(p, q)-string in wrapped membrane on 2-torus and matrix regularization

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基研研究会「弦理論と場の理論 – 量子と時空の最前線」
2007 年 8 月 6 日 (月) 近畿大学

based on a work with

S.Uehara (Utsunomiya U) and S.Yamada (Nagoya U), to appear

Introduction

- M-theory in 11-dimensions includes the membrane which is expected to play an important role to understand the fundamental d.o.f. of the theory.

  Bergshoeff-Sezgin-Townsend (1987).

- The membrane on $\mathbb{R}^9 \times T^2$ is related to type IIB string in 10-dimensions by means of the double dimensional reduction. The wrapped membrane is the (p, q)-strings in the reduced theory.

  Duff-Howe-Inami-Stelle (1987), J. H. Schwarz (1995,96), E. Witten (1996), Okagawa-Uehara-Yamada, (2006)

- On the other hands, the matrix regularized membrane on $\mathbb{R}^{11}$ and $\mathbb{R}^{10} \times S^1$ are Matrix theory and matrix string theory which give nonperturbative definitions of M-theory and type IIA superstring theory, respectively.

  Banks-Fischler-Shenker-Susskind (1997), Dijkgraaf-Verlinde-Verlinde (1997), de Wit-Hoppe-Nicolai (1988), Sekino-Yoneya (2001),

- The matrix regularization procedure of wrapped membrane on $\mathbb{R}^9 \times T^2$ are introduced.

  Uehara-Yamada, (2004)

Plan of this talk

- Introduction
- Procedure of matrix regularization of wrapped membrane on $\mathbb{R}^9 \times T^2$.
- T-dual to the standard form of Yang-Mills theory
- Duality of the theory
- Summary

Light-cone membrane action in curved background

Our starting point:

$$S_b = \frac{LT}{2}\int d\tau \int_0^{2\pi} d\sigma^1 d\sigma^2 \Big[(D_\tau X^M)^2$$

$$- \frac{1}{2L^2}\{X^M, X^N\}^2 + \frac{1}{L} D_\tau X^M A_{MNP}\{X^N, X^P\} \Big] ,$$

$$D_\tau X^M = \partial_\tau X^M - \frac{1}{L}\{A, X^M\},$$

$$\{A, B\} \equiv \epsilon^{ij}\partial_{\sigma^i} A \partial_{\sigma^j} B ,$$

where $\partial_\tau = \partial/\partial\tau$, $\partial_{\sigma^i} = \partial/\partial\sigma^i$, $i, j = 1, 2$, $M, N = 1, 2, \cdots, 9$, X^M is the target-space coordinates and A is the gauge field, T is the membrane tension and L is an arbitrary parameter, $[L]_m = -1$.

de Wit-Peeters-Plefka (1997)

Boundary conditions

We chose the boundary conditions as

$$\begin{aligned}
\sqrt{G_{99}}\, X^9(\sigma^1, \sigma^2 + 2\pi) &= 2\pi w_1 L_1 p + \sqrt{G_{99}}\, X^9(\sigma^1, \sigma^2)\,, \\
\sqrt{G_{88} - (G_{89})^2/G_{99}}\, X^8(\sigma^1, \sigma^2 + 2\pi) &= 2\pi w_1 L_2 q + \sqrt{G_{88} - (G_{89})^2/G_{99}}\, X^8(\sigma^1, \sigma^2)\,, \\
\sqrt{G_{99}}\, X^9(\sigma^1 + 2\pi, \sigma^2) &= 2\pi w_2 L_1 r + \sqrt{G_{99}}\, X^9(\sigma^1, \sigma^2)\,, \\
\sqrt{G_{88} - (G_{89})^2/G_{99}}\, X^8(\sigma^1 + 2\pi, \sigma^2) &= 2\pi w_2 L_2 s + \sqrt{G_{88} - (G_{89})^2/G_{99}}\, X^8(\sigma^1, \sigma^2)\,,
\end{aligned}$$

or

$$\begin{aligned}
X^9(\sigma^1, \sigma^2) &= R_1\,(w_1 p \sigma^2 + w_2 r \sigma^1) + Y^1(\sigma^1, \sigma^2)\,, \\
X^8(\sigma^1, \sigma^2) &= R_2\,(w_1 q \sigma^2 + w_2 s \sigma^1) + Y^2(\sigma^1, \sigma^2)\,,
\end{aligned}$$

where

$$pr + qs = 0, \quad ps - qr \equiv n_c, \quad R_1 \equiv \frac{L_1}{\sqrt{G_{99}}}\,, \quad R_2 \equiv \frac{L_2}{\sqrt{G_{88} - (G_{89})^2/G_{99}}}\,.$$

Procedure of matrix regularization: 3 steps

- Introduce the non-commutativity on the membrane spacesheet (σ^1, σ^2), or replace the product of functions on the spacesheet to the star-product.
- If possible, find the central elements of the star-commutator algebra and truncate the generators of the algebra consistently.
- Give a matrix representation of the (truncated) star-commutator algebra.

1st and 2nd steps

- Star-commutators algebra for the set of generators $\{e^{ik_1\sigma^1+ik_2\sigma^2}, \sigma^1, \sigma^2 \,|\, k_1, k_2 \in \mathbb{Z}\}$;

$$
\begin{aligned}
[\, e^{ik_i\sigma^i}, e^{ik'_j\sigma^j} \,]_* &= -2i \sin\left(\frac{\pi}{N}\,\epsilon^{ij} k_i k'_j\right)\, e^{i(k_i+k'_i)\sigma^i} , \\
[\, \sigma^1, e^{ik_i\sigma^i} \,]_* &= -\frac{2\pi k_2}{N}\, e^{ik_i\sigma^i} , \\
[\, \sigma^2, e^{ik_i\sigma^i} \,]_* &= \frac{2\pi k_1}{N}\, e^{ik_i\sigma^i} , \\
[\, \sigma^1, \sigma^2 \,]_* &= i\frac{2\pi}{N} .
\end{aligned}
$$

- Since we can find no central element of the algebra, the truncation is not possible in this case.

3rd step

- General matrix representation

$$
\begin{aligned}
e^{ik_1\sigma^1+ik_2\sigma^2} &\to e^{ik_i T^i{}_j \theta^j/N} \lambda^{-v_1 v_2/2} V^{v_2} U^{v_1}, \\
\sigma^2 &\to c^i \partial_{\theta^i} I_N + d_i \theta^i I_N, \\
\sigma^1 &\to e^i \partial_{\theta^i} I_N + f_i \theta^i I_N .
\end{aligned}
$$

where $k_i = u_i N + v_i$ and U, V are the $N \times N$ clock and shift matrices, respectively.
This gives the representation of the algebra with following constraints

$$
e^i d_i - c^i f_i = \frac{2\pi i}{N}, \quad T^i{}_j = \frac{2\pi i}{(c^1 e^2 - c^2 e^1)} \begin{pmatrix} -e^2 & e^1 \\ -c^2 & c^1 \end{pmatrix} .
$$

Our choice

$$(c^1, c^2, e^1, e^2, d_1, d_2, f_1, f_2) = \Big(-\frac{2\pi i s}{n_c}, \frac{2\pi i r}{n_c}, \frac{2\pi i q}{n_c}, -\frac{2\pi i p}{n_c}, -\frac{r}{N}, 0, \frac{p}{N}, 0 \Big)$$

Then we represented the fields as

$$\begin{aligned}
X^9(\sigma^1, \sigma^2) &\rightarrow -2\pi i R_1 \partial_{\theta^1} I_N + Y^1(\theta^1, \theta^2)\,, \\
X^8(\sigma^1, \sigma^2) &\rightarrow -2\pi i R_2 \partial_{\theta^2} I_N + (n_c R_2/N)\,\theta^1 I_N + Y^2(\theta^1, \theta^2)\,, \\
X^m(\sigma^1, \sigma^2) &\rightarrow X^m(\theta^1, \theta^2)\,, \\
A(\sigma^1, \sigma^2) &\rightarrow A(\theta^1, \theta^2) \\
\Xi(\theta^1, \theta^2) &= \sum_{u_1, u_2 \in \mathbb{Z}} \sum_{v_1, v_2 = -M}^{M} \Xi_{(k_1, k_2)}\, e^{iK_1\theta^1/N} e^{-iK_2\theta^2/N}\, \lambda^{-v_1 v_2/2}\, V^{v_2} U^{v_1}\,,
\end{aligned}$$

where $K_1 = pk_1 - rk_2\,, K_2 = -qk_1 + sk_2\,.$

Poisson bracket and integral

Poisson bracket

$$\{\,\cdot\,,\,\cdot\,\} \to -i\,\frac{N}{2\pi}\,[\,\cdot\,,\,\cdot\,]\,,$$

Integral

$$\int_0^{2\pi} d\sigma^1 d\sigma^2 \,* \to \frac{n_c}{N}\int_F d\theta^1 d\theta^2 \;\mathrm{Tr}[\,*\,]$$

where F is a parallelogram generated by the two vectors, $(2\pi s/n_c, -2\pi r/n_c)$ and $(-2\pi q/n_c, 2\pi p/n_c)$.

Matrix regularized wrapped membrane

$$
\begin{aligned}
S_{2+1} &= \frac{n_c LT}{2} \int d\tau \int_F d\theta^1 d\theta^2 \ \mathrm{Tr}\Big[G_{99}(F_{\tau\theta^1})^2 + 2G_{89} F_{\tau\theta^1} F_{\tau\theta^2} \\
&\quad + G_{88}(F_{\tau\theta^2})^2 - V_{T^2}(F_{\theta^1\theta^2})^2 + (D_\tau X^m)^2 - G_{99}(D_{\theta^1} X^m)^2 \\
&\quad - 2G_{89}\hat{G}_{mn} D_{\theta^1} X^m D_{\theta^2} X^n - G_{88}(D_{\theta^2} X^m)^2 + \frac{1}{2(2\pi L)^2}[X^m, X^n]^2 \Big],
\end{aligned}
$$

where

$$
\begin{aligned}
F_{\tau\theta^i} &= \partial_\tau Y^i - \frac{R_i}{L}\partial_{\theta^i} A + \frac{i}{2\pi L}[A, Y^i], \\
F_{\theta^1\theta^2} &= \frac{n_c R_1 R_2}{NL} I_N + \frac{R_1}{L}\partial_{\theta^1} Y^2 - \frac{R_2}{L}\partial_{\theta^2} Y^1 + \frac{i}{2\pi L}[Y^1, Y^2], \\
D_\tau X^m &= \partial_\tau X^m + \frac{i}{2\pi L}[A, X^m], \\
D_{\theta^i} X^m &= \frac{R_i}{L}\partial_{\theta_i} X^m + \frac{i}{2\pi L}[Y^i, X^m]
\end{aligned}
$$

To Yang-Mills theory

Redefinitions of the fields and paremeters;

$$
\begin{aligned}
Y^i(\theta^1, \theta^2) &\rightarrow \hat{\alpha} A_i(x^1, x^2)\,, \\
X^m(\theta^1, \theta^2) &\rightarrow \hat{\alpha} \phi^m(x^1, x^2)\,, \\
A(\theta^1, \theta^2) &\rightarrow \hat{\alpha} A_0(x^1, x^2)\,, \\
\theta^i &\rightarrow x^i/\hat{\Sigma}_i\,, \\
\tau &\rightarrow x^0/\hat{\Sigma}\,,
\end{aligned}
$$

where $[\hat{\alpha}]_m = -2$ and $[\hat{\Sigma}_1]_m = [\hat{\Sigma}_2]_m = [\hat{\Sigma}]_m = -1$. Then, in order to have the standard form of the Yang-Mills action, we should set

$$
\hat{\Sigma} = \frac{\hat{\alpha}}{2\pi L}\,, \qquad \hat{\Sigma}_i = \frac{\hat{\alpha}}{2\pi R_i}
$$

Standard form of Yang-Mills action

Finally we obtain

$$\begin{aligned} S_{2+1} &= \frac{n_c}{g_{YM}^2}\int dx^0 \int_{\mathcal{F}} dx^1 dx^2 \sqrt{-\det \mathcal{G}_{\alpha\beta}}\ \mathrm{Tr}\Big[-\frac{1}{4}\mathcal{G}^{\alpha\beta}\mathcal{G}^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} \\ &\quad -\frac{1}{2}\mathcal{G}^{\alpha\beta}D_\alpha\phi^m D_\beta\phi^n \hat{G}_{mn} + \frac{1}{4}\hat{G}_{mp}\hat{G}_{nq}[\,\phi^m,\phi^n\,][\,\phi^p,\phi^q\,]\Big], \\ F_{\alpha\beta} &= f_{\alpha\beta} + \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[\,A_\alpha, A_\beta\,], \\ D_\alpha\phi^m &= \partial_\alpha\phi^m + i[\,A_\alpha,\phi^m\,], \quad (\alpha,\beta,\gamma,\delta = 0,1,2) \end{aligned}$$

The worldvolume metric $\mathcal{G}_{\alpha\beta}$:

$$\mathcal{G}_{\alpha\beta} = \left(\mathcal{G}^{\alpha\beta}\right)^{-1}, \quad \mathcal{G}^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & G_{99} & G_{89} \\ 0 & G_{89} & G_{88} \end{pmatrix}$$

The constant magnetic flux $f_{\alpha\beta}$:

$$f_{01} = f_{02} = 0, \quad f_{12} = \frac{n_c}{2\pi N \hat{\Sigma}_1 \hat{\Sigma}_2} I_N$$

The (dimensionless) Yang-Mills coupling:

$$\tilde{g}^2_{YM} \equiv g^2_{YM} (2\pi\hat{\Sigma}_1 2\pi\hat{\Sigma}_2)^{1/2} = \frac{2\pi l^3_{11}}{(L_1 L_2)^{3/2}}$$

Fischler-Halyo-Rajaraman-Susskind (1997)

The boundary conditions

$$\begin{aligned} \Xi(x^1 + 2\pi\hat{\Sigma}_1,\, x^2) &= U^r V^p\, \Xi(x^1,\, x^2)\, (U^r V^p)^\dagger\,, \\ \Xi(x^1,\, x^2 + 2\pi\hat{\Sigma}_2) &= U^s V^q\, \Xi(x^1,\, x^2)\, (U^s V^q)^\dagger\,. \end{aligned}$$

Duality

The Yang-Mills action is invariant under the transformation:

$$\begin{aligned}
x^i &\to \tilde{x}^i = f^i(x)\,, \\
\mathcal{G}^{ij}(x) &\to \tilde{\mathcal{G}}^{ij}(\tilde{x}) = M^i{}_k M^j{}_l\, \mathcal{G}^{kl}(x)\,, \\
A_i(x) &\to \tilde{A}_i(\tilde{x}) = A_j(x)(M^{-1})^j{}_i\,, \\
A_0(x) &\to \tilde{A}_0(\tilde{x}) = A_0(x)\,, \\
\begin{pmatrix} A_{mn9} \\ A_{mn8} \end{pmatrix}(x) &\to \begin{pmatrix} \tilde{A}_{mn9} \\ \tilde{A}_{mn8} \end{pmatrix}(\tilde{x}) = M \begin{pmatrix} A_{mn9} \\ A_{mn8} \end{pmatrix}(x), \\
A_{m89}(x) &\to \tilde{A}_{m89}(\tilde{x}) = (\det M)\, A_{m89}(x)\,, \\
A_{mnp}(x) &\to \tilde{A}_{mnp}(\tilde{x}) = A_{mnp}(x)\,, \\
\phi^m(x) &\to \tilde{\phi}^m(\tilde{x}) = \phi^m(x)\,,
\end{aligned}$$

where

$$M^i{}_j \left(= \frac{\partial \tilde{x}^i}{\partial x^j} \right) \in GL(2,\mathbb{R})\,.$$

$SL(2,\mathbb{R})$-subgroup

$SL(2,\mathbb{R})$-subgroup of the symmetry;

$$\tilde{\mathcal{G}}^{ij}(x) = \Lambda^i{}_k \Lambda^j{}_l \mathcal{G}^{kl}(\Lambda^{-1}x) , \quad \tilde{A}_i(x) = (\Lambda^{-1})^j{}_i A_j(\Lambda^{-1}x) , \quad \text{etc.} ,$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) .$$

These can be rewritten as

$$\tilde{\tau} = \frac{c+d\tau}{a+b\tau} , \quad \tilde{\jmath}_{IJ} = |a+b\tau| \, \jmath_{IJ} , \quad \begin{pmatrix} \tilde{B}^{(1)}_{IJ} \\ \tilde{B}^{(2)}_{IJ} \end{pmatrix} = \Lambda \begin{pmatrix} B^{(1)}_{IJ} \\ B^{(2)}_{IJ} \end{pmatrix} , \quad \tilde{D}_{mnp8} = D_{mnp8} , \quad \text{etc.},$$

where $\tau \equiv l + ie^{-\varphi}$ and $I, J = 1, \cdots, 8$. The wrapping numbers are mapped as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \to \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{r} & \tilde{s} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \Lambda^{-1}$$

Summary

- We have introduced the general matrix representation for the wrapped membrane on $\mathbb{R}^9 \times T^2$.
- We constructed the $GL(2,\mathbb{R})$-invariant (2+1)-dimensional Yang-Mills action from the wrapped membrane in curved space through the matrix regularization.
- We identified the type IIB superstring $SL(2,\mathbb{R})$ duality as the target-space rotation in the matrix regularized wrapped membrane theory.

  It was shown that the type IIB $SL(2,\mathbb{R})$ duality (at the tree level) can be realized as the $SL(2,\mathbb{R})$ target-space rotation in 11-dimensional supergravity. Bergshoeff-Hull-Ortin (1995)
- We can show that the double dimensional reduced membrane wrapped around the (p,q)-cycle is Green-Schwarz (p,q)-strings.