量子統計力学の幾何学的アプローチと最小面積原理*

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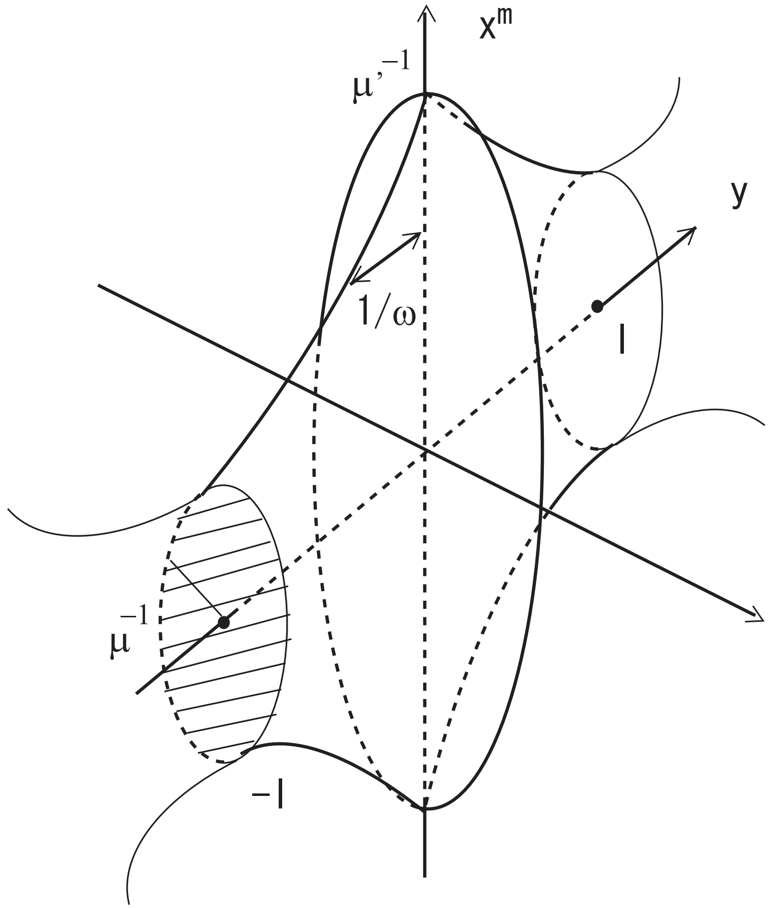
1. Introduction

Figure 1: IR-regularized geometry of 5D warped space(1)



Warped Metric $\quad ds^2 = \frac{1}{\omega^2 z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) = \mathrm{e}^{-2\omega|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad ,$

$$\left[(X^M) \equiv (x^\mu, z) \text{ or } (x^\mu, y) \quad , \quad M, N = 0,1,2,3,5(\text{or } z);\ \mu,\nu = 0,1,2,3 \quad .\right]$$

$$(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1) \quad , \quad -l \leq y \leq l \quad , \quad |z| = \frac{1}{\omega}\mathrm{e}^{\omega|y|} \quad , \quad \frac{1}{\omega} < |z| < \frac{1}{T} \quad ,$$

$$(T \equiv \omega \mathrm{e}^{-\omega l}) \qquad R_{MN} = 4\omega^2 G_{MN} \quad , \quad R = 20\omega^2 \quad (1)$$

Flat Metric $\quad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad , \quad (X^M) = (x^\mu, y) \quad , \quad -l \leq y \leq l \quad , \quad (2)$

$$E^{Warp}_{Cas}/\Lambda T^{-1} = -\alpha\omega^4 \left(1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)\right) = -\alpha(\omega_r)^4 \quad ,$$

$$\omega_r = \omega\sqrt[4]{1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)} \quad , \qquad (3)$$

$$|c| \ll 1 \quad , \quad |c'| \ll 1 \quad , \quad \omega_r = \omega(1 - c\ln(\Lambda/\omega) - c'\ln(\Lambda/T)) \quad ,$$

$$\beta\text{-function} \equiv \frac{\partial}{\partial(\ln\Lambda)} \ln\frac{\omega_r}{\omega} = -c - c' \quad . \tag{4}$$

Numerically $\alpha = 1.2$, $4c = -0.11$, $4c' = 0.10$ using the elliptic suppression factor.

Old Casimir Energy, Flat

$$\mathrm{e}^{-l^4 E_{Cas}} = \int \mathcal{D}A \exp\left[i \int d^4x dy (\mathcal{L}^{5D}_{EM} + \mathcal{L}_{gauge}) \right] \Bigg|_{\text{Euclid}} \quad (5)$$

$$E_{Cas}(l) = \int \frac{d^4p}{(2\pi)^4} \int_0^l dy (F^-(\tilde{p}, y) + 4F^+(\tilde{p}, y)) \quad ,$$

$$F^-(\tilde{p}, y) \equiv \int_{p^2}^{\infty} dk^2 G_k^-(y, y) = \int_{\tilde{p}}^{\infty} d\tilde{k} \frac{\cosh \tilde{k}(2y - l) - \cosh \tilde{k}l}{2\sinh(\tilde{k}l)} \quad ,$$

$$F^+(\tilde{p}, y) \equiv \int_{p^2}^{\infty} dk^2 G_k^+(y, y) = \int_{\tilde{p}}^{\infty} d\tilde{k} \frac{-\cosh \tilde{k}(2y - l) - \cosh \tilde{k}l}{2\sinh(\tilde{k}l)} \quad . \quad (6)$$

Old Casimir Energy, Warped

$$\mathrm{e}^{-T^{-4}E_{Cas}} = \int \mathcal{D}\Phi \;\exp\left[i \int d^5X\sqrt{-G}(-\frac{1}{2}\nabla^M\Phi\nabla_M\Phi - \frac{1}{2}m^2\Phi^2)\right]\Big|_{\text{Euclid}} . \quad (7)$$

$$-E^{\mp}_{Cas}(\omega,T) = \int \frac{d^4p_E}{(2\pi)^4}\int_{1/\omega}^{1/T} dz\;\, s(z)\int_{p_E^2}^{\infty}\{G_k^{\mp}(z,z)\}dk^2$$
$$\equiv \int \frac{d^4p_E}{(2\pi)^4}\Big|_{\tilde{p}\leq\Lambda}\int_{1/\omega}^{1/T} dz\;\, F^{\mp}(\tilde{p},z) \quad (8)$$

$$G_p^{\mp}(z,z') = \mp\frac{\omega^3}{2}z^2z'^2\frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z)\mp\mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\}\{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z')\mp\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega})-\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})} . \quad ($$

Proposed Casimir Energy of 5D Systems

For Flat Geometry

$$-\mathcal{E}_{Cas}(l,\Lambda)=\int_{1/\Lambda}^{l}d\rho\int_{r(0)=r(l)=\rho}\prod_{a,y}\mathcal{D}x^a(y)F_1(\frac{1}{r},y)\ \exp\left[-\frac{1}{2\alpha'}\int_0^l\sqrt{r'^2+1}\ r^3dy\right],$$

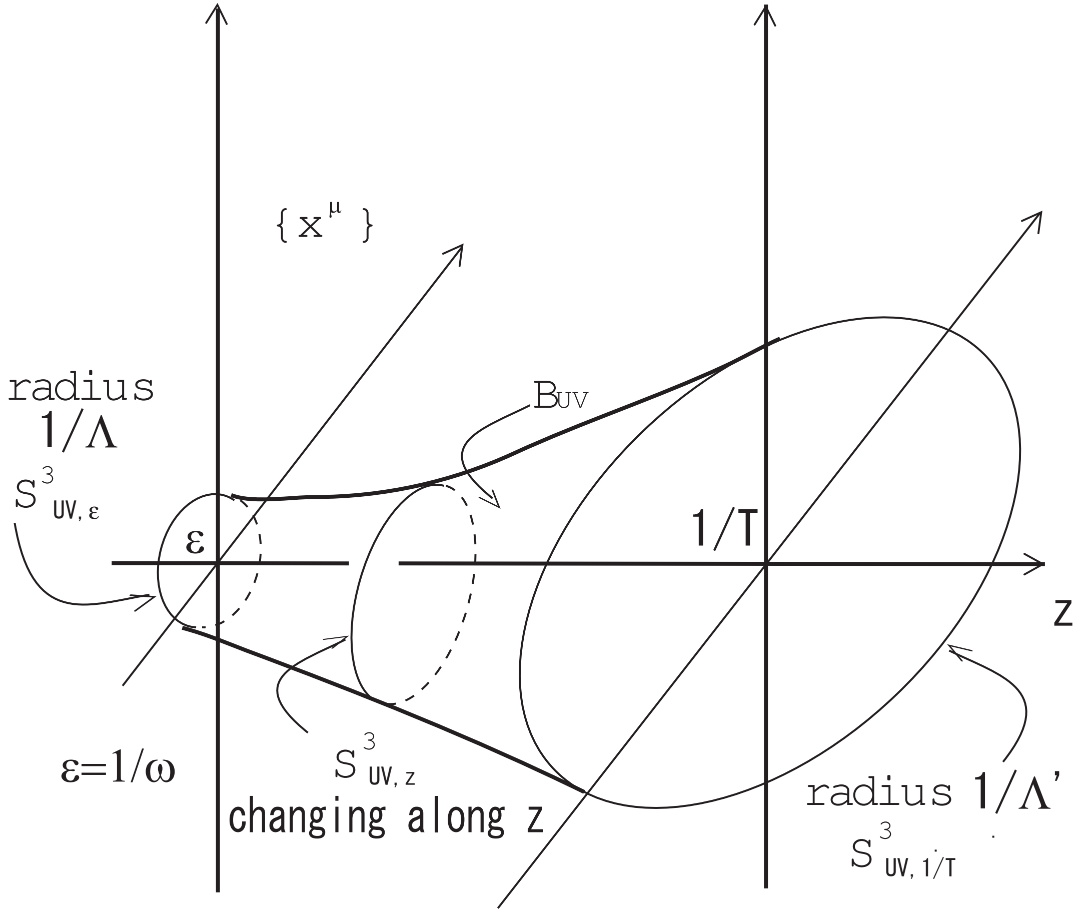
For Warped Geometry

$$-\mathcal{E}_{Cas}(\omega,T,\Lambda)=\int_{1/\Lambda}^{1/\mu}d\rho\int_{r(1/\omega)=r(1/T)=\rho}\prod_{a,z}\mathcal{D}x^a(z)F_2(\frac{1}{r},z)$$

$$\times\exp\left[-\frac{1}{2\alpha'}\int_{1/\omega}^{1/T}\frac{1}{\omega^4z^4}\sqrt{r'^2+1}\ r^3dz\right],($$

where $r=\sqrt{\sum_{a=1}^4(x^a)^2}$.

Figure 2: Warped geometry. S^3 radius $r(z)$ changes along the extra axis.



2.Quantum Statistical System of Harmonic Oscillator

'Dirac' Type

$$ds^2 = dX^2 + \omega^2 X^2 d\tau^2 = G_{AB} dX^A dX^B \quad , \tag{11}$$

$$(X^A) = (X^1, X^2) = (X, \tau) \quad , \quad (G_{AB}) = \text{diag}(1, \omega^2 X^2) \quad ,$$

$$R_{AB} = 0 \quad , \quad R = G^{AB} R_{AB} = 0 \quad , \tag{12}$$

where $A, B = 1, 2$. Periodicity:

$$\tau \to \tau + \beta \quad . \tag{13}$$

The induced metric on the line

$$X = x(\tau) \quad , \quad dX = \dot{x} d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$
$$ds^2 = (\dot{x}^2 + \omega^2 x^2) d\tau^2 \quad . \tag{14}$$

Then the length L of the path $x(\tau)$

$$L = \int ds = \int_0^\beta \sqrt{\dot{x}^2 + \omega^2 x^2} d\tau \quad . \tag{15}$$

We take the half of the length ($\frac{1}{2}L$) as the system Hamiltonian (minimal length principle). Free energy F:

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_{\tau} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^{\beta} \sqrt{\dot{x}^2+\omega^2 x^2} d\tau\right] \quad , \quad (16)$$

Normal Type

$$ds^2 = \frac{1}{d\tau^2}(dX^2)^2 + \omega^4 X^4 d\tau^2 + 2\omega^2 X^2 dX^2 = \frac{1}{d\tau^2}(dX^2+\omega^2 X^2 d\tau^2)^2 \quad , \quad (17)$$

where we have the following condition.

$$d\tau^2 \sim O(\epsilon^2) \quad , \quad dX^2 \sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2} dX^2 \sim O(1) \quad , \quad (18)$$

Note that we do not have 2D metric in this case ('primordial' geometry ?). We impose the periodicity (period: β):(13). The induced metric on the line:

$$X = x(\tau) \quad , \quad dX = \dot{x} d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$
$$ds^2 = (\dot{x}^2 + \omega^2 x^2)^2 d\tau^2 \quad . \tag{19}$$

On the path, we have this induced metric. The length L is given by

$$L[x(\tau)] = \int ds = \int_0^\beta (\dot{x}^2 + \omega^2 x^2) d\tau \quad . \tag{20}$$

Taking $\frac{1}{2}L$ as the Hamiltonian (minimal length principle), the free energy F:

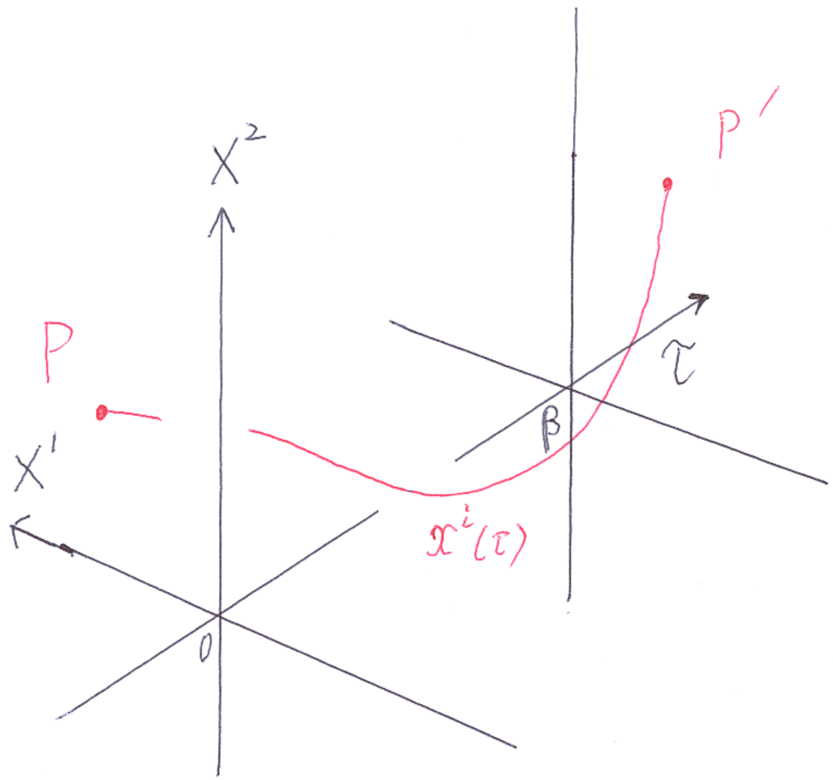
$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_{\tau} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^{\beta}(\dot{x}^2 + \omega^2 x^2)d\tau\right] \quad , \quad (21)$$

This is exactly the free energy of the harmonic oscillator.

3.Quantum Statistical System of N Harmonic Oscillators and O(N)-Nonlinear Generalization

'Dirac ' Type

Figure 3: A path $\{x^i(\tau)|i = 1, 2, \cdots, N\}$ in N(=2)+1 dim space. It starts at P=$(\rho_1, \rho_2, \cdots, \rho_N, 0)$ and ends at P′=$(\rho'_1, \rho'_2, \cdots, \rho'_N, \beta)$.



N+1 dim Euclidean space $(X^i, \tau), i = 1, 2, \cdots, N$.

$$ds^2 = \sum_{i=1}^{N} (dX^i)^2 + \omega^2 d\tau^2 \sum_{i=1}^{N} (X^i)^2 = G_{AB} dX^A dX^B \quad , \tag{22}$$

$$A, B = 1, 2, \cdots, N, N+1; \quad X^{N+1} \equiv \tau \quad ,$$

$$(G_{AB}) = \text{diag}(1, 1, \cdots, 1, \omega^2 r^2) \quad , \quad r^2 \equiv \sum_{i=1}^{N} (X^i)^2 \quad . \tag{23}$$

Explicitly for N=2,

$$ds^2 = dx^2 + dy^2 + \omega^2 (x^2 + y^2) d\tau^2 \quad ,$$

$$(R_{AB}) = \frac{1}{(r^2)^2}\begin{pmatrix} -y^2 & xy & 0 \\ yx & -x^2 & 0 \\ 0 & 0 & -\omega^2(r^2)^2 \end{pmatrix} \quad , \quad R = -\frac{2}{r^2} \quad , \quad r^2 = x^2 + y^2 \quad ,$$

$$\sqrt{G} = \omega\sqrt{x^2 + y^2} \quad , \quad \sqrt{G}R = -\frac{2\omega}{\sqrt{x^2 + y^2}} \tag{24}$$

where $(X^1, X^2, X^3) = (x, y, \tau)$. Periodicity (13). A path $\{X^i = x^i(\tau)|\ 0 \leq \tau \leq \beta,\ i = 1, 2, \cdots, N\}$, Induced metric on this line:

$$X^i = x^i(\tau) \quad , \quad dX^i = \dot{x}^i d\tau \quad , \quad \dot{x}^i \equiv \frac{dx^i}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$

$$ds^2 = \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau^2 \quad . \tag{25}$$

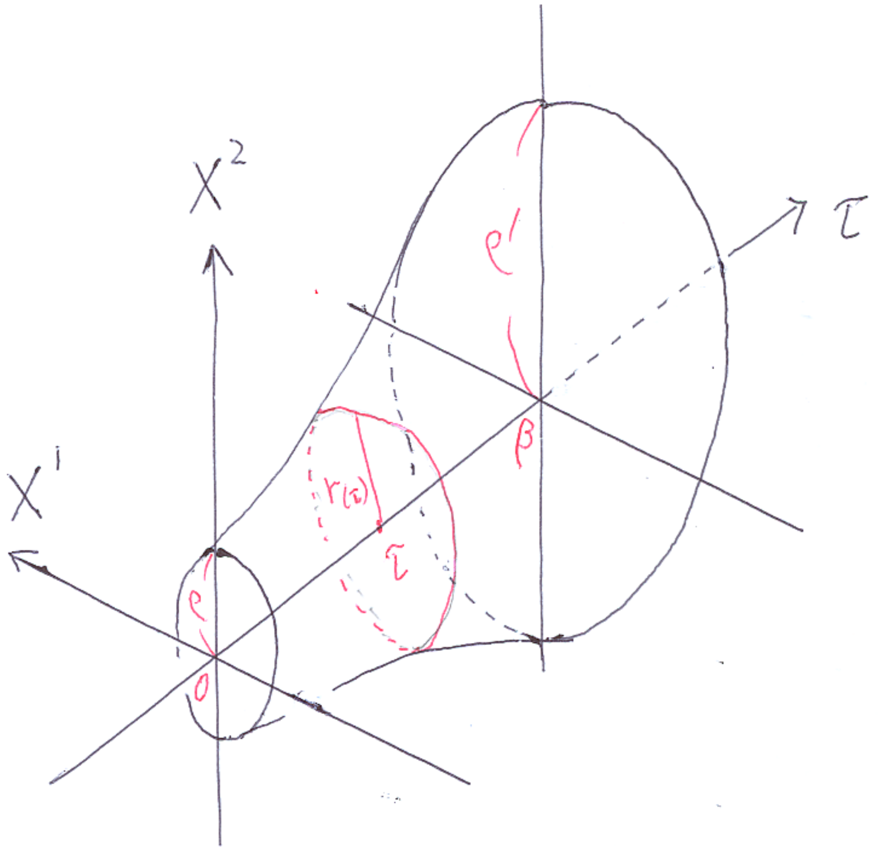
See Fig.3. Length L

$$L = \int ds = \int_0^\beta \sqrt{\sum_{i=1}^N ((\dot{x}^i)^2 + \omega^2 (x^i)^2)\ d\tau} \quad . \tag{26}$$

Free energy F: (Hamiltonian $\frac{1}{2}L$, minimal length principle)

$$\mathrm{e}^{-\beta F} = (\prod_i \int_{-\infty}^{\infty} d\rho_i) \int_{\substack{x^i(0) = \rho_i \\ x^i(\beta) = \rho_i}} \prod_{\tau, i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sqrt{\sum_{i=1}^N ((\dot{x}^i)^2 + \omega^2 {x^i}^2) d\tau}\right]$$

Figure 4: N(=2) dim hypersurface in N+1 dim space $(X^1, X^2, \cdots, X^N, \tau)$. S^{N-1} radius $r(\tau)$ starts by $r(0) = \rho$ and ends by $r(\beta) = \rho'$.



Another geometrical quantity (,instead of the length L). N dim *hypersurface* in N+1 dim space (a closed string). See Fig.4.

$$\sum_{i=1}^{N}(X^i)^2 = r^2(\tau) \quad , \quad \sum_{i=1}^{N} X^i dX^i = r\dot{r}d\tau \quad . \tag{28}$$

$r(\tau)$ describes a path which is *isotropic* in the N dim space (the 'brane' at τ). The induced metric on this N dim hypersurface:

$$ds^2 = \sum_{i,j}(\delta_{ij} + \frac{\omega^2}{\dot{r}^2}X^i X^j)dX^i dX^j \equiv \sum_{i,j} g_{ij} dX^i dX^j \quad ,$$

$$g_{ij} = \delta_{ij} + \frac{\omega^2}{\dot{r}^2}X^i X^j \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad , \quad \det(g_{ij}) = 1 + \frac{\omega^2 r^2}{\dot{r}^2} \quad . \tag{29}$$

$- - - - - - >$ Metric of the O(N) nonlinear sigma model. Area of this N dim hypersurface:

$$A_N = \int \sqrt{\det g_{ij}} d^N X = \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + \omega^2 r^2} r^{N-1} d\tau \quad . \tag{30}$$

Free energy F: (Hamiltonian $\frac{1}{2}A_N$, minimal area principle)

$$e^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + \omega^2 r^2} r^{N-1} d\tau\right] \quad . (3$$

Compare this result with the proposed 5D Casimir energy for the *flat* geometry

(10). We recognize, if we start with *** NHO9c

$$ds^2 = \sum_{i=1}^{N} (dX^i)^2 + d\tau^2 \quad \text{(N+1 dim Euclidean flat)} \quad , \tag{32}$$

instead of (23), the integration measure becomes exactly the same as (10).

Normal Type

Another type of N+1 dim Euclidean space (X^i, τ); $i = 1, 2, \cdots N$:

$$ds^2 = d\tau^{-2}\{\sum_{i=1}^{N}(dX^i)^2\}^2 + \omega^4\{\sum_{i=1}^{N}(X^i)^2\}^2 d\tau^2 + 2\omega^2\{\sum_{i=1}^{N}(X^i)^2\}\{\sum_{j=1}^{N}(dX^j)^2\}$$

$$= \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2 + \omega^2 d\tau^2 \sum_{i=1}^{N}(X^i)^2\}^2 \quad , \tag{33}$$

where we assume

$$d\tau^2 \sim O(\epsilon^2) \quad , \quad (dX^i)^2 \sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2\} \sim O(1) \quad , \tag{34}$$

Again, in the above case, we do not have N+1 dim (bulk) metric ('primordial' geometry). Periodicity β (13). A path $\{x^i(\tau)|\ 0 \leq \tau \leq \beta, i = 1, 2, \cdots, N\}$. Induced metric:

$$X^i = x^i(\tau) \quad , \quad dX^i = \dot{x}^i d\tau \quad , \quad \dot{x}^i \equiv \frac{dx^i}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$
$$ds^2 = [\sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)]^2 d\tau^2 \quad . \tag{35}$$

Length L:

$$L[x^i(\tau)] = \int ds = \int_0^\beta \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau \quad . \tag{36}$$

Free energy F: (Hamiltonian $\frac{1}{2}L$ minimal length principle)

$$e^{-\beta F} = \left(\prod_i \int_{-\infty}^{\infty} d\rho_i\right) \int_{\substack{x^i(0)=\rho_i \\ x^i(\beta)=\rho_i}} \prod_{i,\tau} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau\right]$$

This is exactly the free energy of N harmonic oscillators.

Instead of (33), a slightly modified metric.

$$ds^2 = \omega^4 \{\sum_{i=1}^{N} (X^i)^2\}^2 d\tau^2 + 2\omega^2 \kappa \{\sum_{i=1}^{N} (X^i)^2\}\{\sum_{j=1}^{N} (dX^j)^2\} \quad , \tag{38}$$

where we need not the last condition in (34). We have the metric G_{AB} in this case (ordinary geometry). Explicitly for $N = 2$,

$$ds^2 = \omega^4 (x^2 + y^2)^2 d\tau^2 + 2\omega^2 \kappa (x^2 + y^2)(dx^2 + dy^2) \quad ,$$

$$(R_{AB}) = \frac{1}{(r^2)^2} \begin{pmatrix} 4y^2 & -4xy & 0 \\ -4yx & 4x^2 & 0 \\ 0 & 0 & \frac{2\omega^2}{\kappa}(r^2)^2 \end{pmatrix}, R = -\frac{4}{\kappa\omega^2 (r^2)^2} \quad , \quad r^2 = x^2 + y^2,$$

$$\sqrt{G} = 2\kappa\omega^4 r^4 \quad , \quad \sqrt{G}R = 8\omega^2 \quad , \text{(39)}$$

where $(X^1, X^2, X^3) = (x, y, \tau)$. The N dim hypersurface (28). Induced metric:

$$ds^2 = \sum_{i,j=1}^{N} 2\omega^2 r^2 (\kappa\delta_{ij} + \frac{1}{2}\frac{\omega^2}{\dot{r}^2} X^i X^j) dX^i dX^j \equiv \sum_{i,j} g_{ij} dX^i dX^j \quad . \tag{40}$$

Area:

$$A_N = \int \sqrt{\det g_{ij}} d^N X = \frac{(2\pi\kappa)^{N/2}}{\Gamma(\frac{N}{2}+1)} \int_0^{\beta} (\omega r)^N \sqrt{\dot{r}^2 + \frac{r^2\omega^2}{2\kappa}}\, r^{N-1} d\tau \quad . \tag{41}$$

Free energy: (Hamiltonian $\frac{1}{2}A_N$ minimal area principle)

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)\,=\,\rho \\ r(\beta)\,=\,\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau)$$

$$\exp\left[-\frac{1}{2}\frac{(2\pi\kappa)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta (\omega r)^N \sqrt{\dot{r}^2 + \frac{r^2\omega^2}{2\kappa}}\; r^{N-1} d\tau\right] \quad . \qquad (42)$$

Compare this result with the proposed 5D Casimir energy for the *warped* geometry (10). We recognize, if we start with

$$\text{Euclidean } (\mathrm{AdS})_{N+1}: \quad ds^2 = \frac{1}{\tau^2}\{d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\} \quad , \qquad (43)$$

instead of (38), the integration measure exactly becomes the same as that in (10): $\tau^{-N}\sqrt{\dot{r}^2+1}r^{N-1}d\tau$.

4. Comparison with the Ordinary (1+3 EM) Casimir Energy Calculation

1+1 Dim Model

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\psi(x,t)=0 \quad , \quad -\infty<x<\infty \quad , \quad -\infty<t<\infty \quad . \tag{44}$$

Stationary States

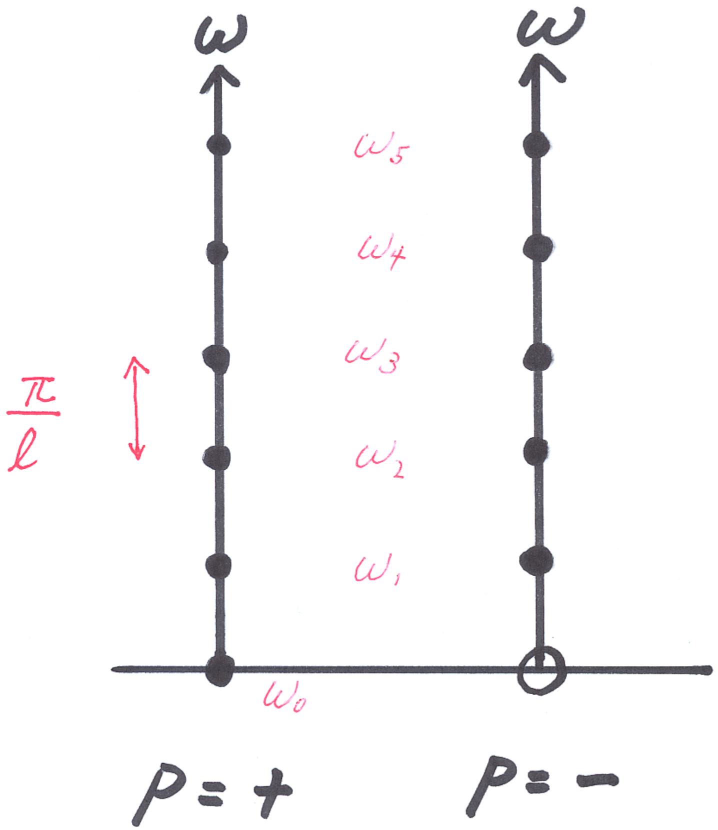
$$\psi(x,t)=\mathrm{e}^{-i\omega t}f(x) \quad , \quad i\frac{\partial\psi}{\partial t}=\omega\psi$$

$$\left(\omega^2+\frac{d^2}{dx^2}\right)f(x)=0 \quad , \quad f(x)=\{\cos\omega x,\sin\omega x\} \quad . \tag{45}$$

periodicity : $x \to x + 2l$ ($2l$: the separation length)
Z_2-symmetry P : $x \leftrightarrow -x$
Eigen Modes (Functions)

$$\left\{ \begin{array}{llll} \mathsf{P}{=}- & \mathrm{e}^{-i\omega_n t}\sin(\omega_n x) & \omega_n = \frac{\pi}{l}n & , \quad n = 1, 2, 3, \cdots \\ \mathsf{P}{=}+ & \mathrm{e}^{-i\omega_n t}\cos(\omega_n x) & \omega_n = \frac{\pi}{l}n & , \quad n = 0, 1, 2, \cdots \end{array} \right. \tag{46}$$

Figure 5: Equal spacing of the energy eigen values



Average Energy of this statistical system at β^{-1}
1) *equal-energy distribution* (Rayleigh-Jeans approximation)

$$E_\beta = \frac{1}{2}\beta^{-1} \times 2 \times \int_0^\Lambda \frac{d\omega}{\frac{\pi}{l}} = \beta^{-1}\frac{l}{\pi}\Lambda \quad , \qquad \mathrm{DoF} = 2 \times \int_0^\Lambda \frac{d\omega}{\frac{\pi}{l}} \quad , \tag{47}$$

linearly divergent
2) *Planck's distribution* (correct one)

$$E_\beta = \int_0^\infty \frac{\omega/2}{\mathrm{e}^{\beta\omega}-1} 2 \frac{d\omega}{\frac{\pi}{l}} = * * * * \, l\beta^{-2} \quad , \tag{48}$$

finite , 1+1 dim Stepan-Boltzman's law

Casimir energy (energy at $\beta = \infty$)

$$E^1_{Cas} = \int_\mu^\Lambda \frac{\omega}{2} 2 \frac{d\omega}{\frac{\pi}{l}} = \frac{1}{2}\frac{l}{\pi}(\Lambda^2 - \mu^2) \quad , \tag{49}$$

diverges quadratically
familiar regularization

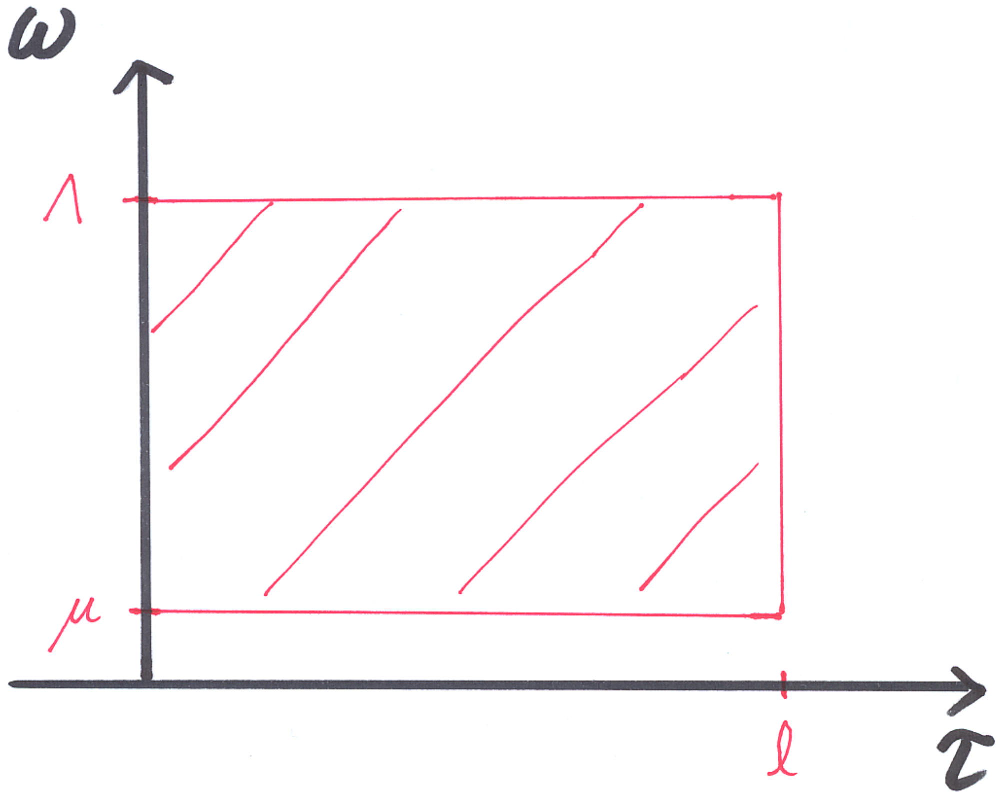
$$\begin{aligned} E^2_{Cas} &= 2 \times \sum_{n=1}^{\infty} \frac{1}{2}\omega_n g(\frac{\omega_n}{\Lambda}) - \int_0^\Lambda 2\frac{\omega}{2}\frac{1}{\frac{\pi}{l}} g(\frac{\omega}{\Lambda}) d\omega \\ &= \sum_{n=1}^{\infty} \frac{n\pi}{l} g(\frac{1}{\Lambda}\frac{n\pi}{l}) - \int_0^{\frac{l\Lambda}{\pi}} \frac{n\pi}{l} g(\frac{1}{\Lambda}\frac{n\pi}{l}) dn \\ &= \sum_{n=1}^{\infty} X(n) - \int_0^\infty dn X(n) = -\frac{1}{2!} B_2 X'(0) = -\frac{1}{12}\frac{\pi}{l} \quad , \end{aligned} \tag{50}$$

Present approach

$$E^1_{Cas} = \int_\mu^\Lambda \frac{\omega}{2} 2 \frac{d\omega}{\frac{\pi}{l}} = \int_\mu^\Lambda \frac{\omega}{2} \int_0^l \frac{d\omega d\tau}{\pi/2} \quad , \tag{51}$$

where we introduce an new axis τ
Integral over the rectangle region $((\Lambda - \mu) \times l)$ in (ω, τ) space.

Figure 6: Rectangle region in 2D (ω, τ) space



This integral should be regularized as folllows
Line element in $(x=\omega^{-1},\tau)$ space.

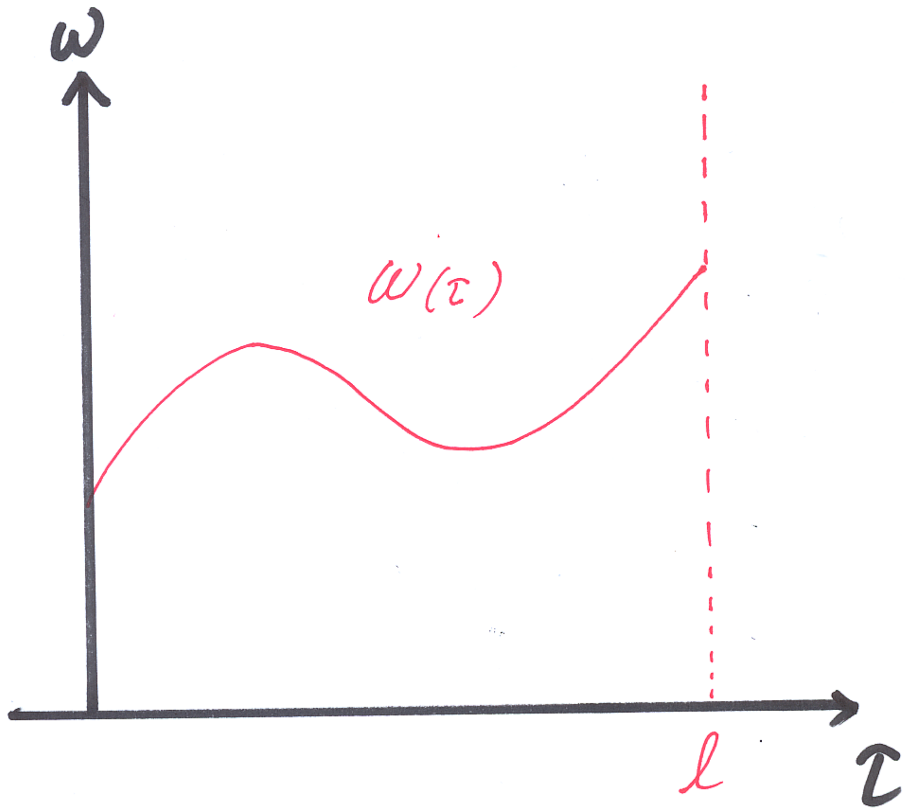
$$ds^2=\frac{1}{d\tau^2}(dx^2+\Omega^2x^2d\tau^2)^2 \tag{52}$$

Ω : a regularization parameter.

$$L=\int_0^l(\dot{x}^2+\Omega^2x^2)d\tau=\int_0^l(\omega^{-4}\dot{\omega}^2+\Omega^2\omega^{-2})d\tau,$$
$$\mathcal{E}_{Cas}=\int_0^\infty d\kappa\int_{\omega(0)=\omega(l)=\kappa}\mathcal{D}\omega(\tau)\frac{\omega(\tau)}{2}\exp\left\{-\frac{1}{2\alpha'}L\right\}\quad, \tag{53}$$

$1/\alpha'$: another regularization parameter (string tension parameter)

Figure 7: Path in the 2D (ω, τ) space



5. Discussion and Conclusion

A possible meaning of the last equation of (18) or (34)

The momentum cut-off parameter Λ for the UV-reglarization and μ for the IR-regularization.

$$\frac{1}{d\tau^2} dX^2 < \frac{\Lambda^2}{\mu^2} = \frac{\omega^2}{T^2} \sim \infty \quad , \tag{54}$$

OR

$$T^2 \; dX^2 < \omega^2 \; d\tau^2 \tag{55}$$

$- - - - - >$ Some kind of uncertainty relation ?

Relation between coordinates (not in the phase space of coordinate and momen-

tum).
We have some constraint, exprssed by (55), on the bulk-space coordinates. Does this suggest new 'quantization' rule for the higher dimensional quantum field theory ?