”Seiberg-Witten Theory and AGT Relation”

Tohru Eguchi

We consider $\mathcal{N} = 2$ supersymmetric gauge theories in 4-dimensions and study the case when the theory possesses the conformal invariance.

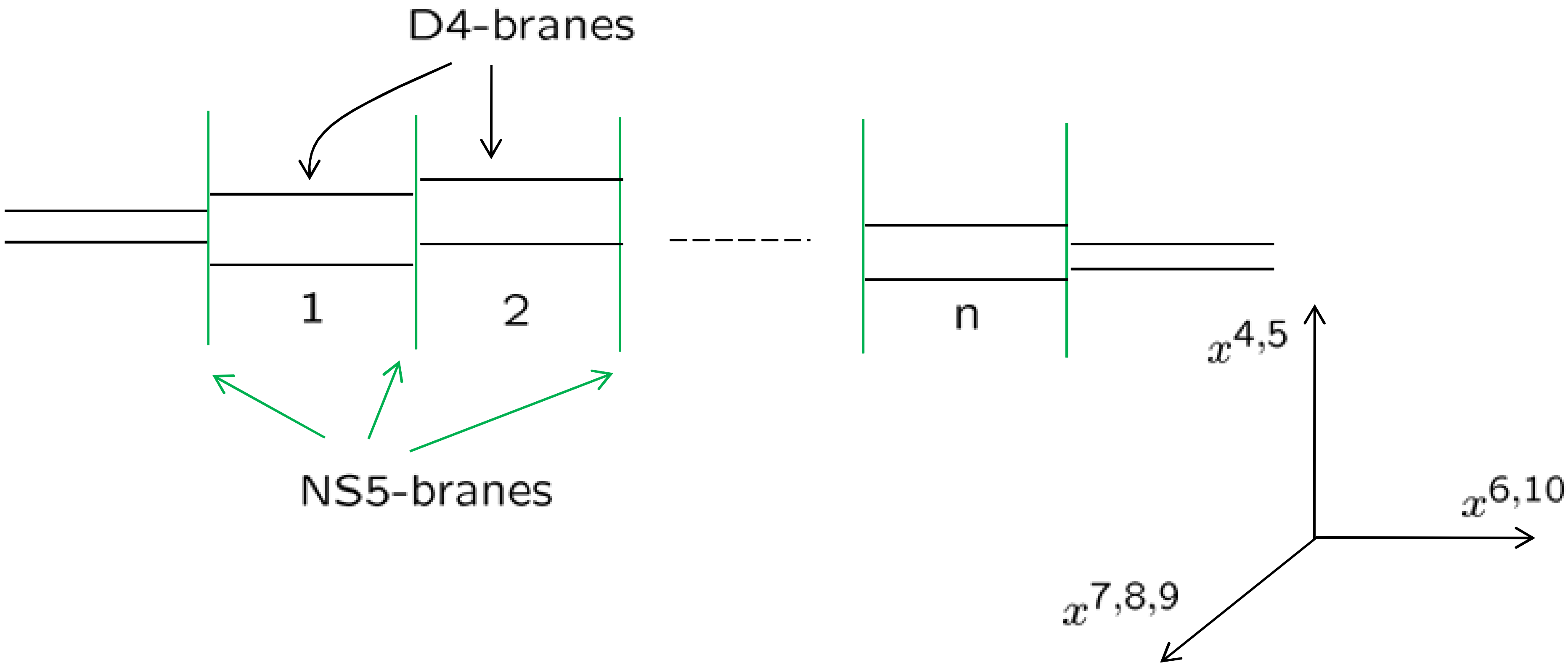
Simplest example of a conformal invariant theory:

$SU(2)$ gauge theory with $N_f = 4$ hypermultiplets

We may consider its generalizations

A chain of $SU(2)$ gauge theories with bifundamentals and fundamental at the ends: quiver gauge theories

As is well-known, such quiver theories are obtained using the brane construction as shown in the figure:

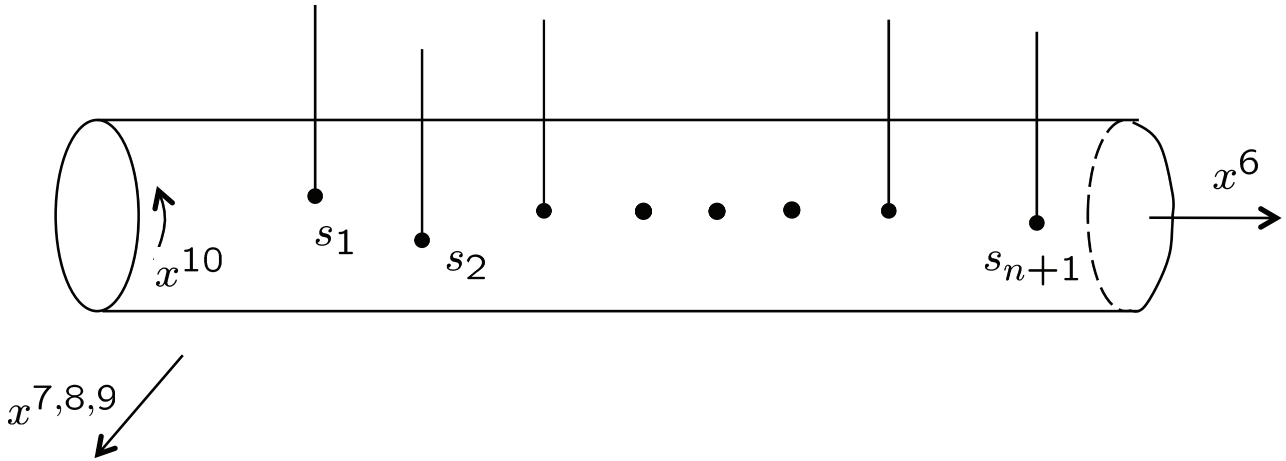


One has $n+1$ NS5 branes and a pair of $D4$ branes are suspended between neighbouring NS5 branes giving rise to $SU(2)_1 \times SU(2)_2 \cdots \times SU(2)_n$ gauge symmetry. Two D4 branes at extreme left and right extend to $x_6 = \pm\infty$ representing fundamental hypermultiplets. In such a configuration each $SU_i(2)$ theory couples to $N_f = 4$ hypermultiplets and is conformally invariant. Thus there exists a set of marginal parameters in the theory

$$\{\tau_i = \frac{\theta_i}{\pi} + \frac{8i\pi}{g_i^2}, \qquad i = 1, , n\}$$

Uplifting this brane configuration to 11 dimensions

$\Longrightarrow$ M theory picture with an M5 brane wrapping a Riemann surface (cylinder) with punctures.



Thus, conformal $\mathcal{N} = 2$ theories

$\approx$ an M5 brane wrapping a Riemann surface C with a number of punctures.

Number of parameters of Riemann surface $C_{g,n}$ of genus g

with n punctures: $3g-3+n$

This agrees with the number of gauge theory parameters $\{\tau_i\}$.

Hence one expects Gaiotto

S-duality group of quiver gauge theory $=$

mapping class group of Riemann surface $C_{g,n}$

Remarkable observation Alday, Gaiotto, Tachikawa

AGT relation

$$\langle \prod V_{m_i}(\tau_i) \rangle = \int [da] \; |Z_{\rm Nek}(\tau; a; m, \epsilon_i)|^2$$

Liouville correlation function | Nekrasov partition function of $SU(2)$ gauge theory in Ω background

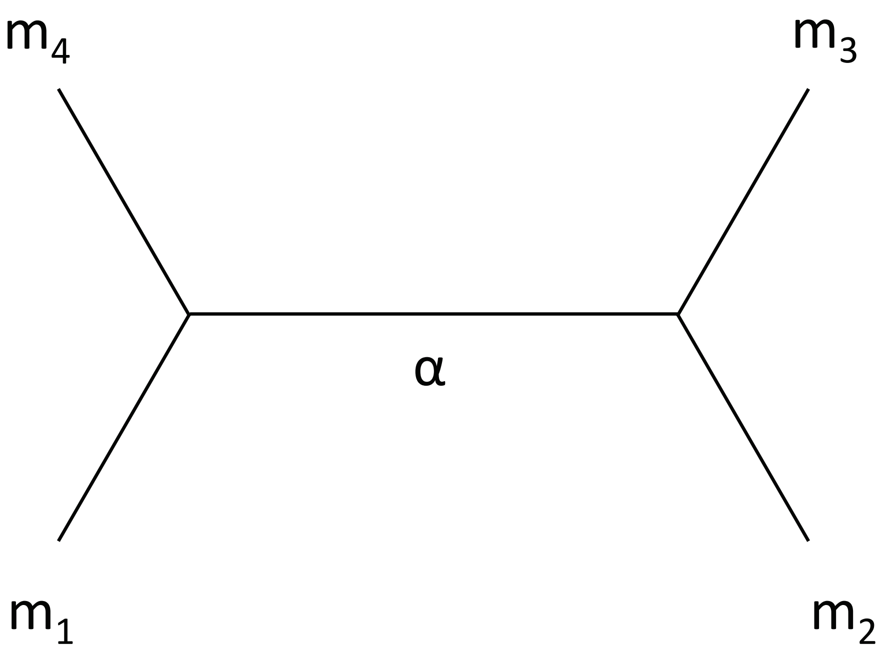
Liouville momentum

$$\begin{cases} \text{external line:} \quad \dfrac{m_i}{\hbar}, \quad \Delta_i = \dfrac{m_i}{\hbar}\left(Q - \dfrac{m_i}{\hbar}\right) \\ \text{interbal line:} \quad \alpha = \dfrac{Q}{2} + \dfrac{a}{\hbar} \end{cases}$$

Background charges

$$Q = b + \frac{1}{b}, \quad c = 1 + 6Q^2$$

$$\epsilon_1 = b\hbar, \quad \epsilon_2 = \frac{\hbar}{b}$$



Nekrasov formula

Sum over Yang tableau; $Y = (\lambda_1 \geq \lambda_2 \geq \cdots)$

$$Z_{\rm Nek} = \sum_{(Y_1,Y_2)} q^{|\vec{Y}|} Z_{\rm vector}(\vec{a},\vec{Y}) Z_{\rm antifund}(\vec{a},\vec{Y},m_1)$$

$$\times Z_{\rm antifund}(\vec{a},\vec{Y},m_2) Z_{\rm fund}(\vec{a},\vec{Y},-m_3) Z_{\rm fund}(\vec{a},\vec{Y},-m_4)$$

Here

$$Z_{\text{vector}}(\vec{a}, \vec{Y}) = \prod_{i,j=1,2} \prod_{s \in Y_i} \Big(a_{ij} - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1) \Big)^{-1}$$
$$\times \prod_{t \in Y_j} \Big(a_{ji} + \epsilon_1 L_{Y_j}(t) - \epsilon_2 (A_{Y_i}(t) + 1) + \epsilon_+ \Big)^{-1}$$
$$Z_{\text{fund}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a_i + \epsilon_1(\ell - 1) + \epsilon_2(m - 1) - \mu + \epsilon_+)$$
$$Z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a_i + \epsilon_1(\ell - 1) + \epsilon_2(m - 1) + \mu)$$

$\epsilon_+ = \epsilon_1 + \epsilon_2$, $a_{ij} = a_i - a_j$. $L_Y(s)$ and $A_Y(s)$ are leg and arm length of the site s.

Nekrasov formula is obtained by summing over contributions from fixed points in the ADHM formula under gauge and Lorenz transformation ($SO(4) = SU(2)_L \times SU(2)_R \in (\epsilon_1, \epsilon_2)$).

First exact relationship between 4-dim CFT and 2-dim CFT.

Higher rank generalization: Toda theories

♠ Attempts at direct proof **Fateev-Litvinov**

Detailed study of the algebraic structure of conformal block in Liouville theory , i.e. the recursion relation by Al.B.Zamolodchikov.

$$\langle V_\alpha \rangle \approx \mathcal{F}_\alpha^\Delta(q), \quad \mathcal{F}_\alpha^\Delta(q) = \sum q^{mn} \frac{R_{n,m}}{\Delta - \Delta_{m,n}} \mathcal{F}_\alpha^{\Delta_{m,-n}}(q)$$

and comparison with the sum over Yang tableaus of gauge theory side.

Conformal block of 1-point function in Liouville theory on a torus = $\mathcal{N} = 4$ gauge theory perturbed by the mass of the adjoint hypermultiplet ($\mathcal{N} = 2^*$ theory)

♠ Exact Integration

Consider Liouville correlation function in free field representation

$$\langle \prod_a e^{im_a \phi(q_a)} \prod_{i=1}^{N} \int e^{b\phi(z_i)} \, dz_i \rangle$$

screening ops.

$$= \prod_{a<b} (q_a - q_b)^{2m_a m_b} \int \prod_{i,a} dz_i \, (z_i - q_a)^{-2ibm_a} \prod_{i<j} (z_i - z_j)^{-2b^2},$$

$$\sum_i im_a + Nb = Q$$

Dotesnko-Fatteev integral

This is an integration of Selberg type.

$$I_N(a,c,\beta) = \int \prod_{i=1}^{N} dx_i \prod_{i<j} (x_i - x_j)^{2\beta} \prod_{i=1}^{N} x_i^a (1 - x_i)^c$$

$$= \prod_{j=0}^{N-1} \frac{\Gamma(a+1+j\beta)\Gamma(c+1+j\beta)\Gamma(1+(j+1)\beta)}{\Gamma(a+c+2+(N+j-1)\beta)\Gamma(1+\beta)}$$

Attempts at exact evaluation and comparison with conformal blocks.

Morozov-Kironov-Shakirov,

Itoyama-Oota$\cdots$

♠ Monodromy transformations

SW curve Σ : $x^2 = \phi_2(z)$,

ϕ_2 has double poles $\approx \dfrac{m_i^2}{(z-z_i)^2}$

$$\frac{1}{2\pi i}\oint_{A_i} x dz = a_i, \; \frac{1}{2\pi i}\oint_{B_i} x dz = a_D^i, \; a_D^i = \frac{1}{4\pi i}\frac{\partial F}{\partial a_i}$$

In the semi-classical limit $\hbar \to 0$, $\epsilon_{1,2} << a_i, m_i$

$$Z \approx \exp\left(-\frac{F(a_i)}{\hbar^2}\right)$$

Liouville stress tensor $T(z)$

$$\langle T(z) V_{m_1}(z_1) \cdots V_{m_n}(z_n) \rangle \approx \frac{-1}{\hbar^2} \phi_2(z) \langle V_{m_1}(z_1) \cdots V_{m_n}(z_n) \rangle$$

$$\Downarrow$$

$$\frac{-\frac{m_i^2}{\hbar^2}}{(z-z_i)^2} \approx \frac{\Delta_i}{(z-z_i)^2}$$

Degenerate field

Consider a field $\Phi_{2,1}(z) = e^{\frac{-b}{2}\phi(z)}$ which possesses a degeneracy at level 2

$$\partial_z^2 \Phi_{2,1}(z) = -b^2 : T(z) \Phi_{2,1}(z) :$$

Correlation function with an extra insertion of $\Phi_{2,1}$

$$Z(a_i; z) = \langle \Phi_{2,1}(z) V_{m_1}(z_1) \cdots V_{m_n}(z_n) \rangle$$

In the semi-classical limit

$$Z(a_i; z) \approx \exp\left(-\frac{F(a_i)}{\hbar^2} + \frac{bW(a_i; z)}{\hbar} + \cdots \right)$$

One finds

$$(\partial W)^2 = \phi_2(z) = x(z)^2$$

Hence

$$W_\pm(z) = \pm \int_{z^*}^{z} x dz$$

shift around A, B cycles gives

$$Z(a_i; z + A_j) = \exp(\frac{2\pi i b}{\hbar} a_j) Z(a_i; z)$$

$$Z(a_i; z + B_j) = \exp(\frac{2\pi i b}{\hbar} a_D^j) Z(a_i; z)$$

Similarly we may consider the process

1. Insert identity operator inside the Liouville correlator
2. $\Phi_{2,1} \otimes \Phi_{2,1} \approx 1$
3. Transport one of $\Phi_{2,1}$'s around A, B cycle
4. Pair annihilate two Φ's into identity

$$\Longrightarrow \mathcal{L}(\gamma)\mathcal{F}_\alpha = \frac{\cos(\pi b(2\alpha - Q))}{\cos(\pi b Q)}\mathcal{F}_\alpha$$

These processes give monodromy factors corresponding to the action of Wilson loop, 't Hooft loop and surface operators.

Alday-Gaiotto-Gukov-Tachikawa-Verlinde

Drukker-Gomis-Okuda-Teschner

♠ Matrix Model

Dotsenko-Fatteev integral when $b = i$ suggests a matrix model interpretation with an action

$$S = \sum_a m_a \log(M - q_a)$$

and $\{z_i\}$ are identified as matrix eigenvalues.

Dijkgraaf-Vafa

We find that this model in fact reproduces Seiberg-Witten theory (also for the asymptotically free cases $N_f = 2, 3$). But it still has mysterious features. T.E.-Maruyoshi

Let us consider the simple case of 4 hypermultiplets with masses $m_\pm, \tilde{m}_\pm$. Define

$$m_0 = \frac{1}{2}(m_+ - m_-),\ m_1 = \frac{1}{2}(\tilde{m}_+ - \tilde{m}_-)$$
$$m_2 = \frac{1}{2}(m_+ + m_-),\ m_3 = \frac{1}{2}(\tilde{m}_+ + \tilde{m}_-)$$

Condition:

$$\sum_i m_i = 2g_s N$$

M theory curve is given by

$$
\begin{aligned}
C_M: \; & (v-m_+)(v-m_-)z^2 \\
& +c_1(v^2+Mv-U)z + c_1(v-\tilde{m}_+)(v-\tilde{m}_-) = 0
\end{aligned}
$$

For convenience, set $c_1 = -(1+q)$, $c_2 = q$. By shifting v to eliminate the linear term and setting $v = xz$

$$
\begin{aligned}
C_M : x^2 = & \left(\frac{m_2 z^2 + (1+q)\frac{M}{2}z + m_3 q}{z(z-1)(z-q)}\right)^2 \\
& + \frac{(m_0^2 - m_2^2)z^2 - (1+q)Uz + (m_1^2 - m_3^2)q}{z^2(z-1)(z-q)}
\end{aligned}
$$

Seiberg-Witten differential behaves at a pole as

$$\lambda_{SW} = \frac{x dz}{2\pi i} \approx \frac{m_*}{z - z_*}$$

Mass appears at residues.

Pole at $z = 0, z = \infty$; residue $\pm m_1, \pm m_0$.

Require pole at $z = 1$ with residue $\pm m_2$ and $z = q$ with residue $\pm m_3 \Longrightarrow$

$$M = \frac{-2q}{1+q}(m_2 + m_3)$$

♣ UV and IR gauge coupling constant

Standard SW curve of $N_f = 4$ in massless case

$$C_{SW} : y^2 = 4x^3 - g_2 u x^2 - g_3 u^3$$

Here

$$\begin{aligned} g_2(\omega_1, q) &= \left(\frac{\pi}{\omega_1}\right)^4 \frac{1}{24}\left(\vartheta_3(q)^8 + \vartheta_2(q)^8 + \vartheta_4(q)^8\right), \\ g_3(\omega_1, q) &= \left(\frac{\pi}{\omega_1}\right)^6 \frac{1}{432}\left(\vartheta_4(q)^4 - \vartheta_2(q)^4\right) \\ &\quad \times \left(2\vartheta_3(q)^8 + \vartheta_4(q)^4\vartheta_2(q)^4\right) \end{aligned}$$

On the other hand M theory curve in the masssless limit is given by

$$C_M : x^2 = -\frac{(1+q)U}{z(z-1)(z-q')}$$

Here U is related to $u = tr\phi^2$ as

$$U = Au$$

and we have used q' in order to distinguish it from q of C_{SW}.

By comparing the periods we find

$$q' = \frac{\vartheta_2(q)^4}{\vartheta_3(q)^4}, \qquad A = \frac{1}{\vartheta_2(q)^4 + \vartheta_3(q)^4}$$

We regard q in SW curve as the gauge coupling in the infra-red regime $q = q_{IR}$ and q' in M theory curve as the ultra-violet gauge coupling constant $q' = q_{UV}$. Relation

$$q_{UV} = \frac{\vartheta_2(q_{IR})^4}{\vartheta_3(q_{IR})^4}$$

has been obtained by various authors.

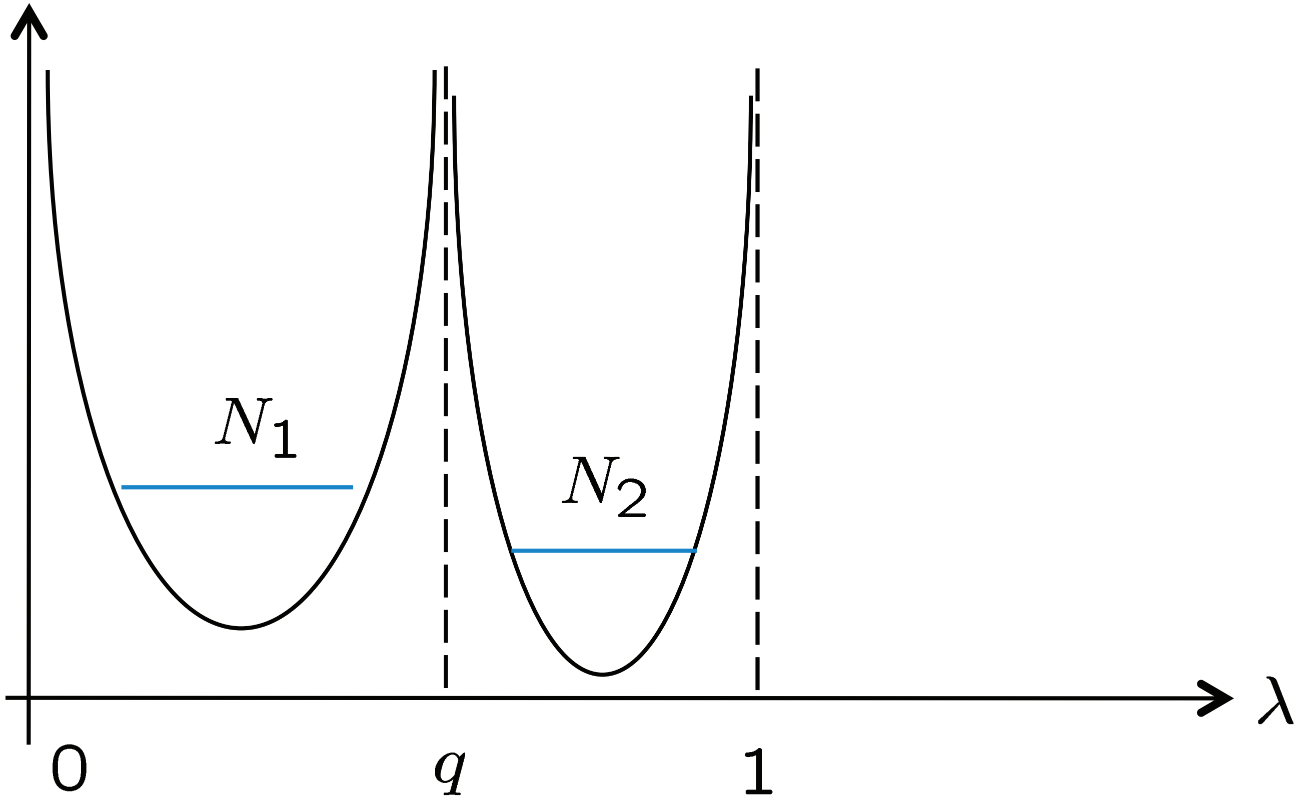
Grimm et al, Marshakov et al

♠ Matrix model and modular invariance

Equation of motion

$$\sum \frac{m_i}{\lambda_I - q_i} + 2g_s \sum_{I \neq J} \frac{1}{\lambda_I - \lambda_J} = 0$$

We have $q_1 = 0, q_2 = 1, q_3 = q_{UV}$. Eigenvalue distribution is as given in the figure.



Resolvent of the theory is defined by

$$R_m(z) = g_s Tr \frac{1}{z - M}$$

and satisfies the loop equation

$$\langle R_m(z) \rangle^2 = -\langle R_m(z) \rangle W'(z) + \frac{f(z)}{4}$$

$$f(z) = 4 g_s Tr \left\langle \frac{W'(z) - W'(M)}{z - M} \right\rangle = \sum_{i=1}^{3} \frac{c_i}{z - q_i}$$

Matrix model curve (spectral curve) is defined by the discriminant of the loop eq.

$$C_{spec.curve} : x^2 = W'(z)^2 + f(z)$$
$$= \left(\frac{m_1}{z} + \frac{m_2}{z-1} + \frac{m_3}{z-q}\right)^2 + \frac{(m_0^2 - \sum_i m_i^2)z + qc_1}{z(z-1)(z-q)}$$

Eq. of motion $\Longrightarrow \sum_i c_i = 0$

Residue at ∞ being $\pm m_0 \Longrightarrow c_2 + qc_3 = m_0^2 - (\sum m_i)^2$

Then

$$qc_1 = (1+q)m_1^2 + (1-q)m_3^2 + 2qm_1m_2 - 2qm_2m_3$$
$$+2m_1m_3 - (1+q)U$$
$$\Longrightarrow C_W = C_{spec.curve}$$

• Modular invariance

Consider the massless limit of spectral curve

$$x^2 = -\frac{(1+q)U}{z(z-1)(z-q)} = -\frac{\frac{u}{\theta_3^4}}{z(z-1)(z-q)}$$

This is invariant under

$$\begin{aligned} I &: (z,x) \longrightarrow (1-z,x), \qquad q \longrightarrow 1-q,\ u \longrightarrow -u,\ S \\ II &: (z,x) \longrightarrow (\frac{1}{z}, -z^2 x), \qquad q \longrightarrow \frac{1}{q}, \quad u \longrightarrow u,\ STS \end{aligned}$$

Recall $q = \dfrac{\theta_2^4}{\theta_3^4}$.

Consider massive case. Under the S- and STS-transformations mass parameters are transformed into each other

$$
\begin{aligned}
I &: (0,1,q,\infty) \longrightarrow (1,0,1-q,\infty), \quad m_1 \leftrightarrow m_2 \\
II &: (0,1,q,\infty) \longrightarrow (\infty,1,\frac{1}{q},0), \quad m_0 \leftrightarrow m_1
\end{aligned}
$$

Under these transformations, the spectral curve should be invariant. By imposing the conditions

$$
\begin{aligned}
x^2(z;m_0,m_1,m_2,m_3;q) &= x^2(1-z;m_0,m_2,m_1,m_3;1-q) \\
x^2(z;m_0,m_1,m_2,m_3;q) &= \frac{1}{z^4}x^2(\frac{1}{z};m_1,m_0,m_2,m_3:\frac{1}{q})
\end{aligned}
$$

one can completely fix the mass dependence of the parameter U. Solution to the above conditions is given by

$$(1+q)U = \frac{u}{\vartheta_3^4} - q(m_2+m_3)^2 + \frac{1+q}{3}\left(\sum_{i=0}^{3} m_i^2\right)$$

- Asymptotically free theory with $N_f = 3$

precise relationship between u and $Tr\phi^2$

$$u = \langle Tr\phi^2 \rangle - \frac{1}{6}(\vartheta_4^4 + \vartheta_3^4) \sum_{i=0}^{3} m_i^2.$$

Recall

$$m_\pm = m_2 \pm m_0, \quad \tilde{m}_\pm = m_3 \pm m_1,$$

We take the limit

$$\tilde{m}_- \to \infty, \quad q \to 0,$$

with

$$\tilde{m}_- q = \Lambda_3 \qquad \text{fixed}$$

Matrix action reduces to

$$W(M) = \tilde{m}_+ \log M - \frac{\Lambda_3}{2M} + m_2 \log(M-1).$$

the spectral curve for $N_f = 3$ theory becomes

$$x^2 = \frac{\Lambda_3^2}{4z^4} - \frac{\tilde{m}_+ \Lambda_3}{z^3(z-1)} - \frac{u - (m_2 + \frac{1}{2}\tilde{m}_+)\Lambda_3}{z^2(z-1)} + \frac{m_0^2}{z(z-1)} + \frac{m_2^2}{z(z-1)^2} - \frac{m_2 \Lambda_3}{z^2(z-1)}.$$

Predicts the same free energy and discriminant as that of the

standard SW curve

$$
\begin{aligned}
x^2 \;=\; & z^2(z-u) - \frac{1}{4}\Lambda_3^2(z-u)^2 \\
& -\frac{1}{4}(m_+^2 + m_-^2 + \tilde{m}_+^2)\Lambda_3^2(z-u) + m_+ m_- \tilde{m}_+ \Lambda_3 z \\
& -\frac{1}{4}(m_+^2 m_-^2 + m_-^2 \tilde{m}_+^2 + \tilde{m}_+^2 m_+^2)\Lambda_3^2
\end{aligned}
$$

- Asymptotically free theory with $N_f = 2$

Matrix action:

$$
W(M) = \tilde{m}_+ \log M - \frac{\Lambda_2}{2M} - \frac{\Lambda_2 M}{2}
$$

Spectral curve:

$$x^2 = \frac{\Lambda_2^2}{4z^4} + \frac{\tilde{m}_+\Lambda_2}{z^3} + \frac{u}{z^2} + \frac{m_+\Lambda_2}{z} + \frac{\Lambda_2^2}{4}$$

Computation of free energy

$$g_s N_1 = \frac{1}{4\pi i}\oint x(u)dz$$

(N_1 denotes the filling fraction of the first cut "1").

Derivative of free energy in Λ_2 is given by

$$\Lambda_2 \frac{\partial F}{\partial \Lambda_2} = -\Lambda_2 \frac{g_s}{2}\langle \sum_I \left(\frac{1}{\lambda_I} + \lambda_I \right) \rangle = 2u + \Lambda_2^2 - m_+^2 - \tilde{m}_+^2$$

On the other hand

$$4\pi i \frac{\partial(g_s N_1)}{\partial u} = \oint_{C_1} \frac{dz}{\sqrt{P_4(z)}}$$

where

$$P_4(z) = z^4 + \frac{4\tilde{m}_+}{\Lambda_2} z^3 + \frac{4u}{\Lambda_2^2} z^2 + \frac{4m_+}{\Lambda_2} z + 1$$

This is a complete elliptic integral and we can expand u in terms of $a = 2g_s N_1$ (we put $m_+ = \tilde{m}_+ \equiv m$ for simplicity)

$$u = a^2 + \frac{m^2}{2a^2}\Lambda_2^2 + \frac{a^4 - 6m^2a^2 + 5m^4}{32a^6}\Lambda_2^4 + \cdots$$

Then by integrating over Λ_2 we finally obtain

$$4F_m = 2(a^2 - m^2)\log\Lambda_2 + \frac{a^2+m^2}{2a^2}\Lambda^2 + \frac{a^4 - 6a^2m^2 + 5m^4}{64a^6}\Lambda^4 + \cdots$$

This gives the same free energy as the standard SW curve

$$x^2 = (z^2 - \frac{1}{4}\Lambda_2^4)(z-u) + m_+\tilde{m}_+\Lambda_2^2 z - \frac{1}{4}(m_+^2 + \tilde{m}_+^2)\Lambda_2^4$$

♠ Discussions

1. We want a much wider class of correspondences:

Liouville, Toda

$\Longrightarrow$ WZW, cosets, parafermions etc.

$\mathcal{N} = 2$ Yang-Mills on R^4

$\Longrightarrow$ on ALE spaces, rational surfaces?

2. Want five-dimensional version of AGT. It is known that 5-dimensional Nekrasov formula counts the number of holomorphic curves in non-comapct CY manifolds (geometric engineering). 5-dim. AGT $\Longrightarrow$ CFT acting on the space of Gromov-Witten invariants.