theory

Topological B-model

$$Z_k = \int \mathcal{D}X\mathcal{D}\Psi e^{-\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}[X_\mu,X_\nu]^2+\frac{1}{2}\bar{\Psi}\Gamma^\mu[X_\mu,\Psi]\right)}$$

$$(\mu,\nu=1,\ldots,6)$$

$$Z_{2\mathrm{dYM}} \simeq |\sum_k Z_k q^k|^2$$

T-dual

$$\tilde{Z}_k = \int \mathcal{D}Y\mathcal{D}\Psi e^{-\frac{1}{g^2}\int_0^\beta dt\mathrm{Tr}\left(\frac{1}{2}D_tY^iD_tY_i+\frac{1}{4}[Y^i,Y^j]^2+\mathrm{fermions}\right)}$$

$$(i,j=1,\ldots,5)$$

$$Z_{q-2\mathrm{dYM}} \simeq |\sum_k \tilde{Z}_k q^k|^2$$

[Hoppe-Kazakov-Kostov]

Topological A-model

Non-Critical Strings

$$\mathcal{F}_{c=1}(\mu;R) = 2\gamma(-i\mu + \frac{1}{2} - \frac{1}{2R}|\epsilon_1 = 1, \epsilon_2 = -\frac{1}{R})$$

Self-dual radius $\longleftrightarrow$ Self-dual Ω-background

Why? We need to understand the duality relations

Calabi-Yau

Near horizon of NS5

Little string

Liouville theory

Non-Critical M-theory?

We expect [Alexandrov-Kostov 2004, Horava-Keeler 2005]:

$$\mathcal{F}_{\rm NCM}(\mu;R) = 2\tilde{\gamma}(-i\mu + \frac{1}{2} - \frac{1}{2R}|\epsilon_1 = 1, \epsilon_2 = \frac{1}{R}, \epsilon_3 = \frac{i}{\beta})$$

In the $\beta \to 0$ limit, we get $c=1$ string

- q-deformed discrete matrix model
- 3d Young diagram
- Multiple gamma function
- Non-relativistic Fermi liquid in 2+1 dimensions

would be important to understand the non-perturbative dynamics of $c=1$ strings!

Conclusion

We have seen the relations between:

- BPS state counting
- Counting Young diagrams
- 2d Yang-Mills / 3d Chern-Simons
- Discrete matrix models
- Topological string theory
- Non-critical string theory

Future Directions

- Elliptic extension
  (6d gauge theory, Topological F-theory…)
- Counting monopoles, vortexes, domain walls
  (Effective theory on solitons)
- Quantum foam and quantum gravity
  (Quantum theory of form gravity, CS gravity)
- Relation to integrable systems
- AdS bubbling
- Landscape of SUSY vacua