Two-dimensional lattice for four-dimensional N =4 supersymmetric Yang-Mills

~ Hybrid Discretization ~

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arXiv:1004.5513

§ 1 Introduction

Supersymmetric Gauge Theory

- Natural extension of flat space-time. (philosophical importance)
- Necessary to unify the interactions. (phenomenological importance)
- Exact results in quantum field theory. (theoretical importance)
  (Seiberg-Witten theory, Dijkgraaf-Vafa theory, Nekrasov's formula, etc…)
- Gauge/Gravity duality
- Connection to superstring theory

We need non-perturbative analysis.

Possible two ways for non-perturbative analysis

1. SUSY algebra

- strong algebraic constraint by supersymmetry
- exact analysis (Seiberg-Witten, Dijkgraaf-Vafa, Nekrasov's formula etc…)

Powerful enough to get exact results

but

We can see (usually) only a part of the theory.

2. Numerical computation by **non-perturbative regularization**

Typical example: Lattice QCD

Once SUSY gauge theory is regularized non-perturbatively・・・

① We can compute any physical quantity numerically in principle.

② **We obtain a "definition" of the theory.**

<u>Difficulty</u>

It is difficult to keep *all* supersymmetry on a lattice.

⇩

We do not have enough symmetry to single out the continuum theory.

⇩

Especially for 4d theory, we need many fine tunings to take the continuum limit.
(almost impossible to carry it out)

non-perturbative formulations for SUSY gauge theories

1) 1d theories (matrix quantum mechanics)

- lattice formulations S. Catterall, T. Wiseman (2008)
- non-lattice formulations (momentum cutoff)

M. Hanada, J. Nishimura, S. Takeuchi (2007)

2) 2d theories

- **Sugino's lattice model** for
  - ✓ N=(2,2), (4,4) SYM F. Sugino (2002-2005)
  - ✓ N=(2,2) theory with matter F. Sugino (2008)
- Suzuki-Taniguchi lattice model for N=(2,2) SYM H. Suzuki, Y. Taniguchi (2005)

2d N=(8,8) theory (matrix string theory) is missing.

3) 3d theories

- lattice formulation for 3d N=1 SYM N. Maru, J. Nishimura (1997)
- 3d N=8 SYM on $R \times fuzzy\ S^2$

J.M. Maldacena, M.M. Sheikh-Jabbari, M. Van Raamsdonk (2003)

4) 4d theories

- lattice formulation for N=1 pure SYM e.g. J. Nishimura (1997)
- 4d large N (planar) N=4 SYM on $R \times S^3$

T. Ishii, G. Ishiki, S. Shimasaki, A. Tsuchiya (2008)

4d N=2 and 4 SYM with finite rank gauge group is missing.

In this talk, I will give

1. a new lattice formulation for 2d N=(8,8) U(N) SYM (matrix string theory)

2. a possible scenario to obtain 4d N=4 U(k) SYM from fuzzy S^2 background of the 2d lattice theory.

Plan of this talk

§ 2 continuum 2d N=(8,8) SYM

Euclidean action

$$S_0 = \frac{2}{g_{2d}^2} \int d^2x \, Tr\left(\frac{1}{2} F_{12}^2 + \frac{1}{2} \left(D_\mu X^I \right)^2 - \frac{1}{4} [X^I, X^J]^2 + \frac{1}{2} \Psi^T (D_1 + \gamma_2 D_2) \Psi + \frac{i}{2} \Psi^T \gamma_I [X^I, \Psi] \right)$$

where $\mu = 1,2, \quad I, J = 3,4, \cdots, 10.$

fields

- $\boldsymbol{A_\mu}$: gauge field
- $\boldsymbol{X^I}$: 8 scalar fields
- $\boldsymbol{\Psi}$: 16-component spinor

symmetries

16 supersymmetries
SO(8) R-symmetry

REWRITE

2 SUSY $\boldsymbol{Q_\pm}$
and
$\boldsymbol{SU(2)_R}$
become manifest

Ffield redefinition (BTFT form)

$$X^I \Rightarrow \begin{cases} X_i & (i=3,4) \\ B_A & (A=1,2,3) \\ C, \phi_+, \phi_- & \end{cases} \qquad \Psi \Rightarrow \begin{cases} \psi_{+\mu}, \rho_{+i}, \chi_{+A}, \eta_+ \\ \psi_{-\mu}, \rho_{-i}, \chi_{-A}, \eta_- \end{cases}$$

$\begin{pmatrix} \psi_{+\mu} \\ \psi_{-\mu} \end{pmatrix}, \quad \begin{pmatrix} \chi_{+A} \\ \chi_{-A} \end{pmatrix}, \quad \begin{pmatrix} \eta_+ \\ -\eta_- \end{pmatrix}, \quad \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix}$: SU(2) doublets $\quad \begin{pmatrix} \phi_+ \\ C \\ -\phi_- \end{pmatrix}$: SU(2) triplet

$$\left.\begin{aligned}
&Q_\pm A_\mu = \psi_{\pm\mu}, \quad Q_\pm \psi_{\pm\mu} = \pm i D_\mu \phi_\pm, \quad Q_\mp \psi_{\pm\mu} = \frac{i}{2} D_\mu C \mp \tilde{H}_\mu, \\
&Q_\pm \tilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2}[C, \psi_{\pm\mu}] \mp \frac{i}{2} D_\mu \eta_\pm, \\
&Q_\pm X_i = \rho_{\pm i}, \quad Q_\pm \rho_{\pm i} = \mp [X_i, \phi_\pm], \quad Q_\mp \rho_{\pm i} = -\frac{1}{2}[X_i, C] \mp \tilde{h}_i, \\
&Q_\pm \tilde{h}_i = [\phi_\pm, \rho_{\mp i}] \mp \frac{1}{2}[C, \rho_{\pm i}] \pm \frac{1}{2}[X_i, \eta_\pm], \\
&Q_\pm B_A = \chi_{\pm A}, \quad Q_\pm \chi_{\pm A} = \pm [\phi_\pm, B_A], \quad Q_\mp \chi_{\pm A} = -\frac{1}{2}[B_A, C] \mp H_A, \\
&Q_\pm H_A = [\phi_\pm, \chi_{\mp A}] \pm \frac{1}{2}[B_A, \eta_\pm] \mp \frac{1}{2}[C, \chi_{\pm A}], \\
&Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm [\phi_\pm, C], \quad Q_\mp \eta_\pm = \mp \left[\phi_+, \phi_-\right], \\
&Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm.
\end{aligned}\right\} \text{nilpotent up to gauge trans.}$$

Action in Q_+Q_--exact form

$$S_0 = Q_+Q_-\mathcal{F}^{(0)}$$

$$\mathcal{F}^{(0)} = \frac{1}{g_{2d}^2}\int d^2x\ \mathrm{Tr}\Big\{-iB_A\Phi_A - \frac{1}{3}\epsilon_{ABC}B_A[B_B,B_C]$$
$$-\psi_{+\mu}\psi_{-\mu} - \rho_{+i}\rho_{-i} - \chi_{+A}\chi_{-A} - \frac{1}{4}\eta_+\eta_-\Big\},$$

where

$$\Phi_1 = 2(-D_1X_3 - D_2X_4), \quad \Phi_2 = 2(-D_1X_4 + D_2X_3),$$
$$\Phi_3 = 2(-F_{12} + i[X_3,X_4])$$

§ 3 Plane-wave like mass deformation

We deform the theory s.t.

① $\boldsymbol{Q_{\pm}}$ and $\boldsymbol{SU(2)_R}$ are still symmetries of the deformed theory

② add **mass terms** to all scalars

③ add a **Myers term** to the triplet $(\phi, \bar{\phi}, C)$

STEP 1: deform $Q_\pm$ SUSY as

$$
\text{(A)}\begin{cases} Q_\pm A_\mu = \psi_{\pm\mu}, \quad Q_\pm \psi_{\pm\mu} = \pm i D_\mu \phi_\pm, \quad Q_\mp \psi_{\pm\mu} = \frac{i}{2} D_\mu C \mp \tilde{H}_\mu, \\ Q_\pm \tilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2}[C, \psi_{\pm\mu}] \mp \frac{i}{2} D_\mu \eta_\pm + \frac{M}{3}\psi_{\pm\mu}, \end{cases}
$$

$$
\text{(X)}\begin{cases} Q_\pm X_i = \rho_{\pm i}, \quad Q_\pm \rho_{\pm i} = \mp [X_i, \phi_\pm], \quad Q_\mp \rho_{\pm i} = -\frac{1}{2}[X_i, C] \mp \tilde{h}_i, \\ Q_\pm \tilde{h}_i = [\phi_\pm, \rho_{\mp i}] \mp \frac{1}{2}[C, \rho_{\pm i}] \pm \frac{1}{2}[X_i, \eta_\pm] + \frac{M}{3}\rho_{\pm i}, \end{cases}
$$

$$
\text{(B)}\begin{cases} Q_\pm B_A = \chi_{\pm A}, \quad Q_\pm \chi_{\pm A} = \pm[\phi_\pm, B_A], \quad Q_\mp \chi_{\pm A} = -\frac{1}{2}[B_A, C] \mp H_A, \\ Q_\pm H_A = [\phi_\pm, \chi_{\mp A}] \pm \frac{1}{2}[B_A, \eta_\pm] \mp \frac{1}{2}[C, \chi_{\pm A}], + \frac{M}{3}\chi_{\pm A} \end{cases}
$$

$$
\text{(C)}\begin{cases} Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm[\phi_\pm, C] + \frac{2M}{3}\phi_\pm, \\ Q_\mp \eta_\pm = \mp[\phi_+, \phi_-] \pm \frac{M}{3} C, \quad Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm \end{cases}
$$

<u>Nilpotency</u>

$$
Q_\pm^2 = (\text{infinitisimal gauge transformation by } \pm\phi_\pm) \ \pm \frac{M}{3} J_{\pm\pm},
$$

$$
\{Q_+, Q_-\} = (\text{infinitisimal gauge transformation by } C) \ -\frac{M}{3} J_0.
$$

STEP 2: add mass terms

$$S = \left(Q_+ Q_- - \frac{M}{3}\right)(\mathcal{F}_0 + \Delta\mathcal{F}) = S_0 + \Delta S$$

$$\Delta\mathcal{F} = \frac{1}{g_{2\mathrm{d}}^2}\int d^2x \mathrm{Tr}\left[\sum_{A=1}^{3}\frac{a_A}{2}B_A^2 + \sum_{i=3}^{4}\frac{c_i}{2}X_i^2\right]$$

$$\Delta S = \frac{1}{g_{2d}^2}\int d^2x\ \mathrm{Tr}\Big\{\frac{2M^2}{81}(B_A^2 + X_i^2) - \frac{M}{2}C[\phi_+,\phi_-] + \frac{M^2}{9}\left(\frac{C^2}{4} + \phi_+\phi_-\right) + \frac{2M}{3}\psi_{+\mu}\psi_{-\mu} + \frac{2M}{9}\rho_{+i}\rho_{-i} + \frac{4M}{9}\chi_{+A}\chi_{-A} - \frac{M}{6}\eta_+\eta_- - \frac{4iM}{9}B_3(F_{12} + i[X_3, X_4])\Big\}.$$

N.B

- S is $Q_\pm$- invariant: $\boldsymbol{Q_\pm S = 0}$.
- When $a_A, c_i \in (-\frac{2M}{3}, 0)$, the scalars B_A, X_i have positive mass terms.

  We set $\boldsymbol{a_1 = a_2 = a_3 = -\frac{2M}{9}}$, $\boldsymbol{c_3 = c_4 = -\frac{4M}{9}}$ in order to cancel as many terms as possible.

Important result

① **fuzzy S^2 is a classical configuration**

$$[\phi_+, \phi_-] = \frac{M}{3}C, \quad [C, \phi_\pm] = \pm\frac{2M}{3}\phi_\pm, \quad B_A = X_i = 0$$

② There is no scalar flat direction after the deformation.

N.B

- This configuration is $Q_\pm$-invariant:

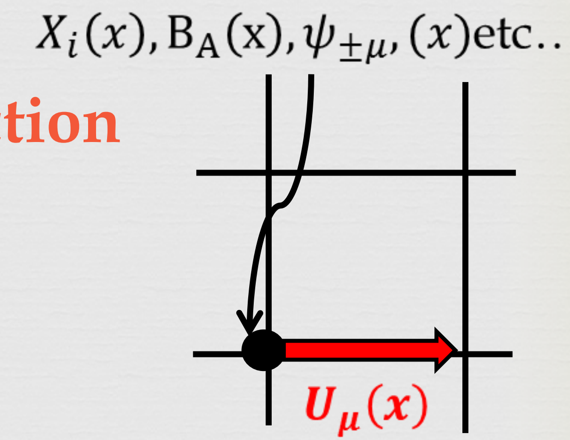
$$Q_\pm \eta_\pm = \pm[\phi_\pm, C] + \frac{2M}{3}\phi_\pm, \quad Q_\mp \eta_\pm = \mp[\phi_+, \phi_-] \pm \frac{M}{3}C,$$

- This deformation **softly** breaks the other 14 of 16 supersymmetries.

§ 4 Lattice formulation of the mass deformed 2d N=(8,8) SYM

$Q_\pm$-exact form of the continuum action

Sugino's construction
of
SUSY lattice



Lattice Action in $Q_\pm$-exact form

Obtained lattice action:

$$||A|| \equiv \sqrt{Tr(AA^\dagger)}$$

$$S_{\rm lat} = \begin{cases} \left(Q_+Q_- - \frac{M_0}{3}\right)\mathcal{F}_{\rm lat} & ||1 - U_{12}(x)|| < \epsilon \text{ for any } x \\ +\infty & \text{otherwise} \end{cases}$$

Trivial latticization of $\mathcal{F}$:

$$\frac{1}{g_{2d}^2}\int d^2x \to \frac{1}{g_0^2}\sum_x, \quad M \to M_0$$

necessary to single out the trivial vacuum

cf)

$$\mathcal{F}^{(0)} = \frac{1}{g_{2d}^2}\int d^2x \ \mathrm{Tr}\Big\{ - iB_A\Phi_A - \frac{1}{3}\epsilon_{ABC}B_A[B_B, B_C] - \psi_{+\mu}\psi_{-\mu} - \rho_{+i}\rho_{-i} - \chi_{+A}\chi_{-A} - \frac{1}{4}\eta_+\eta_- + \sum_{A=1}^{3}\frac{a_A}{2}B_A^2 + \sum_{i=3}^{4}\frac{c_i}{2}X_i^2\Big\}$$

This action clearly preserves $Q_\pm$ supersymmetries.

Important properties

① fuzzy sphere configuration is still classical configuration keeping $Q_\pm$:

$$\text{(potential terms)} \ni \left(i[X_p(x), X_q(x)] + \frac{M_0}{3}\epsilon_{pqr}X_r(x) \right)^2$$

with $C = 2X_8,\ \phi_\pm = X_9 + iX_{10}$ $(p, q, r = 8,9,10)$.

② The flat directions of the scalar fields are lifted up with keeping $Q_\pm$.

③ We do not need any fine tuning in taking 2d continuum limit:

We can show that all possible relevant and marginal operators are forbidden by the $Q_\pm$-symmetry and $SU(2)_R$ symmetry.

We can now simulate Matrix String Theory on computer!

N.B.

We have to keep M finite in taking the continuum limit.
We take $M \to 0$ after that.

§ 5 4d N=4 U(k) SYM theory

Let us recall **deformed continuum 2d theory**

- The deformed theory has fuzzy sphere solution.
- Let us set $N = (2n+1)k$. If we expand the action around the solution,

  $$C = \frac{2M_0}{3}\hat{L}_3 \otimes 1_k,\ \phi_\pm = \frac{M_0}{3}(\hat{L}_1 \pm i\hat{L}_2) \otimes 1_k$$

  we obtain **4d N=4 SYM on $\mathbb{R}^2 \times \boldsymbol{fuzzy\ S^2}$**.

  (UV cutoff = $nM \equiv \Lambda$, IR cutoff = M)
- We can obtain the 2d continuum theory without any fine tuning from the lattice theory.

idea

lattice theory $\xrightarrow{\textbf{step 1}}$ 2d continuum theory around fuzzy sphere solution $\xrightarrow{\textbf{step 2}}$ 4d N=4 SYM on $\mathbb{R}^2 \times fuzzy\, \mathbb{R}^2$

continuum limit of the lattice theory

$\Lambda \to \infty$ with θ: fixed

$(M \to 0, n \to \infty$ with $M \sim n^{-\frac{1}{2}})$

$\Leftrightarrow$

4d theory with
momentum cutoff $\Lambda = Mn$
non-commutative parameter $\theta = Mn^2$

- ✓ There is no problem in **step 1**.
- ✓ **Step 2** is OK at tree level.
- ✓ **Only the problem is whether quantum corrections appear or not.**

CLAIM: **Any quantum correction disappears in STEP 2.**

that is

We can obtain the 4d theory without any fine-tuning.

Radiative correction

Superficial degrees of divergence of a graph

$E_B \cdots$ # of bosonic external lines
$E_F \cdots$ # of fermionic external lines

$$D = 4 - E_B - \frac{3}{2} E_F$$

The most severe UV divergences come from $E_B = 2\ (\Lambda^2)$
(1-point function is forbidden by the $Q_\pm$ symmetry.)

point **The deformation parameter M is IR origin and is soft in 4d sense.**

possible structure of the divergent terms:

$$A \cdot \Lambda^2 + O\left(M^p \left(\log\frac{\Lambda}{M}\right)^q\right) \qquad (p, q = 1,2,\cdots)$$

- The leading term is canceled because of the original 16 SUSY.
- The next leading terms vanish in the continuum limit:

$$M^p \left(\log\frac{\Lambda}{M}\right)^q \sim M^p (\log N)^q \to 0 \qquad \text{since } M \propto N^{-\frac{1}{2}} \to 0.$$

We obtain 4d N=4 U(k) SYM on $\mathbb{R}^2 \times$ noncommutative $\mathbb{R}^2$ without any fine tuning !

FINAL STEP: Smooth commutative limit

In 4d N=4 SYM, it is believed that $\theta \to 0$ limit is smooth.

Matusis-Susskind-Toumbas

If we believe this conjecture, we obtain commutative 4d N=4 SYM on $\mathbb{R}^4$ without any fine tuning!

§ 6 Conclusion and Future Works

1. We constructed a new lattice formulation of 2d N=(8,8) SYM theory(matrix string theory).
   - ✓ numerical approach to matrix string theory
   - ✓ application to black hole physics
2. We discuss a possible scenario to obtain 4d N=4 U(k) SYM from the 2d lattice theory.
   - ✓ We have to check the validity of the step 2 at least perturbatively.
   - ✓ AdS/CFT beyond SUGRA level !
3. We can use the same method to regularized 4d N=2 SYM.
   - ✓ Many other applications to numerical analysis of SUSY gauge theories

Strategy

① Lattice gauge fields are on links: $A_\mu(x) \Rightarrow \boldsymbol{U_\mu(x) = e^{iaA_\mu}}$ and all other fields are on sites.

② Make all the lattice fields and coupling constants dimensionless by

$$(\text{scalars})^{lat} = a(\text{scalars})^{\text{cont}}, \qquad (\text{fermions})^{\text{lat}} = a^{\frac{3}{2}}(\text{fermions})^{\text{cont}}$$

$$Q_\pm^{lat} = a^{\frac{1}{2}} Q_\pm^{cont}, \qquad g_0 \equiv a g_{2d}, \qquad M_0 \equiv aM$$

③ change the $Q_\pm$-transformation consistently.

④ The action is obtained through the Q-exact form of the continuum action.