Quantum Master Equation for QED in Exact RG

— 基研研究会（近畿大学), Aug. 10, 2007 —

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Abstract

厳密くりこみ群：ゲージ対称性をどう扱うか

- 正則化で（一見）破れたゲージ対称性に対するWT恒等式の導出
- 反場の導入とBRS変換の nilpotency の回復 $\Rightarrow$ ゲージ対称性の維持
- Polchinski方程式とBRS対称性

Motivation : One of the most important subjects in ERG

How to realize (gauge) symmetries, naively not compatible with reg. scheme ?

♣ Two different approaches:

1) finding symmetry preserving reg. (Morris *et al.* '00-)

2) using "broken Ward-Takahashi identities" or "modified Slavnov-Taylor identities"

for • S : generator of connected cutoff Green functions

or • Γ : generator of its 1PI part

to control symmetry breaking effects

(Becchi '93, Ellwanger, Bonini *et al.*. '94-, Morris *et al.* '00-, Freire *et al.* '01-)

Does exact symmetry exist in the latter approach ?

♢ A generic argument: Batalin-Vilkovisky (BV) antifield formalism

the presence of local (as well as global) symmetries

$\Leftrightarrow$ Quantum Master Equation (QME) $\Sigma_{\phi,\phi^*} = 0$ (for S)

- A general argument, $\Sigma_{\phi,\phi^*} = 0$ for cutoff- removed action

  $\Rightarrow \Sigma_{\Phi,\Phi^*} = 0$ for Wilson action. (Igarashi, Itoh and So '01)

- QME: hard to solve.

  — So far, it was shown for lattice chiral symmetry

  Ginsparg-Wilson relation $\equiv$ QME, and solved it for self-interacting fermions.

  — How about gauge theory ?

WT identity for Wilson action $S[\phi]$ in QED discussed by Sonoda:

suitable for study of its exact BRS symmetry

(cf. Becchi, Bonini-D'Attanasio-Marchesini for pure YM)

- BRS tr. depends on the Wilson action $S[\phi]$
- $\delta S[\phi] + \delta \mathcal{D}\phi = 0$
- $\delta^2 \neq 0$ (absence of nilpotency)

♢ We discuss, in this talk,

1) Path-integral derivation of WT identity for the Wilson action ($\Sigma_\Phi = 0$).
   (following Becchi, Bonini *et al.*)

2) Extension to the QME ($\Sigma_{\Phi,\Phi^*} = 0$)
   $\Rightarrow$ Non-trivial antifields dependence in our master action

3) A proof of BRS inv. of the Polchinski eq. using "quantum BRS tr."($\delta_Q^2 = 0$).

♢ Publications

On QED and ERG

- H.Sonoda, hep-th/0703167
- Y.Igarashi, K.Itoh, and H. Sonoda, PTP **118** (2007) 121

Related papers

- PTP **106** (2000) 149 (hep-th/0101101) $\longrightarrow$ presence of symmetries along RG flow
- PL **B526** (2002) 164 (hep-th/0111112) $\longrightarrow$ global symmetry
- PL B535 (2002) 363: NP B640(2002)95 (hep-lat/0206006)
  $\longrightarrow$ Lattice chiral symmetry in fermionic interacting theories

Ward-Takahashi (WT) identity for the Wilson action

◇ Consider generic gauge-fixed theory described by

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right), \qquad J\cdot\phi = J_A\phi^A$$

$$\mathcal{S}[\phi] = \frac{1}{2}\phi\cdot D\cdot\phi + \mathcal{S}_I[\phi], \qquad \phi\cdot D\cdot\phi = \phi^A D_{AB}\phi^B$$

and introduce momentum cutoff function

$$K(p/\Lambda) \approx \begin{cases} 1 & \text{for } p^2 < \Lambda^2 \\ 0 & \text{for } p^2 > \Lambda^2 \end{cases}$$

to decompose ϕ with propagator $D^{-1}(p)$

$\Rightarrow$ IR fields Φ with $K(p)D^{-1}(p)$ $\oplus$ UV field χ with $(1-K(p))D^{-1}(p)$

- To this end, insert gaussian integral for new fields θ^A (cf. Wetterich, Bonini *et al.*, Morris)

$$\int \mathcal{D}\theta \exp - \left\{\frac{1}{2}\left(\theta - J(1-K)D^{-1}\right)\cdot\frac{D}{K(1-K)}\cdot\left(\theta - (-)^J D^{-1}(1-K)J\right)\right\} = const$$

Introducing new set of fields: $\phi^A = \Phi^A + \chi^A, \qquad \theta^A = (1-K)\Phi^A - K\chi^A,$

we obtain $\mathcal{Z}_\phi[J] = N_J Z_\Phi[J]$, where regularized theory described by

$$
\begin{aligned}
Z_\Phi[J] &= \int \mathcal{D}\Phi \exp\left(-S[\Phi] + J \cdot K^{-1}\Phi\right) \\
S[\Phi] &\equiv \frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + S_I[\Phi] \qquad (\text{Wilson action}) \\
\exp\Big(-S_I[\Phi]\Big) &\equiv \int \mathcal{D}\chi \exp -\Big(\frac{1}{2}\chi \cdot (1-K)^{-1} D \cdot \chi + \mathcal{S}_I[\Phi + \chi]\Big) \\
N_J &= \exp -\frac{1}{2}\left((-)^J J \ \cdot K^{-1}(1-K) D^{-1} \cdot J\right) \qquad \left((-)^J \ J = (-)^{\epsilon_A} \ J^A\right)
\end{aligned}
$$

◇ Assume gauge-fixed action $\mathcal{S}[\phi]$ to be inv. under BRS tr. $\delta\phi^A = R^A[\phi]\ \lambda$

$$
\delta\mathcal{S} = \frac{\partial^r \mathcal{S}}{\partial \phi^A}\delta\phi^A = \Sigma_\phi \ \lambda = 0, \qquad \Sigma_\phi \equiv \frac{\partial^r \mathcal{S}}{\partial \phi^A} R^A[\phi] \qquad (\lambda : \text{anti} - \text{commuting const.})
$$

The invariance of the part. func. under $\phi^{A\prime} = \phi^A + \delta\phi^A$,

$$\mathcal{Z}_{\phi'}[J] = \mathcal{Z}_\phi[J] + \int \mathcal{D}\phi(-\delta\mathcal{S} + J\cdot\delta\phi)\exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right) = \mathcal{Z}_\phi[J]$$

This gives the standard WT identity for $\mathcal{Z}_\phi$ (BRS inv. of $\mathcal{D}\phi$ assumed):

$$\begin{aligned}
\langle\Sigma_\phi\rangle_\phi &= \mathcal{Z}_\phi{}^{-1}[J]\int\mathcal{D}\phi\ \Sigma_\phi \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right) \\
&= \mathcal{Z}_\phi{}^{-1}[J]\int\mathcal{D}\phi J_A R^A[\phi]\exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right) \\
&= \mathcal{Z}_\phi{}^{-1}[J]\cdot J_A R^A\left[\partial^l/\partial J\right]\mathcal{Z}_\phi[J] = 0
\end{aligned}$$

♢ Find the WT op. Σ_Φ using the relation $\mathcal{Z}_\phi[J] = N_J Z_\Phi[J]$:

$$\mathcal{Z}_\phi{}^{-1}[J]J_A R^A\left[\partial^l/\partial J\right]\mathcal{Z}_\phi[J] = Z_\Phi^{-1}[J]N_J^{-1}J_A R^A\left[\partial^l/\partial J\right]N_J Z_\Phi[J]$$

$$= Z_\Phi^{-1}[J]\int\mathcal{D}\Phi\ \Sigma_\Phi\ \exp\left(-S[\Phi] + J\cdot K^{-1}\Phi\right) = \langle\Sigma_\Phi\rangle_\Phi$$

to obtain the WT identity for regularized theory Z_Φ. (Becchi, Bonini *et al.*)

Derivation of WT identity in QED (cf. Sonoda hep-th/0703167)

♢ Consider QED with $\phi^A = \{A_\mu, B, c, \bar{c}, \psi, \bar{\psi}\}$ and $J_A = \{J_\mu, J_B, J_c, J_{\bar{c}}, J_\psi, J_{\bar{\psi}}\}$.

The action $\mathcal{S}[\phi] = \phi \cdot D \cdot \phi/2 + \mathcal{S}_I[\phi]$ is given by

$$\begin{aligned}\frac{1}{2}\phi^A D_{AB}\phi^B &= \int_k \Big[\frac{1}{2}A_\mu(-k)(k^2\delta_{\mu\nu} - k_\mu k_\nu)A_\nu(k) + \bar{c}(-k)ik^2 c(k) \\ &\qquad - B(-k)\big(ik_\mu A_\mu(k) + \frac{\alpha}{2}B(k)\big)\Big] + \int_p \bar{\psi}(-p)(\not{p} + im)\psi(p) \\ \mathcal{S}_I[\phi] &= -e\int_{p,\ k} \bar{\psi}(-p-k)\not{A}(k)\psi(p)\end{aligned}$$

It is inv. under the BRS tr.

$$\begin{aligned}\delta A_\mu(k) &= -ik_\mu\ c(k), \qquad \delta\bar{c}(k) = iB(k), \qquad \delta c(k) = \delta B(k) = 0 \\ \delta\psi(p) &= -ie\int_k \psi(p-k)\ c(k), \qquad \delta\bar{\psi}(-p) = ie\int_k \bar{\psi}(-p-k)\ c(k)\end{aligned}$$

This fixes $J_A R^A \left[\partial^l/\partial J\right] \equiv J \cdot R$ and the factor N_J:

$$\ln N_J = \int_p \Big(\frac{1-K}{K}\Big)(p) J_\psi(-p) \frac{1}{\not p + im} J_{\bar\psi}(p) + \cdots$$

♢ Now compute
$\mathcal{Z}_\phi{}^{-1}[J]\ (J\cdot R)\ \mathcal{Z}_\phi[J] = Z_\Phi^{-1}[J] N_J^{-1}\ (J\cdot R)\ N_J Z_\Phi[J]$
$= \langle \Sigma_\Phi \rangle_\Phi = 0$ to find Σ_Φ.
(We use the same notation for the IR fields: $\Phi^A = \{A_\mu, B, c, \bar c, \psi, \bar\psi\}$.)

♣ Nontrivial deformation from the standard WT identity generated by
1) the presence of N_J
2) the scale factor K^{-1} in $J\cdot K^{-1}\Phi$ and in $\Phi\cdot K^{-1} D\cdot\Phi/2$.

In particular, we have bilinear source terms in $\mathcal{Z}_\phi^{-1}(J\cdot R)_{\mathrm{matter}}\mathcal{Z}_\phi$:

$$J_\psi \cdots J_{\bar\psi} \to \exp(S)\frac{\partial}{\partial\bar\psi}\cdots\frac{\partial}{\partial\psi}\exp(-S) \to \left\langle \frac{\partial S}{\partial\bar\psi}\cdots\frac{\partial S}{\partial\psi} - \frac{\partial}{\partial\bar\psi}\cdots\frac{\partial}{\partial\psi}S \right\rangle_\Phi$$

We obtain

$$
\begin{aligned}
\Sigma_\Phi \quad = \quad & \int_k \Big\{ \frac{\partial S}{\partial A_\mu(k)}(-ik_\mu)c(k) + \frac{\partial^r S}{\partial \bar{c}(k)} iB(k) \Big\} \\
& -ie \int_{p,\ k} \Big\{ \frac{\partial^r S}{\partial \psi(p)} \frac{K(p)}{K(p-k)} \psi(p-k) - \frac{K(p)}{K(p+k)} \bar{\psi}(-p-k) \frac{\partial^l S}{\partial \bar{\psi}(-p)} \Big\} c(k) \\
& -ie \int_{p,\ k} \Big\{ \frac{\partial^l S}{\partial \bar{\psi}(-p+k)} \frac{\partial^r S}{\partial \psi(p)} - \frac{\partial^l \partial^r S}{\partial \bar{\psi}(-p+k) \partial \psi(p)} \Big\} U(-p, p-k) c(k)
\end{aligned}
$$

The matrix U (regularized both in IR and UV regions) is given by

$$
U(-p, p-k) = \frac{1 - K(p-k)}{\not{p} - \not{k} + im} K(p) - \frac{1 - K(p)}{\not{p} + im} K(p-k)
$$

◇ We may put the quadratic functional derivative term $(\partial S / \partial \psi)(\partial S / \partial \bar{\psi})$ as

$$
\Big[\frac{\partial^r S}{\partial \psi(p)}\ c(k) \Big\{ \frac{K(p)}{K(p-k)} \psi(p-k) - U(-p, p-k) \frac{\partial^l S}{\partial \bar{\psi}(-p+k)} \Big\} \Big]
$$

to define BRS tr. for the fields Φ^A:

$$
\begin{aligned}
\delta A_\mu(k) &= -ik_\mu\, c(k), \quad \delta\bar{c}(k) = iB(k), \quad \delta c(k) = \delta B(k) = 0 \\
\delta\psi(p) &= ie\int_k c(k)\Big\{\frac{K(p)}{K(p-k)}\psi(p-k) - U(-p, p-k)\frac{\partial^l S}{\partial\bar{\psi}(-p+k)}\Big\} \\
\delta\bar{\psi}(-p) &= ie\int_k \frac{K(p)}{K(p+k)}\bar{\psi}(-p-k)\; c(k)
\end{aligned}
$$

Then, Σ_Φ takes the form

$$\Sigma_\Phi = \frac{\partial^r S}{\partial \Phi^A} \delta \Phi^A + ie \frac{\partial^l \partial^r S}{\partial \bar{\psi} \partial \psi} \, c \, U$$

The last term can be interpreted as the Jacobian factor associated with $\psi(p) \to \psi(p) + \delta\psi(p)$.

- Remarks on the BRS tr. described above

  1) It depends on Wilson action $S[\Phi]$: $U \; (\partial S / \partial \bar{\psi}) \; c$ term in $\delta\psi$.

  2) It is not unique: We may modify $\delta\bar{\psi}$ instead of $\delta\psi$ or both.

  3) It is not nilpotent: $\delta^2 \psi \neq 0$.

  How do we construct nilpotent BRS tr. where the contribution from nontrivial Jacobian factor is included ?

BV formalism and the QME in QED

◇ For gauge fixed BRS inv. action $S_0[\phi]$, $\delta S_0 = \frac{\partial^r S_0}{\partial \phi^A}\delta\phi^A = 0$

introduce "extended" action

$$S[\phi, \phi^*] = S_0[\phi] + \phi^*_A \delta\phi^A \qquad (\phi^*_A : \text{anti} - \text{fields for fields } \phi^{\mathrm{A}})$$

- "canonical structure" defined by the anti-bracket

$$(X,\ Y) = \frac{\partial^r X}{\partial \phi^A}\frac{\partial^l Y}{\partial \phi^*_A} - \frac{\partial^r X}{\partial \phi^*_A}\frac{\partial^l Y}{\partial \phi^A}$$

Then, $\delta X = (X,\ S)$, so that (classical) BRS inv. of the action is expressed by the classical master equation

$$(S,\ S) = 0$$

♣ For the partition function $\int \mathcal{D}\phi\mathcal{D}\phi^* \exp - S[\phi, \phi^*]$, consider the change of variables with anti-commuting const. λ

$$\phi^A \to \phi^A + (\phi^A, S)\lambda = \phi^A + \frac{\partial^l S}{\partial \phi^*_A}\lambda, \qquad \phi^*_A \to \phi^*_A + (\phi^*_A, S)\lambda = \phi^A - \frac{\partial^l S}{\partial \phi^A}\lambda$$

1) change of the action $\quad S \to S + (S,\ S) \cdot \lambda$

2) change of the measure $\quad \log(\mathcal{D}\phi\mathcal{D}\phi^*) \to \log(\mathcal{D}\phi\mathcal{D}\phi^*) - 2\Delta S \cdot \lambda$

where $\quad \Delta \equiv (-)^{A+1} \frac{\partial^r}{\partial \phi^A} \frac{\partial^r}{\partial \phi^*_A}$

The part.func. remains inv. if the action $S[\phi]$ obeys the QME

$$\Sigma_{\{\phi,\phi^*\}} \equiv \frac{1}{2}(S,\ S) - \Delta S = 0 \qquad (S[\phi] \Rightarrow S_M[\phi] : \text{master action})$$

♢ Remarks on the anti-fields:

1) to be eliminated at final stage in the path-integral, e.g., $\int \mathcal{D}\phi\mathcal{D}\phi^* \delta(\phi^*)$

2) structure of the gauge algebra related degrees in power series expansion w.r.t. antifields

♢ Construction of master action S_M for regularized QED

We begin with an extended action being linear in the anti-fields

$$\begin{aligned} S_{\rm lin}[\Phi, \Phi^*] &= S[\Phi] + \int_k \Big\{ A^*_\mu(-k) i k_\mu c(k) + \bar{c}^*(-k) i B(k) \Big\} \\ &+ie \int_{p,\ k} \psi^*(-p) c(k) \Big\{ \frac{K(p)}{K(p-k)} \psi(p-k) - U(-p, p-k) \frac{\partial^l S}{\partial \bar{\psi}(-p+k)} \Big\} \\ &+ie \int_{p,\ k} \bar{\psi}(-p-k) c(k) \frac{K(p)}{K(p+k)}\ \bar{\psi}^*(p) \end{aligned}$$

We find

$$\frac{1}{2}(S_{\rm lin},\ S_{\rm lin}) - \Delta S_{\rm lin} \propto c\ c\ \psi^*\ U\ U \times (\text{functional} \ \ \text{derivative} \ \ \text{terms})$$

Add suitable quadratic term $\propto (\psi^*)^2$ to $S_{\rm lin}$ to find $S_{\rm quad}$, and repeat the same game.

This proceeds infinitely many times, but the infinite series takes the form

$$\sum_n (-ie\psi^* cU)^n \left(\frac{\partial}{\partial\bar{\psi}}\right)^n S,$$

which is nothing but the Taylor expansion of the action where $\bar{\psi}$ is replaced by $\bar{\psi} \to \bar{\psi} - ie\psi^* c\, U$.

♢ Let us introduce shifted fields $\hat{\Phi}^A = \{A_\mu,\ B,\ c,\ \bar{c},\ \psi,\ \bar{\psi} - ie\psi^* c\, U\}$.
Then, we can show

$$S_M[\Phi, \Phi^*] = S[\hat{\Phi}] + \int_k \Big\{ A^*_\mu(-k) i k_\mu c(k) + \bar{c}^*(-k) i B(k) \Big\}$$
$$+ ie \int_{p,\ k} \Big\{ \psi^*(-p-k) c(k) \frac{K(p+k)}{K(p)} \psi(p) + \bar{\psi}(-p-k) c(k) \frac{K(p)}{K(p+k)}\ \bar{\psi}^*(p) \Big\}$$

satisfies the QME $\quad \Sigma_{\{\Phi,\Phi^*\}} = \Big(S_M,\ S_M\Big)/2 - \Delta S_M = 0.$

♢ Remarks

- $\Delta S_M = \dfrac{\partial^2 S_M}{\partial\psi\partial\psi^*} \to \dfrac{\partial^2 S_M}{\partial\psi\partial\bar{\psi}}\, c\, U$

- This term is regularized both in IR and UV regions because of U.

♣ Using the divergence op. Δ, we define quantum BRS tr. (Lavrov-Tyutin) for any X, which is nilpotent because of $\delta_Q S_M = \Sigma_{\{\Phi,\Phi^*\}} = 0$:

$$\begin{aligned} \delta_Q X &\equiv (X,\ S_M) - \Delta X \\ \delta_Q^2 X &= \left(X, \Sigma_{\{\Phi,\Phi^*\}}\right) = 0 \end{aligned}$$

BRS invariance of the Polchinski equation

Two important equations in ERG:

1) QME that controls symmetries

2) the Polchinski equation, the flow equation.

◇ We have obtained the Polchinski eq. for master action $S_M[\Phi, \Phi^*]$.

Here we show its BRS invariance. Under the change of cutoff Λ

$$\partial_t \, \Sigma_{\{\Phi,\Phi^*\}} = (\partial_t S_M, \; S_M) - \Delta \partial_t S_M = \delta_Q \, \partial_t S_M = 0,$$

$$(\; \partial_t \equiv \Lambda \partial_\Lambda \;)$$

which implies $\quad \partial_t S_M = -\delta_Q G.$

Actually, G is the generator of the canonical transformation that generates the flow.

The concrete form of G

$$
\begin{aligned}
G &= G_1 + G_2 + G_3 \\
G_1 &\equiv \int_k A^*_\mu(-k)\Big[\frac{1}{2}\frac{\dot{K}(k)}{k^2}\Big(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\Big)\frac{\partial S_M}{\partial A_\nu(-k)} \\
&\qquad + \frac{\dot{K}(k)}{k^2} i k_\mu \Big(\frac{\partial S_M}{\partial B(-k)} - i\bar{c}^*(k)\Big)\Big] \\
G_2 &\equiv -\int_k \frac{\dot{K}(k)}{K(k)}\Big[A^*_\mu(-k)A_\mu(k) + B^*(-k)B(k) + \bar{c}^*(-k)\bar{c}(k) \\
&\qquad + \psi^*(-k)\psi(k) + \bar{\psi}(k)\bar{\psi}^*(-k)\Big] \\
G_3 &\equiv \int_p \psi^*(-p)\frac{\dot{K}(p)}{\not{p} + im}\Big[\frac{\partial^l S_M}{\partial\bar{\psi}(-p)} + \frac{ie}{K(p)}\int_k c(k)K(p-k)\bar{\psi}^*(p-k)\Big]
\end{aligned}
$$

Summary

$\diamondsuit$ We gave another derivation of WT identity for QED discussed by Sonoda
It is a gauge-fixed version of the QME: $\Sigma_{\{\Phi,\Phi^*\}}|_{\Phi^*\to 0} = \Sigma_\Phi = 0$

$\diamondsuit$ Non-trivial anti-fields dependence in our master action, $S_M(\bar\psi - ie\psi^*\, c\, U)$.

$\diamondsuit$ Start with BV formalism from the beginning.

For the partition function with ϕ^* as a source to the transformation $\delta\phi = R[\phi]$,

$$\mathcal{Z}_\phi[J,\phi^*] \equiv \int D\phi e^{-S[\phi]-\phi^*\cdot\delta\phi + J\cdot\phi}$$

we find the relation, $\quad -\partial^l_{\phi^*}\mathcal{Z}_\phi[J,\phi^*] = R[\partial^l_J]\cdot\mathcal{Z}_\phi[J,\phi^*]$

— that can be solved for QED to give the master action S'_M.

— S_M and S'_M are related via a canonical tr.

(uniqueness of master action up to canonical tr.)

$\diamondsuit$ Subject for future: QME for QCD (Becchi, Bonini *et al.* $\oplus$ Sonoda)