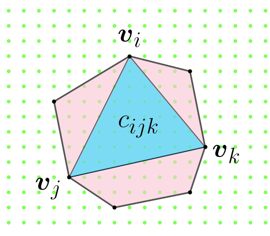
On the concavity of a-functions

University of Tokyo, Mathematical Sciences Akishi KATO
E-mail: akishi@ms.u-tokyo.ac.jp

The principle of a-maximization proposed by Intriligator and Wecht allows us to compute the exact R-charges and the scaling dimensions of the chiral operators of four-dimensional superconformal field theories. But some basic questions are still open: Does a-maximization always lead to a solution? If any, is it unique? In order to test AdS/CFT correspondence, it is desirable to give a definite answer to these questions without assuming the existence of dual geometry or physical stability of SCFT. The problem is not trivial since we can find examples with no critical points or those with several critical points some of which are not local maxima. We solve the problem for a large class of quiver gauge theories specified by a toric diagram. These are the class of quiver gauge theories intensivety studied using brane-tiling technique (Hanany et al.).

A toric diagram $P \subset \mathbb{R}^3$ is a two dimensional convex polygon embedded into height one, with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ from lattice points $\mathbb{Z}^3$. With each P there is associated a quiver gauge theory with $U(N)^l$ gauge groups and chiral bi-fundamental fields. Benvenuti-Zayas-Tachikawa and Lee-Rey showed that the trial a-function is given by $a(\phi) = \frac{9}{32}\mathrm{tr}\, R^3 = \frac{9N^2}{32}\sum_{i,j,k=1}^{n} c_{ijk}\phi^i\phi^j\phi^k$ where $c_{ijk} = \frac{1}{2}|\det(\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k)|$ is the area of the triangle with vertices $\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k$, and ϕ^i are the $U(1)_R$-charges of certain chiral fields satisfying the condition ($\heartsuit$) $\sum_{i=1}^{n}\phi^i = 2, \phi^i > 0$.

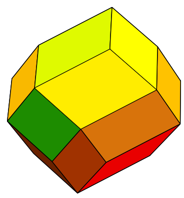
Our main results are as follows:



Theorem 1. If the toric diagram P is non-degenerate (i.e. $\boldsymbol{v}_i$ is an extremal point of P for all $i = 1, \cdots, n$), then $a(\phi)$ as a function on the domain ($\heartsuit$), has a unique critical point ϕ_*, and ϕ_* is the global maximum.

Theorem 2. Under the same assumption, the critical point ϕ_* satisfies $0 < \phi_*^i \leq \frac{2}{3}$, ($i = 1, \cdots, n$). Here, $\phi_*^i = \frac{2}{3}$ holds if and only if $n = 3$. (This does not contradict with the unitarity bound $R \geq \frac{2}{3}$; ϕ^i are R-charges of gauge non-invariant fields.)

The key idea of the proof is to identify a-function with the volume of a three dimensional polytope called *zonotope* generated by the vectors $\phi^i\boldsymbol{v}_i$ ($i = 1, \cdots, n$), and to apply Brunn-Minkowski inequality which asserts that the cubic root of the volume function is concave on the space of polytopes with respect to the operations of Minkowski sum and dilatation. As a biproduct, one can establish a combinatorial version of "a-theorem": the a-function always decreases when a toric diagram gets smaller.



Zonotope

[1] A. Kato, Zonotopes and four-dimensional superconformal field theories. arXiv:hep-th/0610xxx to appear.