Topological Discrete Algebra, Ground State Degeneracy, and Quark Confinement in QCD

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MS, arXiv:0705.2476   + alpha

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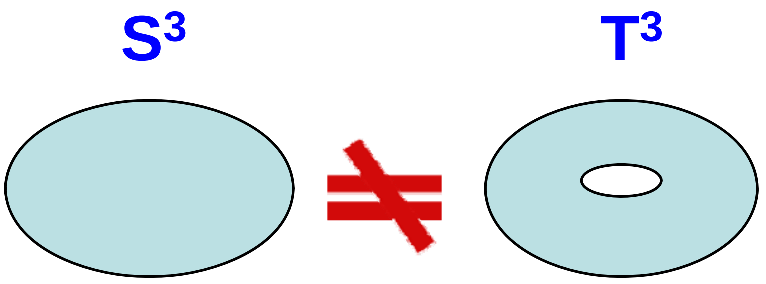
1. Topological Order

2. Quark deconfinement or confinement $\Longleftrightarrow$ Topological order or not

3. New criterion for quark confinement

$$\lim_{\beta\to\infty}\frac{\underset{\mathrm{T}^3}{\mathrm{Tr}}\left(e^{-\beta H}\right)}{\underset{\mathrm{S}^3}{\mathrm{Tr}}\left(e^{-\beta H}\right)}=1$$



Plan of this talk

1. What is topological order ?

2. Topological discrete algebra in QCD

3. Topological degeneracy in QCD

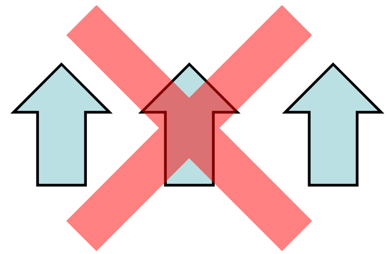
4. Test of our criterion

    - Wilson’s criterion
    - 1-loop analysis
    - Witten index
    - Fradkin-Shenker’s phase diagram

① What's topological order ?

Topological orders = orders which **can not be described by any local order parameter**

$$\cancel{\langle \phi \rangle \neq 0}$$

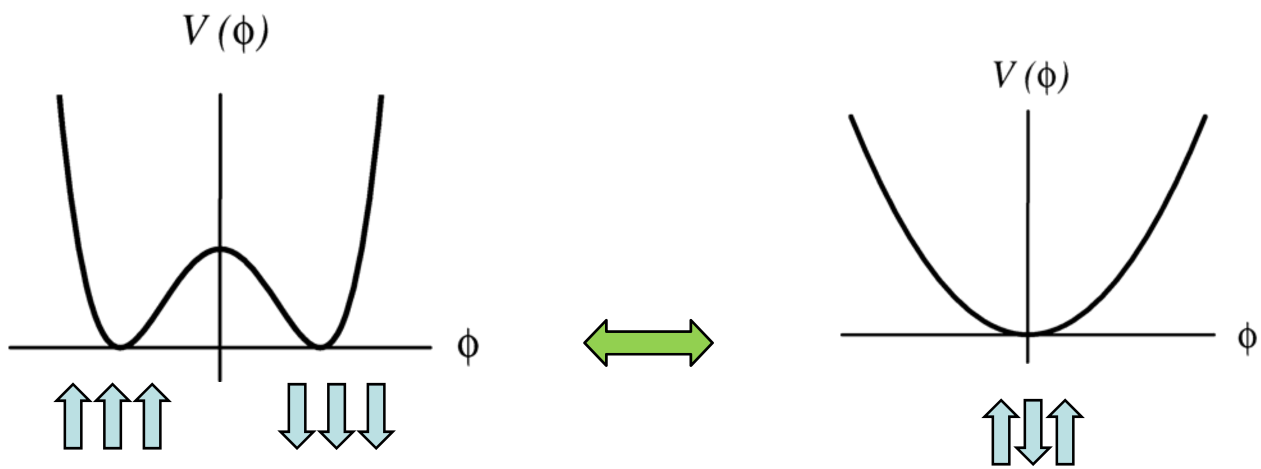


How to detect the topological order ?

Idea

Order = Ground state degeneracy

For symmetry breaking order …



Ordered

2-fold degeneracy

Not ordered

No degeneracy

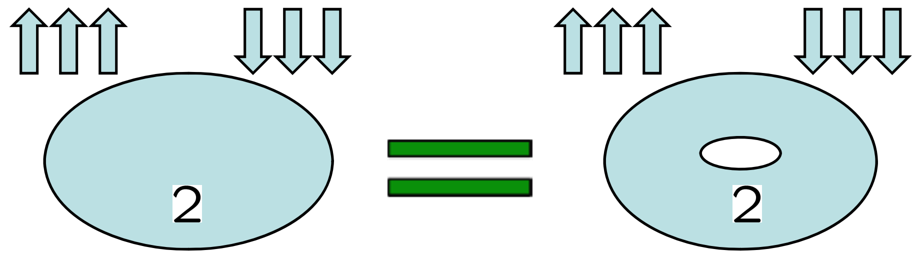
Topological orders

Topological orders = ground-state degeneracy depending on the topology of the space (**Topological degeneracy**)

Wen '90

For ground state degeneracy,

cf.) symmetry breaking orders



- For SSB, the degeneracy is **independent of the topology,**
- For topological orders, the degeneracy **depends on the topology**

ex.) Laughlin state $\nu = \frac{1}{q}$ Wen-Niu '90

degeneracy

What’s the origin of the topological degeneracy ?

For symmetry breaking order

σ_x

The degeneracy is due to the broken generators

For topological orders,

topological discrete algebra

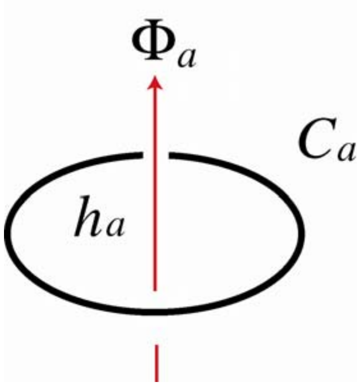
Topological discrete algebra

MS, M.Kohmoto, Y.-S.Wu '06
MS '07

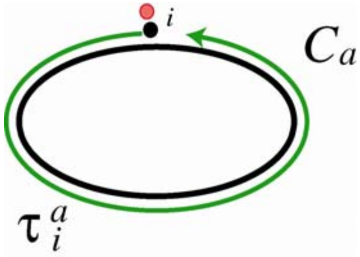
- Hidden algebra in topological orders
- Possible only for the space with non-trivial topology
- Origin of the topological degeneracy

$$T^3 = S^1 \otimes S^1 \otimes S^1$$

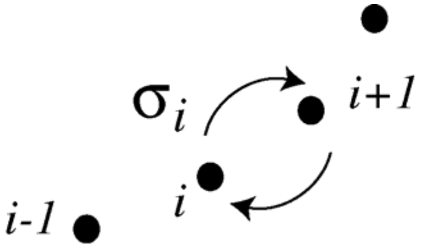
Flux insertion



Winding of excitation



Exchange of excitations

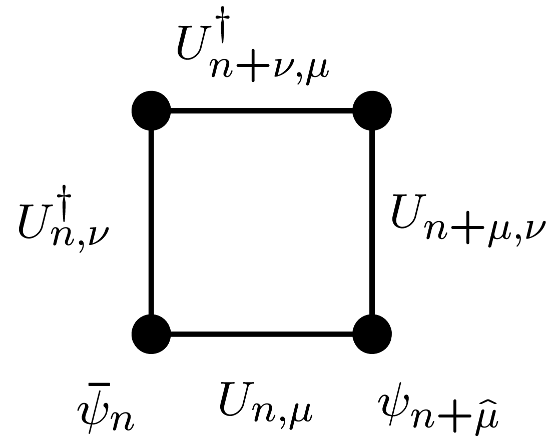


② Topological discrete algebra in QCD

QCD on $\mathrm{T}^3 \times \mathrm{R}$

$$S_{\mathsf{G}} = \sum_{\mathsf{P}} \frac{1}{g^2} \mathrm{Tr}(1 - U_{\mathsf{P}}),$$

$$(U_{\mathsf{P}} = U_{n,\mu} U_{n+\mu,\nu} U^{\dagger}_{n+\nu,\mu} U^{\dagger}_{n,\nu}),$$

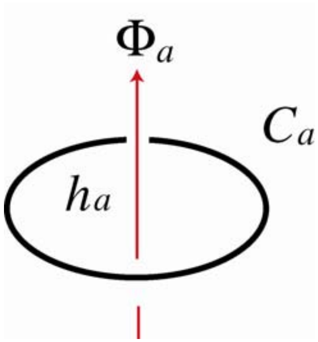


$$\begin{aligned} S_{\mathsf{F}} &= -\frac{1}{2} \sum_{n,\mu} \left(\bar{\psi}_n \gamma_\mu U_{n,\mu} \psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}} \gamma_\mu U^{\dagger}_{n,\mu} \psi_n \right) \\ &- \sum_n m \bar{\psi}_n \psi_n \end{aligned}$$

For QCD,

$$T^3 = S^1 \otimes S^1 \otimes S^1$$

Define flux insertion op. $U_a \ (a = 1, 2, 3)$



Adiabatic electromagnetic flux insertion
by $\Phi_a = 2\pi/3e$
(e : charge of quark)

cf.) for abelian case, $\Phi_0 = 2\pi/e$

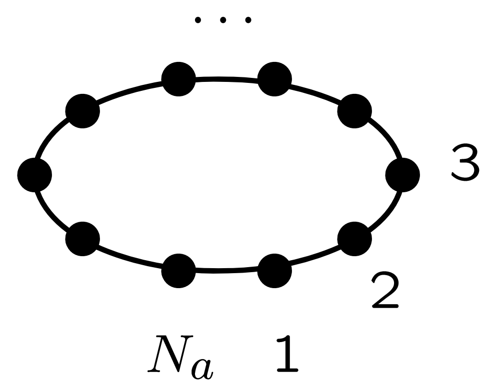
Note that the theory is the same after the flux insertion U_a

After flux insertion,

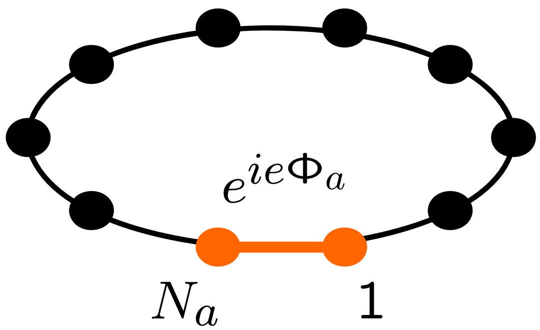
e: charge of quark

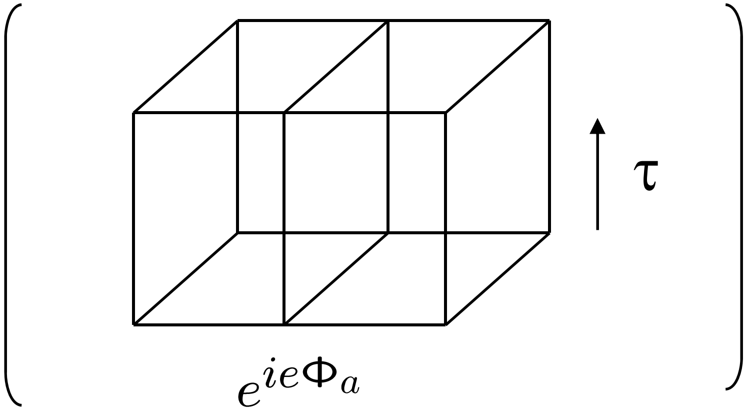
$$\bar{\psi}_n \gamma_a U_{n,a} \psi_{n+a} \to \bar{\psi}_n \gamma_a \underline{e^{ie\Phi_a/N_a}} U_{n,a} \psi_{n+a}$$

induced gauge field



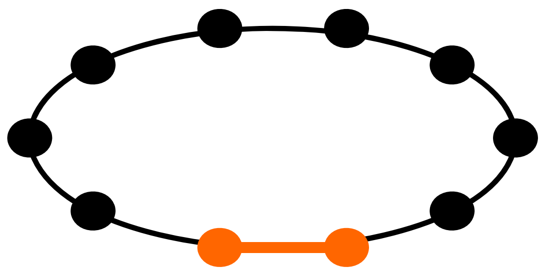
Since the electromagnetic filed is zero on the torus, we can eliminate the induced gauge filed except on the link between N_a and *1*





When $\Phi_a = 2\pi/3e$...

The remaining phase $e^{ie\Phi_a}$ can be removed by the transformation on $U_{n,a}$

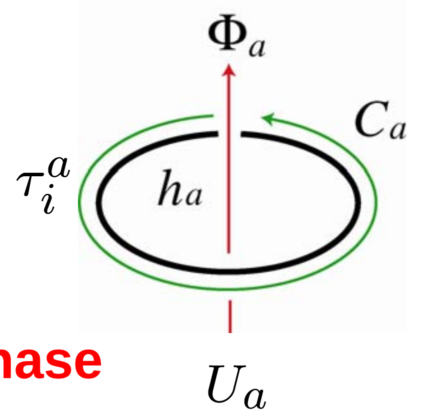


$e^{i2\pi/3} U_{n,a} \in SU(3)$

$$\mathcal{Z}(\Phi_a) = \mathcal{Z}(0)$$

Flux insertion by $2\pi/3e$ U_a

We have different Aharanov-Bohm phases between quark deconfinement phase and confinement one



◆ If the excitation is **quark**, we have

$$\tau_i^a U_a = \underline{e^{-2\pi i/3}} U_a \tau_i^a$$

translation after flux insertion

AB phase

⇕

◆ If the excitation is **hadron,** we have

$$\tau_i^a U_a = U_a \tau_i^a$$

No AB phase

Other commutation relations are determined by ..

② Permutation group

$$\sigma_k^2 = 1, \quad 1 \le k \le N-1,$$
$$(\sigma_k \sigma_{k+1})^3 = 1, \quad 1 \le k \le N-1$$
$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad 1 \le k \le N-3, \quad |l-k| \ge 2,$$
$$\tau_{i+1}^a = \sigma_i \tau_i^a \sigma_i, \quad 1 \le i \le N-1, \quad a = 1,2,3,$$
$$\tau_1^a \sigma_j = \sigma_j \tau_1^a, \quad 2 \le j \le N, \quad a = 1,2,3,$$
$$\tau_i^a \tau_j^b = \tau_j^b \tau_i^a, \quad i,j = 1, \cdots N, \quad a,b = 1,2,3.$$

③ Shur's lemma

- For quark, $\Longrightarrow$ $U_a U_b U_a^{-1} U_b^{-1} = e^{2\pi\lambda_{a,b}} \quad U_a^3 = \text{const.}$
- For hadron, $\Longrightarrow$ $U_a = \text{const.}$

Because the excitations (quarks or hadrons) are boson or fermion,

$$\sigma_i = \pm \mathbf{1}.$$

The unique solution of the permutation group

$$\Rightarrow \quad \boxed{\tau_i^a = T_a \text{ with } T_a T_b = T_b T_a}$$

We have two different topological discrete algebras

For quark deconfinement phase

T_a: quark winding operator

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$$

$$T_a T_b = T_b T_a \qquad U_a^3 = \text{const.}$$

⇕

For quark confinement phase

T_a: hadron winding operator

$$T_a T_b = T_b T_a \qquad U_a = \text{const.}$$

trivial !

③ Topological degeneracy in QCD

If quarks are **deconfined**, the physical states are classified with the permutation group of **quarks**.

Non trivial topological discrete algebra

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$$

$$T_a T_b = T_b T_a \qquad U_a^3 = \text{const.}$$

Topological ground state degeneracy !

On the other hand, …

If quarks are **confined**, the physical states are classified with the permutation group of **hadrons**.

$\Downarrow$

Topological discrete algebra is trivial

$\Downarrow$

No topological degeneracy !

The confinement and deconfinement phases in QCD are discriminated by the topological ground state degeneracy !

For SU(N) QCD on $T^n \times R^{4-n}$

- deconfinement: N^n –fold ground state degeneracy
- confinement: No topological degeneracy

Criterion for confinement

$$\boxed{\lim_{\beta\to\infty}\frac{\operatorname*{Tr}_{T^3}\left(e^{-\beta H}\right)}{\operatorname*{Tr}_{S^3}\left(e^{-\beta H}\right)}=1}$$

No topological degeneracy

④ Test of our criterion

We check this in the following manners

- comparison with the Wilson's criterion
- perturbative calculation of the topological ground state degeneracy
- comparison with Witten index
- consistency check with Fradkin-Shenker's phase diagram

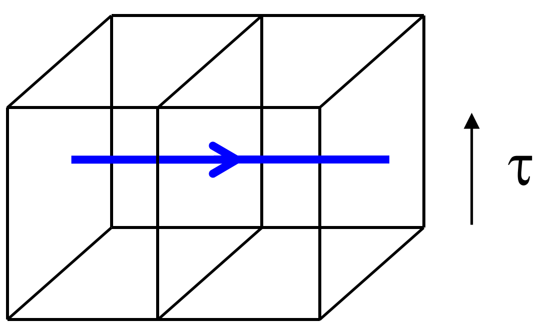
1. comparison with Wilson's criterion

QCD $\rightarrow$ SU(3) YM

heavy quark limit

The pure SU(3) YM has an additional symmetry known as center symmetry

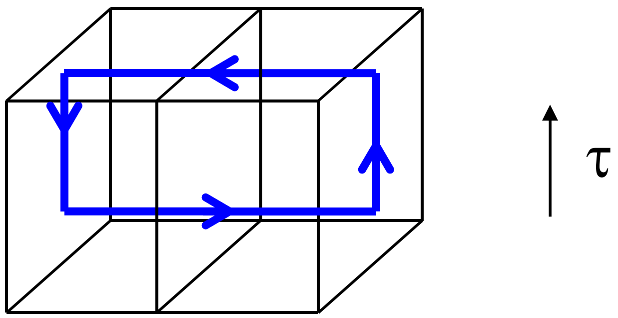
center symmetry



$$\begin{aligned} W(C_a) &\rightarrow B_a W(C_a) B_a^\dagger \\ &= e^{i2\pi/3} W(C_a) \end{aligned}$$

$$B_a : U_a \rightarrow e^{i2\pi/3} U_a$$

confinement phase



① area law

$$\langle W(C_a,\tau)W^\dagger(C_a,\tau')\rangle \sim e^{-\sigma L(\tau-\tau')}$$

temporal gauge

② cluster property

$$\langle W(C_a,\tau)W^\dagger(C_a,\tau')\rangle \overset{|\tau-\tau'|\to\infty}{\longrightarrow} |\langle W(C_a)\rangle|^2$$

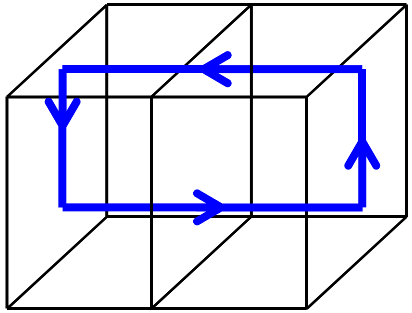
$\Rightarrow$ $$\langle W(C_a)\rangle = 0$$

The center symmetry is not broken

$\Rightarrow$ **No ground state degeneracy**

deconfinement phase

① perimeter law



$$\langle W(C_a,\tau)W^\dagger(C_a,\tau'))\rangle \sim e^{-mL}$$

$$\Rightarrow \quad \langle W(C_a)\rangle \neq 0$$

breaking of the center symmetry $\Rightarrow$ **3^3 degeneracy**

The degeneracy reproduces the one obtained from the topological discrete algebra

In the static limit, our criterion for confinement coincides with the Wilson's.

remark

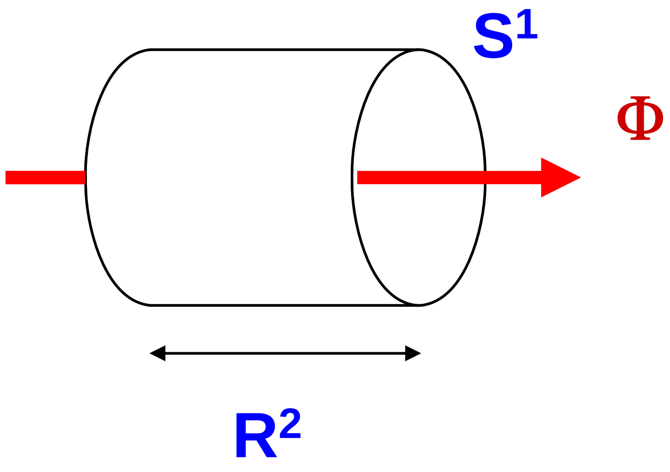
In this limit, topological discrete algebra becomes as follows.

$$
\left\{
\begin{array}{l}
T_a \to W(C_a) \\
U_a \to B_a \\
U_a T_a = e^{i2\pi/3} T_a U_a \to B_a W(C_a) = e^{i2\pi/3} W(C_a) B_a
\end{array}
\right.
$$

2. perturbative calculation of topological degeneracy

- We can calculate the topological degeneracy in the perturbation theory.
- Since the QCD is deconfined in the weak coupling, there must be the topological degeneracy.

SU(2) gauge theory on a $S^1 \times R^{1,2}$



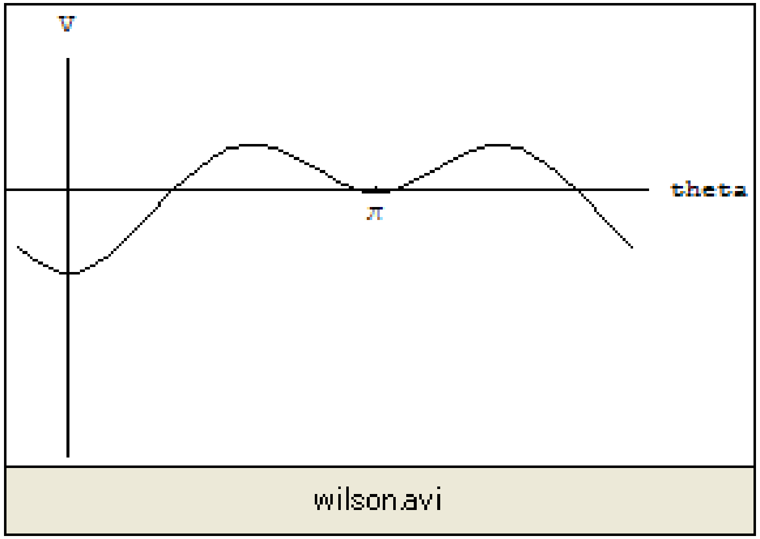
2-fold degeneracy should arises.

The Result

$$W(C) = P\,\mathrm{exp}\left(ig\int_C \langle A_\mu(x)\rangle dx\right) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$V(\theta) = K\{-h(2\theta)+N_f[h(\theta+\Phi/2)-h(\theta-\Phi/2)]\}$$

N.Weiss 81,
Hosotani 83



Again, the result is consistent with the topological degeneracy

4. N=1 SU(N) SYM

- Witten index has information on the ground state degeneracy on the torus T^3
- N=1 SU(N) SYM is in confinement phase, so there must be no topological degeneracy.

We know that there exists degeneracy on T^3

⇨ Witten index $\mathrm{Tr}(-1)^F = N$

$\uparrow$

at least N-degeneracy

But …

- The theory has a discrete chiral symmetry, which is expected to be spontaneously broken.

$$\Psi \to e^{i\pi/2N}\Psi$$

- The rotation symmetry is expected not to broken.

$$\Psi \to -\Psi$$

⇨ N-fold degeneracy is a consequence of SSB of chiral symmetry

No topological degeneracy = quark confinement

3. comparison with Fradkin-Shenker's phase diagram

Fradkin-Shenker's result (79)

- Higgs and the confinement phase are **smoothly connected** when the Higgs fields transform like **fundamental rep (complementarity)**.

- They are **separated by a phase boundary** when the Higgs fields transform like **other than fundamental rep**.

⇓

Our topological argument implies that **no ground state degeneracy** exists when Higgs and the confinement phase are **smoothly connected**.

Z_2 gauge theory $\quad (\mathbf{Z}_2 \in SU(2))$

$$S = K\sum_P U_P + \beta \sum_{n,\mu} \sigma_n U_{n,\mu}\sigma_{n+\mu} \qquad U_{n,\mu} = \pm 1,\ \sigma_n = \pm 1$$

Wilson loop

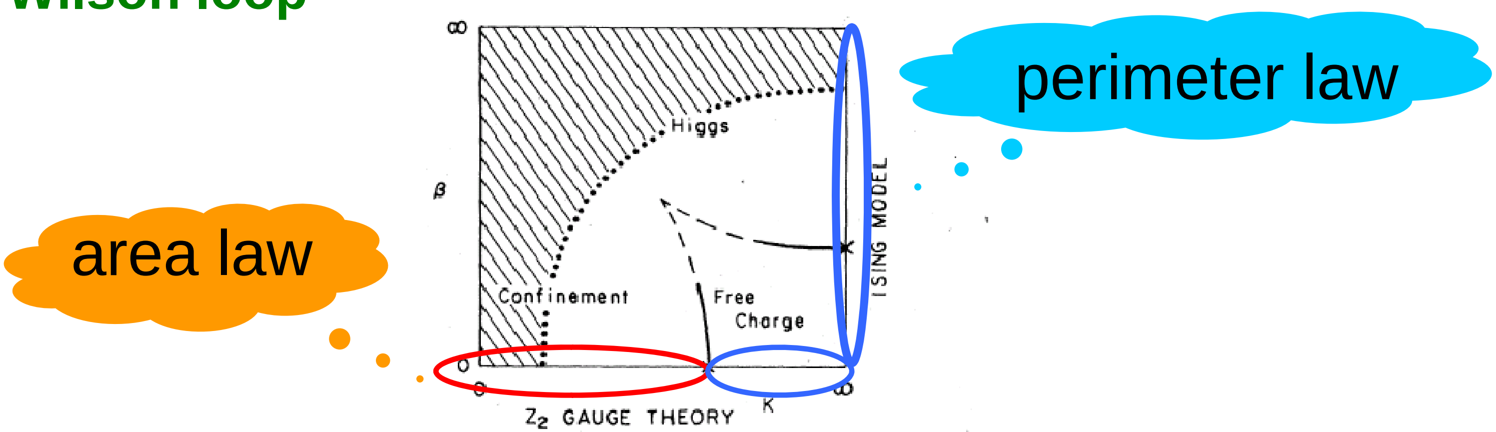
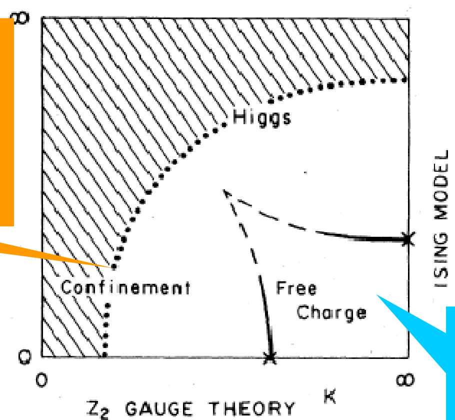


FIG. 1. The phase diagram for the Z_2 model ($d \geq 3$). The shaded region is where the bounds for analyticity hold. The full curves represent lines of second-order transitions given by (2.18). The broken lines are their extrapolation into the diagram. Notice that the analyticity region has a finite width at both the Higgs region ($K=\infty$) and confinement ($\beta=0$). Also note the curvature of the phase transition lines. The phases are described in the text.

Topological degeneracy

no ground state degeneracy



2^3 -fold degeneracy

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Abelian Higgs model

$$S[\phi(\vec{r});U_\mu(\vec{r})]=\frac{K}{2}\sum_{(\vec{r},\mu\nu)}\mathrm{Tr}[U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu^\dagger(\vec{r}+\hat{e}_\nu)U_\nu^\dagger(\vec{r})+\mathrm{h.c.}]+\frac{\beta}{2}\sum_{(\vec{r},\mu)}[\phi(\vec{r})\cdot D\{U_\mu(\vec{r})\}\cdot\phi^\dagger(\vec{r}+\hat{e}_\mu)+\mathrm{c.c.}],$$

1) Higgs charge =1

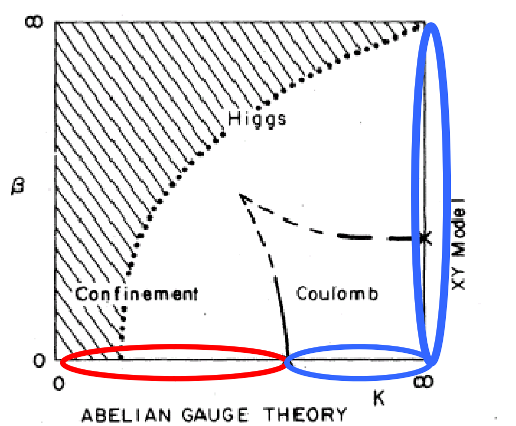


FIG. 2. Phase diagram for the Abelian model with Higgs fields in the fundamental representation ($d=4$). The broken line emerging from the XY transition ($K=\infty$) is a line of first-order transitions. The full line that emerges from the pure gauge transition ($\beta=0$) is a line of transitions of the *same* order as the pure gauge critical point. Notice the curvature of the lines. The phases are described in the text.

2) Higgs charge =2

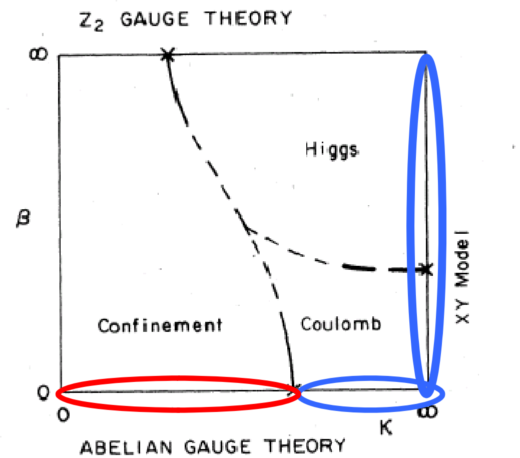


FIG. 3. Phase diagram of the Abelian Higgs model for Higgs fields with two units of charge. The difference with Fig. 2 is that there is a phase with confinement (in the Wilson sense) of static sources in the fundamental representation.

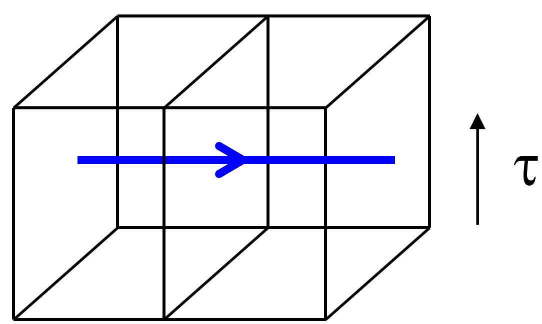
perimeter law

area law

Our topological argument works when the Higgs field has the two unit of charge.

$$S[\phi(\vec{r}); U_\mu(\vec{r})] = \frac{K}{2}\sum_{(\vec{r},\mu\nu)} \mathrm{Tr}[U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu^\dagger(\vec{r}+\hat{e}_\nu)U_\nu^\dagger(\vec{r}) + \mathrm{h.c.}] + \frac{\beta}{2}\sum_{(\vec{r},\mu)} [\underbrace{\phi(\vec{r})\cdot D\{U_\mu(\vec{r})\}\cdot\phi^\dagger(\vec{r}+\hat{e}_\mu)}_{\text{charge 2}} + \mathrm{c.c.}],$$

center symmetry

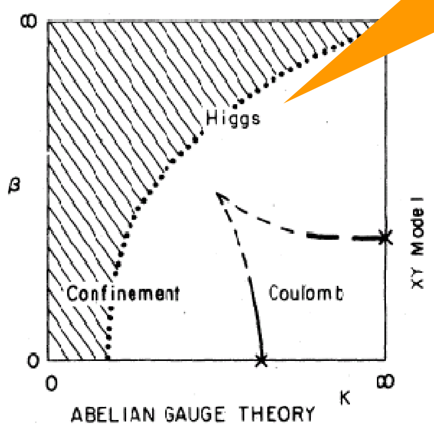


$$U_\mu(\vec{r}) \to -U_\mu(\vec{r})$$

Topological degeneracy

no ground state degeneracy

2^3-fold degeneracy



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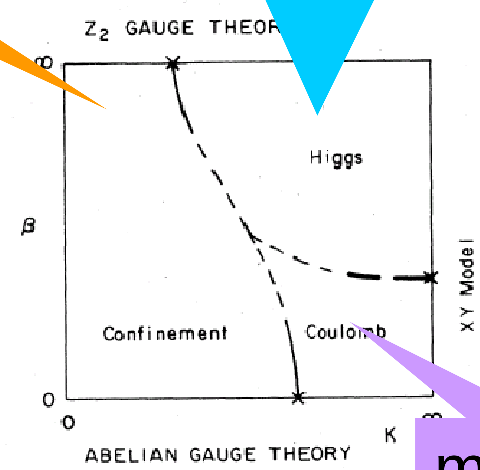

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massless excitation

◆ Summary

◆ New concept of order dubbed as topological order is introduced.

◆ Topological order is characterized by topological degeneracy.

◆ Topological degeneracy gives a gauge-invariant “order parameter” which distinguishes the quark confinement phase and deconfinement one.

◆ Topological criterion of the quark confinement is available even in the presence of the dynamical quarks.

◆ Our topological discrete algebra can be generalized into other systems, (i.e. standard model, SUSY)