Developments in the β-Deformed Matrix Model of Selberg Type

- Method of Generating q-Expansion Coefficients for Conformal Block and $\mathcal{N}=2$ Nekrasov Function by β-Deformed Matrix Model

arXiv:1003.2929 with T. Oota
＝Nucl. Phys. B

・ with T. Oota and T. Yonezawa, in progress,
on the massive scaling limit with the remaining parameters kept finite

・ punch lines : 2d – 4d connection,
0d matrices acting as a bridge

The Jack polynomial and the finite N loop eq.
facilitates the computation with $\varepsilon_i,\ g_s$ finite

I)

6d

$\mathcal{N}=2, \quad SU(N_c)$
$N_f = 2N_c$

folklore in 70's

AGT conjecture '09

2d

4d

abelian,
two-derivative appx.

Liouville
&Toda

DF conformal
block

MMS
⋮

DV
IMO
⋮

curve
& dS_{SW}

$\mathcal{F}^{(eff)}_{SW}(a_i)$

$\cap$

$\mathcal{Z}^{\varepsilon_1,\varepsilon_2}_{\mathrm{Nek}}$

Operator (2nd quantized)
description

0d

loop eq. &
perturbed Selberg
integral

$\mathcal{Z}_{\mathrm{mat}}$

"Matrices"
β-deformed

q expansion
today

IO5
MMM
MS
⋮

Contents:

earlier ref: Dijkgraaf, Vafa 0909.2453; Shoichi Kanno, Yutaka Matsuo, Shotaro Shiba, Yuji Tachikawa 0911.4787; Tohru Eguchi, Kazunobu Maruyoshi 0911.4797; Ricardo Schiappa, Niclas Wyllard 0911.5337; A. Mironov, A. Morozov, Sh. Shakirov 0911.5721; Gaston Giribet 0912.1930; V. Alba, And. Morozov 0912.2535; Mitsutoshi Fujita, Yasuyuki Hatsuda, Ta-Sheng Tai 0912.2988; Masato Taki 0912.4789; Piotr Sulkowski 0912.5476; Sh. Shakirov 0912.5520; A. Mironov, A. Morozov, Sh. Shakirov 1001.0563; A. Popolitov 1001.1407

recent ones: A.Mironov, Al.Morozov, And.Morozov, 1003.5752 , Leszek Hadasz, Zbigniew Jaskolski, Paulina Suchanek, 1004.1841, Can Kozcaz, Sara Pasquetti, Niclas Wyllard, 1004.2025, A.Morozov, Sh.Shakirov, 1004.2917, Hidetoshi Awata, Yasuhiko Yamada, 1004.5122, Dimitri Nanopoulos, Dan Xie, 1005.1350, Ta-Sheng Tai, 1006.0471, Tohru Eguchi, Kazunobu Maruyoshi, 1006.0828, DimitriNanopoulos, Dan Xie, 1006.3486, Shoichi Kanno, Yutaka Matsuo, Shotaro Shiba, 1007.0601

II) A_{n-1} quiver matrix model:

$r = n-1$

IMO 0911.4244
=PTP

constructed s.t. obeying W_n constraints at finite N_a

$$Z \equiv \int \prod_{a=1}^{r} \left\{ \prod_{I=1}^{N_a} \mathrm{d}\lambda_I^{(a)} \right\} \left(\Delta_{A_{n-1}}(\lambda) \right)^{b_E^2} \exp\left(\frac{b_E}{g_s} \sum_{a=1}^{r} \sum_{I=1}^{N_a} W_a(\lambda_I^{(a)}) \right)$$

$$\Delta_{A_{n-1}}(\lambda) = \prod_{a=1}^{r} \prod_{1 \le I < J \le N_a} (\lambda_I^{(a)} - \lambda_J^{(a)})^2 \prod_{1 \le a < b \le r} \prod_{I=1}^{N_a} \prod_{J=1}^{N_b} (\lambda_I^{(a)} - \lambda_J^{(b)})^{(\alpha_a, \alpha_b)}$$

- $\exists n$ spin 1 currents s.t. $\sum_{i=1}^{n} J_i(z) = 0$

$$J_i(z) = \mathrm{i}\partial\varphi_i(z) = \frac{1}{g_s} t_i(z) + b_E \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) \mathrm{Tr} \frac{1}{z - M_a}$$

$$t_i(z) = \sum_{a=i}^{n-1} W'_a(z) - \frac{1}{n} \sum_{a=1}^{n-1} a\, W'_a(z)$$

- : $\det(x - \mathrm{i} g_s \partial\phi(z)) :=: \overleftarrow{\prod_{1 \le i < n}} (x - g_s J_i(z))$: contains W_n generators

- W_n constraints $\left\langle\!\left\langle \det(x - \mathrm{i} g_s \partial\phi(z)) \Big|_{+} \right\rangle\!\right\rangle = 0$

- the curve Σ $\left\langle\!\left\langle \det(x - \mathrm{i} g_s \partial\phi(z)) \right\rangle\!\right\rangle = 0$

Isomorphism with the Witten-Gaiotto curve established in the planar limit this way.

III) · Message

The Detsenko-Fateev multiple integral is an integral representation of the arbitrary 4-point conformal block. $\mathcal{F}(q|c;\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_I)$

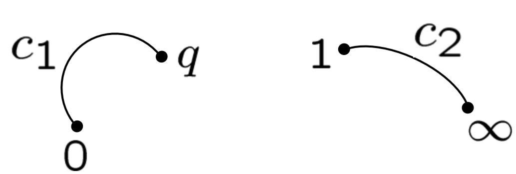
$$c = 1-6Q_E^2,\quad \Delta_i = \frac{1}{4}\alpha_i(\alpha_i-2Q_E),\quad \Delta_I = \frac{1}{4}\alpha_I(\alpha_I-Q_E)$$

We regard this as a version of β-deformed one-matrix model with special attention to the integration domain. Actually, it is a "perturbed double-Selberg matrix model":

$$\begin{aligned} Z_{\text{pert-(Selberg)}^2}(q\,|\,b_E;N_L,\alpha_1,\alpha_2;N_R,\alpha_4,\alpha_3) &= q^{\sigma}(1-q)^{(1/2)\alpha_2\alpha_3} \\ &\times \left(\prod_{I=1}^{N_L}\int_0^1 \mathrm{d}x_I\right)\prod_{I=1}^{N_L} x_I^{b_E\alpha_1}(1-x_I)^{b_E\alpha_2}(1-qx_I)^{b_E\alpha_3}\prod_{1\le I<J\le N_L}|x_I-x_J|^{2b_E^2} \\ &\times \left(\prod_{J=1}^{N_R}\int_0^1 \mathrm{d}y_J\right)\prod_{J=1}^{N_R} y_J^{b_E\alpha_4}(1-y_J)^{b_E\alpha_3}(1-q\,y_J)^{b_E\alpha_2}\prod_{1\le I<J\le N_R}|y_I-y_J|^{2b_E^2} \\ &\times \prod_{I=1}^{N_L}\prod_{J=1}^{N_R}(1-q\,x_Iy_J)^{2b_E^2} \end{aligned}$$

under $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2(N_L+N_R)b_E = 2Q_E$

- originally

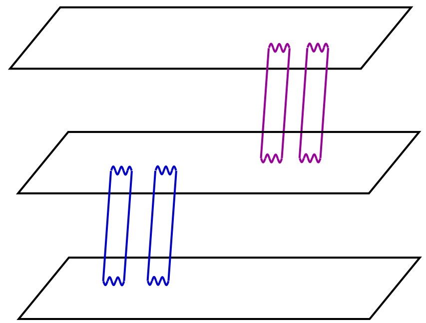


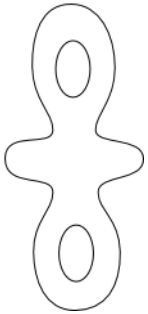
· Original reasoning leading to the computability

$n = 3,\ N_f = 2n = 6,\ n - 1 = 2$ kinds of e.v. distributions

$q \neq 0,\ g = 2$

3 Penner



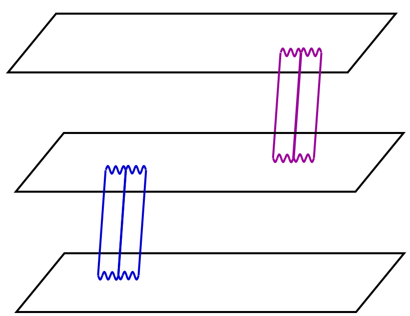


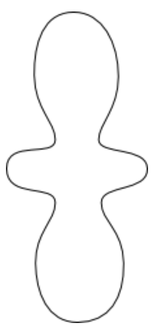
$q \to 0 \quad \Downarrow$

$\Downarrow \quad q \to 0$

$q = 0,\ g = 0$

2 Penner





- This has provided us an important insight
- Need only to take derivatives at $q = 0$

· Back to Z

$$Z_{\text{pert}-(\text{Selberg})^2}(q \,|\, b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3) \\ = q^{\Delta_I - \Delta_1 - \Delta_2} \mathcal{B}_0(b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)\, \mathcal{B}(q \,|\, b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)$$

$$\mathcal{B}_0(b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3) \\ = S_{N_L}(1 + b_E\alpha_1, 1 + b_E\alpha_2, b_E^2)\, S_{N_R}(1 + b_E\alpha_4, 1 + b_E\alpha_3, b_E^2)$$

$$\mathcal{B}(q \,|\, b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3) \\ = (1-q)^{(1/2)\alpha_2\alpha_3} \left\langle\!\!\left\langle \prod_{I=1}^{N_L} (1 - q x_I)^{b_E\alpha_3} \prod_{J=1}^{N_R} (1 - q y_J)^{b_E\alpha_2} \prod_{I=1}^{N_L} \prod_{J=1}^{N_R} (1 - q x_I y_J)^{2b_E^2} \right\rangle\!\!\right\rangle_{N_L, N_R}$$

Here S_{N_L}, S_{N_R} are the celebrated Selberg integral (an extension of Beta fn!!)

$$S_N(\beta_1, \beta_2, \gamma) = \left(\prod_{I=1}^{N} \int_0^1 \mathrm{d}x_I \right) \prod_{I=1}^{N} x_I^{\beta_1 - 1} (1 - x_I)^{\beta_2 - 1} \prod_{1 \le I < J \le N} |x_I - x_J|^{2\gamma} \\ = \prod_{j=1}^{N} \frac{\Gamma(1 + j\gamma)\Gamma(\beta_1 + (j-1)\gamma)\Gamma(\beta_2 + (j-1)\gamma)}{\Gamma(1+\gamma)\Gamma(\beta_1 + \beta_2 + (N + j - 2)\gamma)}$$

and averaging is w.r.t. these.

· Two kinds of generating functions :

$$\mathcal{B}(q) = 1 + \sum_{\ell=1}^{\infty} q^{\ell} \mathcal{B}_{\ell}$$

$$= \left\langle\!\!\left\langle \exp\left[-2 \sum_{k=1}^{\infty} \frac{q^k}{k} \left(b_E \sum_{I=1}^{N_L} x_I^k + \frac{1}{2}\alpha_2 \right) \left(b_E \sum_{J=1}^{N_L} y_J^k + \frac{1}{2}\alpha_3 \right) \right] \right\rangle\!\!\right\rangle_{N_L, N_R}$$

$$= (1-q)^{(1/2)\alpha_2\alpha_3} \mathcal{A}(q)$$

$$\mathcal{A}(q) = 1 + \sum_{\ell=1}^{\infty} q^{\ell} \mathcal{A}_{\ell}$$

$$= \left\langle\!\!\left\langle \exp\left[- \sum_{k=1}^{\infty} \frac{q^k}{k} \left(\alpha_2 + b_E \sum_{I=1}^{N_L} x_I^k \right) \left(b_E \sum_{J=1}^{N_R} y_J^k \right) - \sum_{k=1}^{\infty} \frac{q^k}{k} \left(b_E \sum_{I=1}^{N_L} x_I^k \right) \left(\alpha_3 + b_E \sum_{J=1}^{N_R} y_J^k \right) \right] \right\rangle\!\!\right\rangle_{N_L, N_R}$$

takes the form

$$= \sum_{k=0}^{\infty} q^k \sum_{|Y_1|+|Y_2|=k} \mathcal{A}_{Y_1, Y_2}$$

a pair of partitions (Y_1, Y_2) naturally appears.

· The rest of the plan :

i) some exact cal from special fn

ii) some by solving finite N loop eq. **briefly**

iii) originally from the planar loop eq. and g_s correction

our thought
in chronological
order

omitted today

i)

Jack polynomial $P_\lambda^{(1/\gamma)}(x) \qquad x = (x_1, \cdots, x_N)$

$\lambda = (\lambda_1, \lambda_2, \cdots)$ is a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$

$$\left\langle\!\!\left\langle P_\lambda^{(1/b_E^2)}(x) \right\rangle\!\!\right\rangle_{N_L} = \prod_{i\geq 1} \frac{\left(1 + b_E\alpha_1 + b_E^2(N-i)\right)_{\lambda_i} \left(b_E^2(N_L+1-i)\right)_{\lambda_i}}{\left(2 + b_E(\alpha_1+\alpha_2) + b_E^2(2N_L-1-i)\right)_{\lambda_i}}$$

$$\times \prod_{(i,j)\in\lambda} \frac{1}{(\lambda_i - j + b_E^2(\lambda'_j - i + 1))}$$

conj by Mcdonald ‘87

proven by Kadell ‘97

- From an explicit form of Jack poly. $|\lambda| \leq 2$ we obtain

$$\left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} x_I \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (b_E N_L - Q_E + \alpha_1)}{(\alpha_I - 2Q_E)}$$

$$2\left\langle\!\!\left\langle b_E^2 \sum_{1\leq I<J\leq N_L} x_I x_J \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (b_E N_L - b_E)(\alpha_1 + b_E N_L - Q_E)(\alpha_1 + b_E N_L - Q_E - b_E)}{(\alpha_I - 2Q_E)(\alpha_I - 2Q_E - b_E)}$$

$$\left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} x_I(1-x_I) \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (\alpha_1 + b_E N_L - Q_E)(\alpha_2 + b_E N_L - Q_E)(\alpha_1 + \alpha_2 + b_E N_L - 2Q_E)}{(\alpha_I - 2Q_E)(\alpha_I - 3Q_E + b_E)(\alpha_I - 2Q_E - b_E)}$$

ii)

Back to the model (perturbed double-Selberg $\approx$ 3 Penner).

Recall, at $q=0$, a pair of decoupled Selbergs $\approx$ 2 Penner's .

Build the original model $(q\neq 0)$ through resolvent.

$$Z_{\mathrm{Selberg}}(b_E;N_L,\alpha_1,\alpha_2)=\left(\prod_{I=1}^{N_L}\int_0^1 \mathrm{d}x_I\right)\prod_{1\leq I<J\leq N_L}|x_I-x_J|^{2b_E^2}\exp\left(b_E\sum_{I=1}^{N_L}\widetilde{W}(x_I)\right)$$

$$\tilde{W}(x)=\alpha_1\log x+\alpha_2\log(1-x)$$

- The loop eq. at finite N

$$\left\langle\!\!\left\langle\left(\widehat{w}_{N_L}(z)\right)^2\right\rangle\!\!\right\rangle_{N_L}+\left(\widetilde{W}'(z)+Q_E\frac{\mathrm{d}}{\mathrm{d}z}\right)\left\langle\!\!\left\langle\widehat{w}_{N_L}(z)\right\rangle\!\!\right\rangle_{N_L}-\tilde{f}_{N_L}(z)=0$$

$$\widehat{w}_{N_L}(z):=b_E\sum_{I=1}^{N_L}\frac{1}{z-x_I},\quad \tilde{f}_{N_L}(z):=\left\langle\!\!\left\langle b_E\sum_{I=1}^{N_L}\frac{\widetilde{W}'(z)-\widetilde{W}'(x_I)}{z-x_I}\right\rangle\!\!\right\rangle_{N_L}$$

$$\tilde{w}_{N_L}(z):=\left\langle\!\!\left\langle\widehat{w}_{N_L}(z)\right\rangle\!\!\right\rangle_{N_L}=\left\langle\!\!\left\langle b_E\sum_{I=1}^{N_L}\frac{1}{z-x_I}\right\rangle\!\!\right\rangle_{N_L}$$

- By looking at $O\left(\frac{1}{z}\right), O\left(\frac{1}{z^2}\right), O\left(\frac{1}{z^3}\right)$,

  we obtain exact results

$$\left\langle\!\left\langle b_E\, p_{(1)}(\mu) \right\rangle\!\right\rangle_{N_L} = \left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} x_I \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (b_E N_L - Q_E + \alpha_1)}{(\alpha_1 + \alpha_2 + 2b_E N_L - 2Q_E)},$$

$$\tilde{f}_{N_L}(z) = -\frac{b_E N_L (\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{z(z-1)},$$

$$-\widetilde{w}_{N_L}(0) = \left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} \frac{1}{x_I} \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{\alpha_1},$$

$$\widetilde{w}_{N_L}(1) = \left\langle\!\!\left\langle b_E \sum_{I=1}^{N_L} \frac{1}{1 - x_I} \right\rangle\!\!\right\rangle_{N_L} = \frac{b_E N_L (\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{\alpha_2}$$

The first one agrees with that from i).

· 0d – 4d relation

- matrix side parameters:

  seven parameters with one constraint

  $$b_E,\ \ N_L,\ \ \alpha_1,\ \ \alpha_2,\ \ N_R,\ \ \alpha_4,\ \ \alpha_3$$

- $\mathcal{N}=2,\ SU(2),\ N_f=4$, six parameters

  $$\frac{\epsilon_1}{g_s},\quad \frac{a}{g_s},\quad \frac{m_1}{g_s},\quad \frac{m_2}{g_s},\quad \frac{m_3}{g_s},\quad \frac{m_4}{g_s}$$

- By looking at $\mathcal{B}_1=\mathcal{A}_1-\frac{1}{2}\alpha_2\alpha_3$, and explicit form of $\mathcal{A}_1^{\text{Nek}}=\mathcal{A}^{\text{Nek}}_{[1],[0]}+\mathcal{A}^{\text{Nek}}_{[0],[1]}$,

$$\mathcal{A}^{\text{Nek}}_{[1],[0]}=\frac{(a+m_1)(a+m_2)(a+m_3)(a+m_4)}{2a(2a+\epsilon)g_s^2}$$
$$\mathcal{A}^{\text{Nek}}_{[0],[1]}=\frac{(a-m_1)(a-m_2)(a-m_3)(a-m_4)}{2a(2a-\epsilon)g_s^2}$$

we get

$$b_EN_L=\frac{a-m_2}{g_s},\qquad b_EN_R=-\frac{a+m_3}{g_s},$$
$$\alpha_1=\frac{1}{g_s}(m_2-m_1+\epsilon),\qquad \alpha_2=\frac{1}{g_s}(m_2+m_1),$$
$$\alpha_3=\frac{1}{g_s}(m_3+m_4),\qquad \alpha_4=\frac{1}{g_s}(m_3-m_4+\epsilon).$$

- Splitting of our $\mathcal{A}_1$ into $\mathcal{A}_{[1],[0]}+\mathcal{A}_{[0],[1]}$ done rather nontrivial, $\mathcal{A}_2$ derived

IV)

IOY

- massive scaling limit $N_L,\ N_R,\ b_E,\ (\,a,\ m_2,\ m_3,\ g_s)$ the same as before

1) $q \to 0,\ \ m_1 \to \infty,\ \ qm_1 = \Lambda$: finite

$\alpha_3,\ \alpha_4,\ \alpha_1 + \alpha_2$ the same as before

2) $q \to 0,\ \ m_1, m_4 \to \infty,\ \ qm_1 m_4 = \bar{\Lambda}^2$: finite

$\alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2$ the same as before

The integral representation being developed for

$$\left\langle\!\!\left\langle \prod_{\bar{I}} (1 - q x_{\bar{I}})^{\alpha_3 b_E} \prod_{\bar{I}} (1 - q y_{\bar{I}})^{\alpha_2 b_E} \prod_{\bar{I},\bar{J}} (1 - q x_{\tilde{I}} y_{\tilde{J}})^{2 b_E^2} \right\rangle\!\!\right\rangle_{N_L, N_R}$$

$\longrightarrow$ Λ or $\bar{\Lambda}^2$ expansion

H.W. - More use of the Jack Poly. and the finite N loop eq. both for purely theoretical developments and for physics exploration.