Higher-dimensional Charged Kerr-NUT Black Hole and HKT Structure

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1. Introduction

4-dim. Kerr metric

$$g = -\frac{\triangle_r}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\triangle_r}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2$$

$$\triangle_r = r^2 + a^2 - 2mr, \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

Separation of variables happens:

- Geodesic and Klein-Gordon equations[Carter 1968]
- Maxwell equation and gravitational perturbation [Teukolsky 1972]
- Dirac equation [Chandrasekhar 1976]

Two Killing vectors $\partial_t,\ \partial_\phi$ are not enough to explain such an integrability.

Hidden symmetry is a Killing-Yano symmetry [Penrose and Floyd 1973],

$$f_{ab} = -f_{ba}, \quad \nabla_{(c}f_{a)b} = 0.$$

Several higher-dimensional black holes in supergravity have generalized conformal Killing-Yano symmetry (GCKY).

Plan of talk

1. Introduction (continuation)

Conformal Killing-Yano (CKY) in higher dimensional Kerr BH

2. Charged Kerr-NUT BH in supergravity and Generalized Conformal Killing-Yano (GCKY)

Houri-Kubiznak-Warnick-Y.Y [1002.3616, 1004.1032]

3. HKT and CYT from charged Kerr-NUT BH

Calabi-Yau with torsion (CYT):=hermitian manifold admitting a conncetion with a torsion of SU(n)-holonomy.

1. Conformal Killing-Yano tensor (CKY)

Conformal Killing vector h_a on n-dim. spacetime obeys

$$\nabla_{(a} h_{b)} = \tilde{h} g_{ab}, \quad \tilde{h} = \frac{1}{n} \nabla_b h^b$$

An equivalent equation is

$$\nabla_a h_b = \nabla_{[a} h_{b]} + g_{ab} \tilde{h}$$

Rank-p generalization [Yano-Bochner "Curvature and Betti Number" 1953]

Conformal Killing-Yano (CKY) tensor $h = (h_{a_1 \cdots a_p})$ is anti-symmetric tensor (p-form) satisfying

$$\nabla_a h_{b_1 \cdots b_p} = \nabla_{[a} h_{b_1 \cdots b_p]} + p g_{a[b_1} \tilde{h}_{b_2 \cdots b_p]},$$

$$\tilde{h}_{b_2 \cdots b_p} = \frac{1}{n-p+1} \nabla_c h^c{}_{b_2 \cdots b_p}$$

It can be rewritten as

$$\nabla_X h = \frac{1}{p+1} i(X) dh - \frac{1}{n-p+1} X^* \wedge \delta h$$

where X^* is 1-form dual to X , and $i(X)$ the inner product.

Interesting property of CKY is its conformal invariance:

Let h be a CKY of the metric g_{ab}. Then $\hat{h} := e^{(p+1)\lambda} h$ is a CKY of the coformally equivalent metric $\hat{g}_{ab} := e^{2\lambda} g_{ab}$.

In order to fix the conformal factor we consider the following CKY:

♣ **Co-closed CKY (called Killing-Yano tensor), which imposes $\delta h = 0$:**

$$\nabla_X h - \frac{1}{p+1} i(X) dh = 0$$

♣ **Closed CKY, which imposes $dh = 0$:**

$$\nabla_X h + \frac{1}{n-p+1} X^* \wedge \delta h = 0$$

The Hodge star $*$ maps closed CKY p-forms into Killing-Yano (n-p)-forms and vice versa.

Higher-dimensional black holes admitting a closed CKY 2-form

	mass	rotation	NUT	Λ	parameter
Myers-Perry (1986)	○	○	×	0	1+[(d-1)/2]
Gibbons-Lü-Page-Pope (2004)	○	○	×	non-zero	2+[(d-1)/2]
Chen-Lü-Pope (2006)	○	○	○	non-zero	d

These metrics satisfy vacuum Einstein equation, $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

Kerr-NUT-de Sitter metric [Chen-Lü-Pope 2006]

CKY introduces "canonical coordinate $\{x_\mu, \psi_k\}$" on the Kerr-NUT-de Sitter spacetime:

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu(x)} + \sum_{\mu=1}^{n} Q_\mu(x) \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) d\psi_k \right)^2 + \varepsilon \frac{c}{\sigma_n} \left(\sum_{k=0}^{n} \sigma_k d\psi_k \right)^2$$

The function Q_μ is defined by

$$Q_\mu(x) = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1, \nu\neq\mu}^{n} (x_\mu^2 - x_\nu^2).$$

$X_\mu = X_\mu(x_\mu)$ depends only on single coordinate x_μ. Further, σ_k is given by elementary symmetric functions of x_μ^2 $(\mu = 1, \cdots, n)$.

$d = 4$ Kerr-NUT de Sitter metric:

$$\begin{aligned} g^{(4)} =& \frac{x^2 - y^2}{X(x)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 \\ &+ \frac{X(x)}{x^2 - y^2} (dt + y^2 d\psi)^2 + \frac{Y(y)}{y^2 - x^2} (dt + x^2 d\psi)^2, \end{aligned}$$

where

$$X(x) = (a^2 - x^2)(1 + \lambda x^2) + 2Mx$$

$$Y(y) = (a^2 - y^2)(1 + \lambda y^2) + 2Ly$$

and

$$Ric^{(4)}_{\mu\nu} = 3\lambda g^{(4)}_{\mu\nu}$$

a : angular momentum, M : mass, L : NUT

When we use the canonical coordinate, the separation of variables happens in all dimensions:

♣ Geodesic equation [Frolov-Krtous-Kubiznak 2006]

♣ Klein-Gordon equation [Frolov-Krtous-Kubiznak 2006]

♣ Dirac equation [Oota-Yasui 2007, S.Q.Wu 2008]

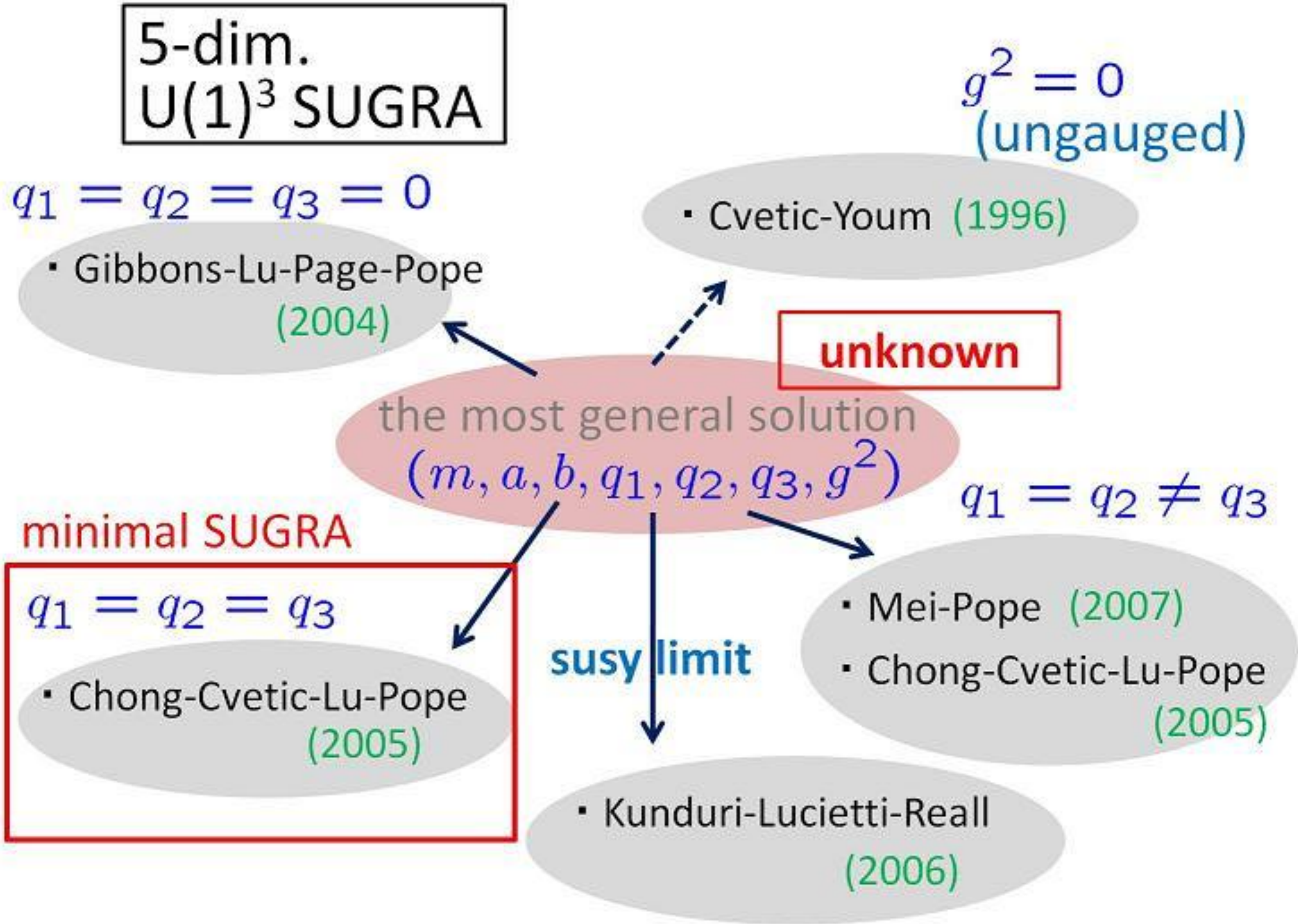
♣ gravitaional perturbation (tensor modes)

[Kunduri-Lucietti-Reall 2006, Oota-Yasui 2008]

♣ Theorem [Houri-Oota-Yasui 2007, Frolov-Krtous-Kubiznak 2008]

Kerr-NUT-de Sitter spacetime is only spacetime admitting a closed CKY 2-form in d-dim. spacetime.

2. Charged Kerr-NUT BH in Supergravity and GCKY



D=5 minimal gauged supergravity

$$S = \int_{M^5} \Big((R + 12g^2) * 1 - (1/2) F \wedge *F + (1/3\sqrt{3}) F \wedge F \wedge A \Big)$$

Chong-Cvetic-Lu-Pope black hole spacetime admits a GCKY, i.e. CKY 2-form with a torsion 3-form $T = *F$ [Kubiznak-Kundri-Y.Y 2009].

GCKY leads to the separability of the geodesic, Klein-Gordon and Dirac equations in the same way as without torsion.

Def. GCKY p-form

$$\nabla^T_X h = \frac{1}{p+1} i(X) d^T h - \frac{1}{n-p+1} X^* \wedge \delta^T h$$

where

$$d^T := \sum_{a=1}^{n} e^a \wedge \nabla^T_{e_a}, \quad \delta^T := -\sum_{a=1}^{n} i(e_a) \nabla^T_{e_a}$$

♣ Hodge star operation

♣ Conformal invariance

♣ d^T-closed GCKY

♣ δ^T-closed GCKY,i.e. Killing-Yano with torsion

Toroidally compactified heterotic supergravity

$$S = \int_{M^d} (R * 1 + *d\phi \wedge \phi + e^{\alpha\phi} * F_{(2)} \wedge F_{(2)} + e^{\beta\phi} * H_{(3)} \wedge H_{(3)})$$

$$F_{(2)} = dA_{(1)}, \quad H_{(3)} = dB_{(2)} - A_{(1)} \wedge F_{(2)}.$$

♣ **d=4 Kerr-Sen black hole**

Burinskii(1995); Okai(1994); Wu-Cai(2003); Hioki-Miyamoto(2008)

♣ **We find a solution admitting a GCKY with torsion $T = H_{(3)}$.**

This is a generalization of Kerr-Sen black hole and charged Kerr-NUT solution in all dimendions [Chow 2008].

d^T-closed GCKY 2-form takes the form

$$h = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{n+\mu} + \sum_{i=1}^{N} \xi_i \prod_{\mu=1}^{n} (x_\mu^2 - \xi_i^2) \omega^{(i)}$$

with $\omega^{(i)}$ Kähler forms.

GCKY has functionally independent eigenvalues x_μ, and constant eigenvalues ξ_i (for simplicity $\xi_i \neq 0$).

The spacetime is the total space of the bundle over Einstein-Kähler manifolds:

- Fiber: charged Kerr-NUT-de Sitter spacetime
- Base: product space of Einstein-Kähler manifolds

$$M_1 \times M_2 \times \cdots \times M_N$$

where $N = \#\{\xi_i\}$, $\dim M_i$ =multiplicity of ξ_i.

- Metric in string frame

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \Big(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu)\theta_k - \frac{1}{H} \sum_{\nu=1}^{n} \frac{N_\nu}{U_\nu} \sum_{k=0}^{n-1} \sigma_k(\hat{x}_\nu)\theta_k \Big)^2 + \sum_{i=1}^{N} \left(\prod_{\mu=1}^{n} (x_\mu^2 - \xi_i^2) \right) g_{Kahler}^{(i)}$$

- Scalar field

$$\phi = \left(\frac{2}{d-2} \right)^{1/2} \log H$$

- 2 and 3-forms (Maxwell fields)

$$F_{(2)} = \sum_{\mu=1}^{n} \partial_\mu \log H e^\mu \wedge e^{\hat{\mu}} + \sum_{i=1}^{N} \Xi_i W_{(i)}, \quad H_{(3)} = -\sum_{\mu=1}^{n} \sqrt{Q_\mu} e^{\hat{\mu}} \wedge F_{(2)}$$

- Torsion

$$T = H_{(3)}$$

In the expression the functions are given by

$$H = 1 + \sum_{\mu=1}^{n} \frac{N_\mu}{U_\mu}, \quad \Xi_i = \frac{2}{H} \sum_{\mu=1}^{n} \frac{N_\mu}{U_\mu} \frac{\xi_i}{x_\mu^2 - \xi_i^2}$$

$$Q_\mu = \frac{X_\mu}{U_\mu}\ , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2)\ ,$$

where X_μ, N_μ are functions of single coordinate x_μ:

$$X_\mu = x_\mu \left(d_\mu + \int \sum_{k=0}^{n} \alpha_k x^{2(k-1)} \prod_{i=1}^{N} (x_\mu^2 - \xi_i^2)^{m_i} dx_\mu \right)$$

$$N_\mu = \alpha X_\mu + \prod_{i=1}^{N} (x_\mu^2 - \xi_i^2)(b_{n-N-1} x_\mu^{2(n-N-1)} + \cdots + b_1 x_\mu^2 + b_0)$$

and θ_k is a one-form defined by

$$d\theta_k = -2 \sum_{i=1}^{N} (-1)^{n-k} \xi_i^{2(n-k)-1} \omega^{(i)}.$$

Symmetry of Dirac operator with torsion 3-form T

$$D = \gamma^a \nabla_a - \frac{1}{24} T_{abc} \gamma^{abc}$$

We prove [Houri-Kubiznak-Warnick-Y.Y, 2010]

Lemma 1. Let h be a δ^T-closed GCKY p-form. Then, the operator K_h graded commutes with the Dirac operator if anomaly vanishes:

$$K_h = h^a{}_{b_1 \cdots b_{p-1}} \gamma^{b_1 \cdots b_{p-1}} \nabla_a + \frac{1}{2(p+1)^2} (dh)_{b_1 \cdots b_{p+1}} \gamma^{b_1 \cdots b_{p+1}} + \frac{1-p}{8(p+1)} T^a{}_{b_1 b_2} h_{a b_3 \cdots b_{p+1}} \gamma^{b_1 \cdots b_{p+1}}$$
$$- \frac{p-1}{4} T^{ab}{}_{b_1} h_{a b b_2 \cdots b_{p-1}} \gamma^{b_1 \cdots b_{p-1}} + \frac{(p-1)(p-2)}{24} T^{abc} h_{abc b_1 \cdots b_{p-3}} \gamma^{b_1 \cdots b_{p-3}}$$

Lemma 2. Let h be a d^T-closed GCKY p-form. Then, the operator L_h graded commutes with the Dirac operator if anomaly vanishes:

$$L_h = h_{b_1\cdots b_p}\gamma^{ab_1\cdots b_p}\nabla_a - \frac{p(d-p)}{2(d-p+1)}(\delta h)_{b_1\cdots b_{p-1}}\gamma^{b_1\cdots b_{p-1}} - \frac{1}{24}T_{b_1b_2b_3}h_{b_4\cdots b_{p+3}}\gamma^{b_1\cdots b_{p+3}}$$

$$+\frac{p}{4}T^a{}_{b_1b_2}h_{ab_3\cdots b_{p+1}}\gamma^{b_1\cdots b_{p+1}} + \frac{p(p-1)(d-p-1)}{8(d-p+1)}T^{ab}{}_{b_1}h_{abb_2\cdots b_{p-1}}\gamma^{b_1\cdots b_{p-1}}$$

Here the anomaly is evaluated as

$$A_{(cl)}(h) = \frac{dd^Th}{p+1} - \frac{T\wedge\delta^T h}{d-p+1} - \frac{1}{2}dT\wedge_1 h$$

$$A_{(q)}(h) = \frac{\delta\delta^T h}{d-p+1} - \frac{T\wedge_3 d^T h}{6(p+1)} + \frac{1}{12}dT\wedge_3 h$$

♣ Anomaly vanishes for all solutions in this talk. So one can construct symmetry operators in the Dirac equation.

4. HKT and CYT from charged Kerr-NUT BH

Def. Hermitian manifold equiped with a torsion-connention ∇^T (Bismut connection) is called Kähler with torsion(KT).

The torsion is a 3-form and the connection satisfies

$$\nabla^T g = 0, \quad \nabla^T J = 0$$

If the holonomy of ∇^T is reduced to SU(n), the manifold is called Calabi-Yau with torsion (CYT).

Such geometry was considered in the study of supersymmetric backgrounds with non-vanishing flux [Strominger, Friedrich, Ivanov et.al]

Bosonic fields of Type II NS-NS sector:

{metric g, dilaton ϕ, 3-form $H_{(3)}$}

From SUSY transformation we require a parallel spinor η satisfying

$$\nabla^T \eta = 0, \quad (d\phi - T/2) \cdot \eta = 0$$

When we put $T = H_{(3)}$, CYT with

- $dT = 0$
- **closed Lee form,** $\theta(X) = \delta\omega(JX)$

gives a SUSY solution [Ivanov-Papadopoulos].

“BPS limit” of even-dim. charged Kerr-NUT BH

- **mass=charge**
- **eigenvalues of GCKY:** $x_\mu \longrightarrow 1 + \epsilon y_\mu, \ \epsilon \longrightarrow 0$

This is an extension of

• Odd dim. Kerr-NUT-de Sitter $\Longrightarrow$ Sasaki-Einstein metrics

Hashimoto-Sakaguchi-Y.Y (2004)

• Even dim. Kerr-NUT-de Sitter $\Longrightarrow$ Calabi-Yau metrics ;Oota-Y.Y (2006)

♣ We find “toric” CYT admitting a GCKY.

The metric is neither Kähler nor Ricci-flat, but the Ricci form with torsion vanishes, i.e. SU(n) holonomy:

$$\rho^T(X,Y) = Ric^T(X, J(Y)) + (\nabla\theta)J(Y) + \lambda(X,Y) = 0,$$

where the Lee form is closed

$$\theta = d\phi$$

with dilaton ϕ and λ is a 2-form defined by

$$\lambda(X,Y) = \sum_{a=1}^{2n} dT(X, Y, e_a, J(e_a)).$$

♣ The 4-dim. solution gives a hyperkähler with torsin (HKT) of Sp(1) holonomy.

The explicit metric is given by

$$g = \sum_{\mu=1}^{n} (e^{\mu} \otimes e^{\mu} + e^{\hat{\mu}} \otimes e^{\hat{\mu}})$$

$$e^{\mu} = \frac{dx_{\mu}}{\sqrt{Q_{\mu}}} \ , \quad e^{\hat{\mu}} = \sqrt{Q_{\mu}} \Big(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_{\mu}) d\psi_k - \frac{1}{H} \sum_{\nu=1}^{n} \frac{N_{\nu}}{U_{\nu}} \sum_{k=0}^{n-1} \sigma_k(\hat{x}_{\nu}) d\psi_k \Big) \ ,$$

where

$$Q_{\mu} = \frac{X_{\mu}}{U_{\mu}}, \quad H = 1 + \sum_{\mu=1}^{n} \frac{N_{\mu}}{U_{\mu}}, \quad U_{\mu} = \prod_{\nu \neq \mu} (X_{\mu} - x_{\nu})$$

and

$$X_{\mu} = \alpha q_{\mu} + \sum_{k=1}^{n-1} c_k x_{\mu}^{k}, \quad N_{\nu} = q_{\mu} + \sum_{k=1}^{n-1} d_k x_{\mu}^{k}$$

with constants $q_{\mu}, c_k, d_k, \alpha$ having a relation

$$c_{n-1} - \alpha(1 + a_{n-1}) = 0$$

which implies $dT = 0$.

- **complex structure**

$$\boldsymbol{J(e^{\mu}) = e^{\hat{\mu}}, \quad J(e^{\hat{\mu}}) = -e^{\mu}}$$

- **Kähler with torsion (Bismut torsion)**

$$\boldsymbol{\omega = \sum_{\mu=1}^{n} e^{\mu} \wedge e^{\hat{\mu}}}$$

$$\boldsymbol{T = -\sum_{\mu}^{n} \sqrt{Q_{\mu}} e^{\hat{\mu}} \wedge \sum_{\nu}^{n} \partial_{\nu} \log H e^{\nu} \wedge e^{\hat{\nu}}}$$

- **GCKY**

$$\boldsymbol{h = \sum_{\mu=1}^{n} (x_{\mu} - \sum_{\nu}^{n} x_{\nu}/2) e^{\mu} \wedge e^{\hat{\mu}}}$$

Summary

♣ Generalized conformal Killing-Yano (GCKY) symmetry of charged Kerr-NUT BH

♣ Spacetime having a structure of the bundle over Einstein-Kähler manifolds

♣ Calabi-Yau with torsion (CYT) from even dimensional charged Kerr-NUT BH, which provides a SUSY solution.

Further problem

♣ Contact manifold from odd dimensional charged Kerr-NUT BH

♣ Global structure and toric data of CYT

♣ Generalization of GCKY symmetry