A lattice study of $\mathcal{N}$=2 Landau-Ginzburg model using a Nicolai map

based on arXiv:1005.4671

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Outline

1.Purpose of this study

2.Lattice formulation of WZ model

3.Simulation Method

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1 Purpose

2d CFT

- critical phenomena of 2d statistical systems
- $\mathcal{N}=2$ minimal models $\overset{\sum c_i = 9}{\Longrightarrow}$ $\mathcal{N}=1$ space-time SUSY (compactified string)

  :

- A problem which remains unsolved is the determination of the correspondence between CFTs and systems (Lagrangians) .

2d $\mathcal{N}$=2 Landau-Ginzburg model (LG model)

$$S = \int d^2x d^4\theta\, K(\Phi, \bar{\Phi}) + \left(\int d^2x d^2\theta\, W(\Phi) + c.c. \right), \qquad \Phi \cdots \text{chiral superfield.}$$

At the IR fixed point, $W(\Phi) = \lambda \Phi^n$ is believed to describe the $\mathcal{N}=2$, $c = 3(1-\frac{2}{n})$ minimal model.

$\hookrightarrow$ check for $K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi$ (WZ model)

$\lambda_{\mathrm{eff}} \to \infty$, lattice !

Why it is believed that LG models describe CFTs ?

- 2d bosonic case '86 A.B.Zamolodchikov

  In the $c = 1 - \frac{6}{n(n+1)}$ minimal model, the fusion rule implies $\cdots\ \phi_{(2,2)}^{2n-3} \propto \partial^2 \phi_{(2,2)}$

  In the 2d bosonic LG model $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \lambda\phi^{2n-2}$, EOM is $\cdots\ \phi^{2n-3} \propto \partial^2\phi$

  $\overset{\text{conjecture}}{\Rightarrow}$ $\phi = \phi_{(2,2)}$ at the IR fixed point. Extending this idea, ...

How to check the conjecture

- early studies

RG flow of c-functions → '89 Kastor, Martinec and Shenker

catastrophe theory → '89 Vafa and Warner → For $W(\Phi) = \lambda\Phi^n$,

ϵ-expansion → '89 Howe and West

elliptic genus → '93 Witten

...

$$\begin{cases} c = 3(1 - \frac{2}{n}) \\ \Phi : (h, \bar{h}) = (\frac{1}{2n}, \frac{1}{2n}) \\ \Phi^2 : (h, \bar{h}) = (\frac{2}{2n}, \frac{2}{2n}) \\ \vdots \\ \Phi^{n-2} : (h, \bar{h}) = (\frac{n-2}{2n}, \frac{n-2}{2n}) \end{cases}$$

- We computed **correlation functions** non-perturbatively for $W(\Phi) \propto \Phi^3$.

susceptibility of CFT:

$$\chi \equiv \int d^2x \langle \phi(x)\phi^*(0)\rangle \xrightarrow{\text{finite volume}} \int_V d^2x \frac{1}{|x|^{2h+2\bar{h}}} \propto V^{1-h-\bar{h}}$$

$$\Rightarrow \log \chi = \underline{(1 - h - \bar{h})} \log \mathrm{V} + \mathrm{const.}$$

For the present $W(\Phi) \propto \Phi^3$, the conjecture expects $1 - h - \bar{h} = 1 - \frac{1}{6} - \frac{1}{6} =$ 0.666...

2 Lattice Formulation of WZ model

Relying on the existence of the Nicolai map as the guiding principle,

'83 Sakai and Sakamoto
'09 Kadoh and Suzuki

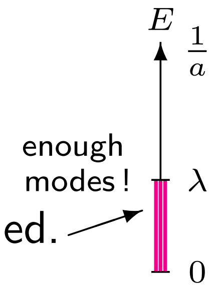
'02 Kikukawa and Nakayama

$$\mathcal{S} = \sum \left\{ \phi^* T \phi + W^* (1 - \frac{a^2}{4} T) W + \left(W'(-S_1 + iS_2)\phi + c.c. \right) \right.$$
$$\left. + \bar{\psi} \left(D + \frac{1+\gamma_3}{2} W'' \frac{1+\hat{\gamma}_3}{2} + \frac{1-\gamma_3}{2} W''^* \frac{1-\hat{\gamma}_3}{2} \right) \psi \right\}$$

where $D = \frac{1}{2}\left[1 + \frac{X}{\sqrt{X^\dagger X}} \right] = T + \gamma_1 S_1 + \gamma_2 S_2, \quad X = 1 - \frac{a}{2}\left[\gamma_\mu (\nabla_\mu^+ - \nabla_\mu^-) - a \nabla_\mu^+ \nabla_\mu^- \right],$

$$W = \frac{\lambda}{3} \Phi^3.$$

- λ is the unique mass parameter (besides a) $\Rightarrow$ $\begin{cases} \text{continuum limit} : a\lambda \to 0 \\ \text{To see CFT, } L \gg (a\lambda)^{-1} \text{ is needed.} \end{cases}$



- no extra fine-tunings $\Leftarrow$ $\begin{cases} \circledcirc \text{ one SUSY } Q & \leftarrow \text{Nicolai map} \\ \circledcirc\ Z_3 \text{ R-symmetry} & \leftarrow \text{overlap fermion} \end{cases}$

- This lattice model faces the sign problem

$|D+F|$ is real, but can be negative. $\Leftarrow \gamma_1 (D+F) \gamma_1 = (D+F)^*$

$$\int (\Pi_n \, d\phi_n \cdots) e^{-S_{lat.}} = \int (\Pi_n \, d\phi_n d\phi_n^*) \underbrace{|D+F|}_{\text{real, but can be negative.}} e^{-S_{\rm B}}$$

3 Simulation Method

Idea '91 Curci et al.

We utilized the Nicolai map : $\eta = W' + (\phi - \frac{a}{2}W')T + (\phi^* - \frac{a}{2}W^{*\prime})(S_1 + iS_2)$.

$$\begin{aligned}
&\int (\Pi_n \, d\phi_n \cdots) e^{-S_{lat.}} \\
&= \int \mathcal{D}\phi\mathcal{D}\phi^* |D+F| \, e^{-S_{\mathrm{B}}}, \qquad \mathcal{D}\phi\mathcal{D}\phi^* \equiv \Pi_n \, d\phi_n d\phi_n^* \\
&= \int \mathcal{D}\phi\mathcal{D}\phi^* \left[\int \mathcal{D}\eta\mathcal{D}\eta^* \delta\Big(\eta - W' - (\phi - \frac{a}{2}W')T - (\phi^* - \frac{a}{2}W^{*\prime})(S_1 + iS_2)\Big) \right] |D+F| \, e^{-S_{\mathrm{B}}} \\
&= \int \mathcal{D}\phi\mathcal{D}\phi^* \left[\int \mathcal{D}\eta\mathcal{D}\eta^* \sum_{i=1}^{N(\eta)} \frac{\delta\big(\phi - \phi_i(\eta)\big)}{||D+F||} \right] |D+F| \, e^{-S_{\mathrm{B}}} \\
&= \int \mathcal{D}\eta\mathcal{D}\eta^* \left[\sum_{i=1}^{N(\eta)} \mathrm{sgn}\, |D + F(\phi_i)| \right] e^{-\sum_n |\eta_n|^2}.
\end{aligned}$$

$$\Rightarrow \quad \langle \mathcal{O} \rangle = \frac{\left\langle \sum_{i=1}^{N(\eta)} \mathcal{O}(\phi_i) \mathrm{sgn} |D + F(\phi_i)| \right\rangle_\eta}{\left\langle \sum_{i=1}^{N(\eta)} \mathrm{sgn} |D + F(\phi_i)| \right\rangle_\eta}, \qquad \text{where } \langle X \rangle_\eta \equiv \frac{\int \mathcal{D}\eta\mathcal{D}\eta^* \, X \, e^{-\sum_n |\eta_n|^2}}{\int \mathcal{D}\eta\mathcal{D}\eta^* \, \underbrace{e^{-\sum_n |\eta_n|^2}}_{\text{positive}}}.$$

Using this expression, we calculated the susceptibility $\chi = \int \mathrm{d}^2 x \langle \phi(x) \phi^*(0) \rangle$.

Algorithm

$$\langle \mathcal{O} \rangle = \frac{\left\langle \sum_{i=1}^{N(\eta)} \mathcal{O}(\phi_i) \text{sgn} \left| D + F(\phi_i) \right| \right\rangle_\eta}{\left\langle \sum_{i=1}^{N(\eta)} \text{sgn} \left| D + F(\phi_i) \right| \right\rangle_\eta} \overset{a \to 0}{\longrightarrow} \textbf{Witten index } \Delta = 2 \text{ (cubic potential)}$$

$$\text{where} \begin{cases} \langle X \rangle_\eta \equiv \dfrac{\int \left(\Pi_n \, d\eta_n d\eta_n^* \right) X(\eta) \, e^{-\sum_x |\eta|^2}}{\int \left(\Pi_n \, d\eta_n d\eta_n^* \right) e^{-\sum_x |\eta|^2}} \\ \\ N(\eta) \text{ counts the solutions of the Nicolai map } \phi_1, .., \phi_{N(\eta)} \end{cases}$$

$$\eta = W' + (\phi - \tfrac{a}{2} W')T + (\phi^* - \tfrac{a}{2} W^{*\prime})(S_1 + iS_2)$$

1. Assigning $\{\eta, \eta^*\}$ as the standard normal distribution,
2. Solving the Nicolai map by the Newton-Raphson algorithm,
3. We sample the configurations of $\{\phi, \phi^*\}$.

- advantage $\cdots$ no sign problem, no autocorrelation
- difficulty $\cdots$ $N(\eta)$

Tests for the configurations

- $\langle \sum_{i=1}^{N(\eta)} \text{sgn}|D+F|\rangle_\eta \xrightarrow{a\to 0}$ **Witten index** $\Delta = 2$ (cubic potential)

  Why Witten index ?

  $\rightarrow$ P.B.C. & For $W(\Phi) = \frac{m}{2}\Phi^2\,(\Delta = 1)$, $(\mathrm{Re}\,\eta, \mathrm{Im}\,\eta) = (\mathrm{Re}\,\phi, \mathrm{Im}\,\phi)\underbrace{\left(D + m(1 - \frac{a}{2}D)\right)}_{\text{positive} \Rightarrow \Delta=1 \text{ is correctly reproduced}}$

  $\rightarrow$ correctly normalized

- **Ward identity for** $\left\langle \eta(x_1)\cdots\eta(x_m)\eta^*(y_1)\cdots\eta^*(y_n)\right\rangle$ **on the lattice**

  From $Q\psi_+ = -\eta^*$, $Q\psi_- = -\eta$, $Q\eta = \frac{\delta}{\delta\psi_+}S_{lat.}$, $Q\eta^* = \frac{\delta}{\delta\psi_-}S_{lat.}$, $\langle Q(\cdots)\rangle = 0$,

  and the Schwinger-Dyson eq. ,

$$\frac{\left\langle \eta(x_1)\cdots\eta^*(y_n)\sum_{i=1}^{N(\eta)}\text{sgn}|D+F|\right\rangle_\eta}{\left\langle \sum_{i=1}^{N(\eta)}\text{sgn}|D+F|\right\rangle_\eta} = \begin{cases} 0 & m \neq n \\ \sum_\sigma \Pi_{k=1}^m \delta_{x_k, y_{\sigma(k)}} & m = n. \end{cases}$$

  For example, $m = n = 1$ provides $\sum_x \langle \eta(x)\eta^*(x)\rangle = \langle S_B\rangle = L^2$.

$\Rightarrow$ If $\sum_{i=1}^{N(\eta)} \text{sgn}|D+F| = 2$ over the η, OK.

4 Numerical Results

Samples with $W(\Phi) = \frac{\lambda}{3}\Phi^3$, $a\lambda = 0.3$, $L = 18, 20, ..., 32$

(Newton iter. from 100 initial config. for each noise) × 320 noises

L	18	20	22	24	26	28	30	32
$(+,+)$	316	319	319	316	316	314	307	316
$(-,+,+,+)$	3	0	1	3	4	6	10	4
$(+)$	1	1	0	0	0	0	1	0
$(+,+,+)$	0	0	0	1	0	0	2	0
Δ	1.997	1.997	2	2.003	2	2	1.994	2
δ [%]	0.3	0.0	0.1	0.4	0.4	0.4	0.4	0.2

test ...

$\sum \mathrm{sgn}|D+F| = 2$

$\sum \mathrm{sgn}|D+F| \neq 2$

,but rare.

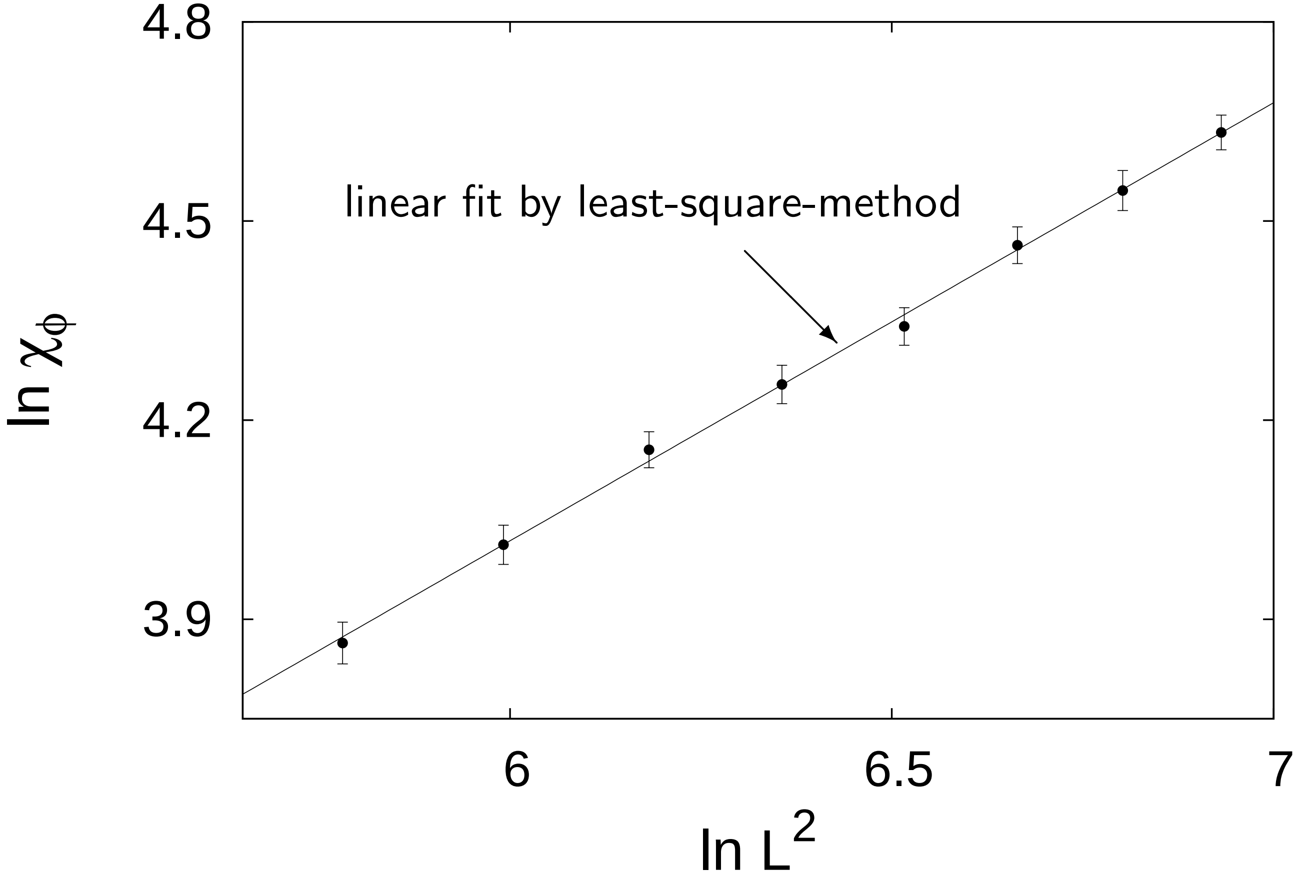
Δ ··· Witten index, δ ··· $\frac{\langle S_B\rangle - L^2}{L^2}$ (a Ward identity)

⇒ For 99% noises, $\sum_{i=1}^{N(\eta)}$ sgn $|D+F| = 2$

⇒ **Witten index $\Delta = 2$ and Ward identities are well reproduced.**

Susceptibility: $\chi_\phi \equiv \sum_{x\geq 3}\langle\phi(x)\phi(0)\rangle$

$W(\Phi)=\frac{\lambda}{3}\Phi^3,\quad a\lambda=0.3,\quad L=18,\ 20,..,\ 32$



$\Rightarrow\ \chi_\phi \propto V^{0.660\pm 0.011}$

$\Rightarrow$ consistent with the conjecture $\chi_\phi \propto V^{0.666\ldots}$ ◎

5 Summary and future plan

Summary

- We observed $\chi = \int_V \mathrm{d}x^2 \langle \phi(x)\phi^*(0) \rangle$ in the cubic potential case, and got the consistent result with the conjecture $\chi \sim V^{0.666\ldots}$.
- We also extracted the effective coupling constant K of the Gaussian model, and obtained $K = 0.242 \pm 0.010$ which is consistent with the $\mathcal{N} = 2$ SUSY point $K = \frac{3}{4\pi} = 0.238...$ This implies the restoration of all supersymmetries in the IR. (see more detail in arXiv:1005.4671)

Future Plan

- further check of the A-D-E classification:

$$
\begin{aligned}
W = \quad & \Phi^4 && \longrightarrow A_3 \text{ model } \textbf{?} \\
& \Phi^3 + \Phi'^4 && \longrightarrow E_6 = A_2 \otimes A_3 \text{ model } \textbf{?} \\
& \Phi^2 + \Phi\Phi'^2 && \longrightarrow D_3 \text{ model } \textbf{?}, \ldots
\end{aligned}
$$

- c-function $\longrightarrow$ central charge, c-theorem
- 2d $\mathcal{N} = 1$ LG model with $W \propto \Phi^3$ ($\overset{\text{infrared}}{\longrightarrow}$ tricritical ising model)

  $\Rightarrow$ dynamical SUSY breaking

Appendix

Lattice formulation of WZ model

- continuum theory

$$S_{cont.} = Q\int \mathrm{d}^2x_E\Big[-H\psi_- + 2\psi_+\bar{\partial}\phi^* - W'\psi_+ - W^{*\prime}\psi_-\Big]$$
$$= \int \mathrm{d}^2x_E\Big[\partial_\mu\phi^*\partial_\mu\phi + |W'|^2 + \bar{\psi}\big(\gamma_\mu\partial_\mu + W''\frac{1+\gamma_3}{2} + W^{*\prime\prime}\frac{1-\gamma_3}{2}\big)\psi\Big], \quad H\text{-onshell.}$$

notation

$\gamma_1 = \sigma_3,\ \gamma_2 = -\sigma_2,\ \gamma_3 = -i\gamma_1\gamma_2 = \sigma_1,$
$\psi = \binom{\psi_1}{\psi_2},\ \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2),\ \psi_\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2),\ \bar{\psi}_\pm = \frac{1}{\sqrt{2}}(\bar{\psi}_1 \mp \bar{\psi}_2),\ \partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and

$$Q^2 = 0 \begin{cases} Q\phi = -\bar{\psi}_-, \quad Q\phi^* = -\bar{\psi}_+, \quad Q\bar{\psi}_\pm = 0, \\ Q\psi_+ = 2\partial\phi + H, \quad Q\psi_- = 2\bar{\partial}\phi^* + H^*, \\ QH = 2\partial\bar{\psi}_-, \quad QH^* = 2\bar{\partial}\bar{\psi}_+, \end{cases} \Rightarrow\ QS_{cont.} = Q^2\int(\cdots) = 0.$$

symmetry

SO(2), translation, $\mathcal{N}=2$ SUSY,
$U(1)_V,\quad U(1)_R\ \ \big(\phi \to e^{-2i\alpha}\phi,\ \psi \to e^{i\alpha\gamma_3}\psi,\ \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_3}\big)$ for $W = \frac{\lambda}{3}\phi^3$

- lattice theory '02 Kikukawa-Nakayama cf. '83 Sakai-Sakamoto, '09 Kadoh-Suzuki

$$S_{lat.} \equiv Q \sum_n a^2 \Big[-H\psi_- + \psi_+\big(-T\phi + (S_1 + iS_2)\phi^*\big) - W'\hat{\psi}_+ - W^{*\prime}\hat{\psi}_- \Big]$$

$$= a^2 \sum_n \Big[\phi^* \frac{2T}{a}\phi + W'^*(1 - \frac{aT}{2})W' + \Big(W'(-S_1 + iS_2)\phi + c.c.\Big)$$

$$+\bar{\psi}\Big(D + \frac{1+\gamma_3}{2}W''\frac{1+\hat{\gamma}_3}{2} + \frac{1-\gamma_3}{2}W''^*\frac{1-\hat{\gamma}_3}{2}\Big)\psi\Big], \quad H\text{-onshell.}$$

lattice Dirac operator

$$D = \tfrac{1}{a}\big[1 + \tfrac{X}{\sqrt{X^\dagger X}}\big], \qquad X = 1 - \tfrac{a}{2}\big[\gamma_\mu(\nabla_\mu^+ - \nabla_\mu^-) - a\nabla_\mu^+\nabla_\mu^-\big].$$

$D\hat{\gamma}_3 + \gamma_3 D = 0$ with $\hat{\gamma}_3 = \gamma_3(1 - aD)$.

notation

$D = T + \gamma_1 S_1 + \gamma_2 S_2$, $\quad \hat{\psi}_\pm = \frac{1}{\sqrt{2}}(1, \pm 1)\frac{1 \pm \hat{\gamma}_3}{2}\psi$ and

$$Q^2 = 0 \begin{cases} Q\phi = -\bar{\psi}_-, \quad Q\phi^* = -\bar{\psi}_+, \quad Q\bar{\psi}_\pm = 0, \\ Q\psi_+ = -T\phi^* + (S_1 - iS_2)\phi + H, \quad Q\psi_- = -T\phi + (S_1 + iS_2)\phi^* + H^*, \\ QH = -T\bar{\psi}_+ + (S_1 - iS_2)\bar{\psi}_-, \quad QH^* = -T\bar{\psi}_- + (S_1 + iS_2)\bar{\psi}_+. \end{cases}$$

symmetry

a-translation, one SUSY Q,

$U(1)_V$, $\quad Z_{3R}$ $\big(\phi \to e^{-2i\alpha}\phi,\ \psi \to e^{i\alpha\hat{\gamma}_3}\psi,\ \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_3},\ \alpha = \frac{n\pi}{3}, n \in \mathbf{Z}\big)$ for $W = \frac{\lambda}{3}\phi^3$.

Desired continuum limit is achieved by $a \to 0$ without extra fine-tunings.

redefinition: $\varphi \equiv \lambda\phi = (\mathrm{mass})^1$, $\quad \chi \equiv \lambda\psi = (\mathrm{mass})^{\frac{3}{2}}$, $\quad \bar\chi \equiv \lambda\bar\psi = (\mathrm{mass})^{\frac{3}{2}}$.

$$S_{lat.} = \frac{1}{\lambda^2}\, a^2 \sum_n \Bigg[\varphi^* \frac{2T}{a}\varphi + \varphi^{*2}(1-\frac{aT}{2})\varphi^2 + \Big(\varphi^2(-S_1+iS_2)\varphi + c.c.\Big)$$
$$+\bar\chi\Big(D + \frac{1+\gamma_3}{2}\,\varphi^2\,\frac{1+\hat\gamma_3}{2} + \frac{1-\gamma_3}{2}\,\varphi^{*2}\,\frac{1-\hat\gamma_3}{2}\Big)\chi\Bigg]$$

same role as $\hbar$

A radiative correction is

$$\delta S = \frac{1}{\lambda^2}\int \mathrm{d}^2\, C\, \mathcal{O}(\varphi,\chi)$$

counting the number of loops l as $\hbar$

$\Rightarrow$ If $\mathcal{O}$ has $(\mathrm{mass})^p$,

$$C = a^{p-4}\sum_{l=0}^{\infty} c_l (a^2\lambda^2)^l \quad \overset{a\to 0}{\longrightarrow} \quad \underbrace{a^{p-4}c_0}_{\text{tree}} + a^{p-2}c_1\lambda^2 + a^p c_2 \lambda^4.$$

$\Rightarrow$ We have to consider $p \le 2$.

$\mathcal{O}_{p\le 2}$ which preserves Z_{3R} and fermion number are a $const.$ and $\varphi^*\varphi$.

But the $const.$ has no effect and $\varphi^*\varphi$ is forbidden by the SUSY Q.

$\Rightarrow$ no extra fine-tunings.

Further Support

It is possible to construct the $\mathcal{N}=2$, $c=1$ SCA by the Gaussian model:

$$S_G = \frac{K}{2}\int \mathrm{d}^2x\,\partial_\mu X\,\partial_\mu X, \qquad X \sim X + 2\pi, \qquad K = \frac{1}{12\pi}, \frac{3}{4\pi}.$$

$\underline{T_{\mathrm{B}}(z)}$

EOM $\partial\bar{\partial}X = 0$ allows $X(z,\bar{z}) = X^L(z) + \theta^R(\bar{z})$, $\langle X^L(z)X^L(0)\rangle = -\frac{1}{4\pi K}\ln z$. Then

$$T_{\mathrm{B}}(z) = -2\pi K :(\partial X^L(z))^2:, \quad T_{\mathrm{B}}(z)T_{\mathrm{B}}(0) \sim \frac{1}{2}\frac{1}{z^4} \ (\Rightarrow c=1).$$

$\underline{G^\pm(z)}$

$$X^L(z) \equiv \frac{1}{\sqrt{4\pi K}}\left[q - ia_0 \ln z + i\sum_{n\neq 0}\frac{a_n}{n}z^{-n}\right], \quad X^R(\bar{z}) \equiv \frac{1}{\sqrt{4\pi K}}\left[\bar{q} - i\bar{a}_0 \ln \bar{z} + i\sum_{n\neq 0}\frac{\bar{a}_n}{n}\bar{z}^{-n}\right].$$

where a_n satisfies the $U(1)$, $k=1$ Kac-Moody algebra.

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_0, q] = -i,$$
$$[\bar{a}_n, \bar{a}_m] = n\delta_{n+m,0}, \quad [\bar{a}_0, \bar{q}] = -i.$$

Then, at only $K = \frac{1}{12\pi}, \frac{3}{4\pi}$, there are two operators of $(h,\bar{h}) = (\frac{3}{2}, 0)$: $G^\pm(z) = e^{\pm 3iX^L(z)}$

$\Rightarrow$ These $T_{\mathrm{B}}(z)$, $G^\pm(z)$, a_n construct the complete $\mathcal{N}=2$, $c=1$ SCA.

On the other hand, in the $\mathcal{N}=2$ LG model ...

- $W \propto \Phi^3$ should provide the $\mathcal{N}=2$, $c=1$ minimal model.
- If one writes $\phi = |\phi| e^{i\theta}$, the R-symmetry is $\theta \to \theta + const.$, which is not to be broken. (Coleman)

$\Rightarrow$ It is natural to identify θ as X in the IR.

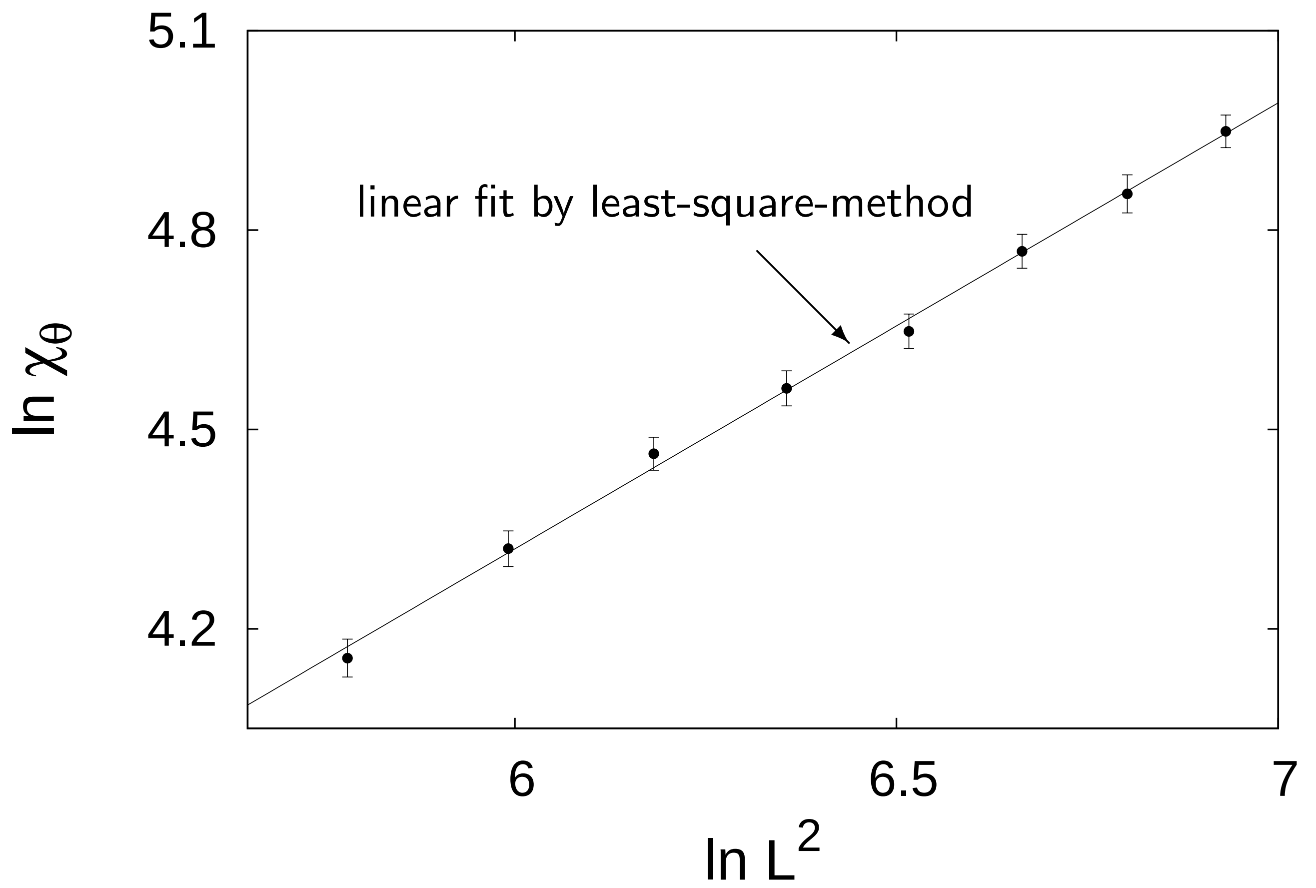
$\Rightarrow$ If this scenario works, the R-charge suggests $K = \frac{3}{4\pi}$.

$$\Rightarrow \chi_\theta \equiv \int \mathrm{d}^2 x \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim V^{1-\frac{1}{4\pi K}}, \qquad K = \frac{3}{4\pi} = 0.238...$$

So we also observed this χ_θ and K to provide the further support for the conjecture.

Susceptibility: $\chi_\theta \equiv \sum_{x\geq 3}\langle e^{i\theta(x)}e^{-i\theta(0)}\rangle$

$W(\Phi)=\frac{\lambda}{3}\Phi^3, \quad a\lambda=0.3, \quad L=18,\ 20, .., \ 32$



$\Rightarrow \ \chi_\theta \propto V^{0.671\pm 0.014}, \quad K = 0.242\pm 0.010$

$\Rightarrow$ consistent with the conjecture $K = 0.238\ldots$ ◎