Topological symmetry of deformed instanton effective action

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July 22, 2010 @ YITP

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1. Introduction

Nekrasov formula [Nekrasov 2002], [Nekrasov-Okounkov], . . .

low-energy effective action of $\mathcal{N} = 2$ SYM is determined by a single holomorphic function $\mathcal{F}(\phi)$ called the prepotential.

$$\mathcal{F}(\phi) = \mathcal{F}(\phi)_{\mathrm{pert}} + \mathcal{F}_{\mathrm{inst}}(\phi), \quad \mathcal{F}^{\mathrm{SU}(2)}_{\mathrm{inst}}(\phi) = \sum_{k=1}^{\infty} \mathcal{F}_k \phi^2 \left(\frac{\Lambda}{\phi}\right)^{4k}.$$

By introducing the deformation ϵ_1, ϵ_2 (Ω-background), Nekrasov defined the instanton partition function

$$Z_{\mathrm{inst}}(\phi, \Lambda, \epsilon_1, \epsilon_2) \equiv \sum_{k=0}^{\infty} \Lambda^{2Nk} \int_{\mathcal{M}_{N,k}} D\mu_{\mathrm{inst}} \exp\left[-S_{\mathrm{eff}}(\phi, \epsilon_1, \epsilon_2)\right]$$

where

$$\mathcal{M}_{N,k} : \text{ instanton moduli space (ADHM construction)}$$

$$D\mu_{\mathrm{inst}} : \text{ integration measure for instanton moduli space}$$

$$S_{\mathrm{eff}}(\phi, \epsilon_1, \epsilon_2) : \text{ instanton effective action deformed by } \epsilon_1 \text{ and } \epsilon_2$$

(effective action for instanton moduli)

$\mathcal{F}_{\mathrm{inst}}(\phi)$ is extracted from Z_{inst} by ϵ-expansion.

$$\log Z_{\mathrm{inst}}(\phi, \Lambda, \epsilon_1, \epsilon_2) = \frac{1}{\epsilon_1 \epsilon_2}\Big(\mathcal{F}_0(\phi) + (\epsilon_1 + \epsilon_2)\mathcal{G}_1(\phi) + O(\epsilon^2)\Big),$$

$$\mathcal{F}_{\mathrm{inst}}(\phi) = \mathcal{F}_0(\phi).$$

When $\epsilon_1 = -\epsilon_2 = \hbar$, $\hbar$ is interpreted as

- coupling constant of topological string theory
  (ϵ-expansion $=$ genus expansion of worldsheet)

  $\rightarrow$ self-dual graviphoton background (closed string approach)
  [Antoniadis-Gava-Narain-Taylor], etc.

  $\rightarrow$ R-R background (open string approach)

  explicit computation: [Billo-Frau-Fucito-Lerda], [Matsuura]

In the case of general ϵ_1 and ϵ_2, Z_{inst} is related to the refined version of topological string.

$\rightarrow$ Searching R-R background corresponding general Ω-background

(cf. [Antoniadis-Hohenegger-Narain-Taylor])

2. $\mathcal{N} = 2$ Super Yang-Mills in Ω-background

Ω-background $=$ 6D gravitational background $+$ Wilson line

- 6D Ω-background metric (on $\mathbb{R}^4 \times \mathbf{T}^2$)

$$
\begin{aligned}
ds^2_{6\mathrm{D}} &= G_{MN} dx^M dx^N \\
&= 2 d\bar{z} dz + (dx^m + \bar{\Omega}^m dz + \Omega^m d\bar{z})^2, \\
\Omega^m &= \Omega^{mn} x_n, \quad \bar{\Omega}^m = \bar{\Omega}^{mn} x_n.
\end{aligned}
$$

Here constant antisymmetric matrix Ω^{mn} and $\bar{\Omega}^{mn}$ are of the form

$$
\Omega^{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & \epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\epsilon}_2 \\ 0 & 0 & \bar{\epsilon}_2 & 0 \end{pmatrix}.
$$

• R-symmetry Wilson line (gauging SU(2) R-symmetry)

$$\boldsymbol{A}^I{}_J = \bar{\mathcal{A}}^I{}_J dz + \mathcal{A}^I{}_J d\bar{z},$$
$$\bar{\mathcal{A}}^I{}_J,\ \mathcal{A}^I{}_J : \text{ constant}, \quad I,\ J : \text{ SU(2) indices.}$$

6D $\mathcal{N} = 1$ super Yang-Mills in Ω-background

$$\mathcal{L}_{6\mathrm{D}} = \sqrt{-G}\,\mathrm{Tr}\left[\frac{1}{4}G^{MP}G^{NQ}F_{MN}F_{PQ} + \frac{i}{2}\bar{\Psi}\Gamma^M\mathcal{D}_M\Psi\right].$$

KK reduction of $(z, \bar{z})$ direction $\to$

4D $\mathcal{N} = 2$ super Yang-Mills in Ω-background

• Lagrangian of $\mathcal{N}=2$ super Yang-Mills in Ω-background

$$
\begin{aligned}
\mathcal{L}_\Omega = \mathrm{Tr}\Big[& \frac{1}{4} F_{mn}F^{mn} + \Lambda^I \sigma^m D_m \bar{\Lambda}_I - \frac{i}{\sqrt{2}} g \Lambda^I [\bar{\varphi}, \Lambda_I] + \frac{i}{\sqrt{2}} g \bar{\Lambda}_I [\varphi, \bar{\Lambda}^I] \\
& + (D_m\varphi - g F_{mn}\Omega^n)(D^m\bar{\varphi} - g F^{mp}\bar{\Omega}_p) \\
& + \frac{1}{\sqrt{2}} g \bar{\Omega}^m \Lambda^I D_m \Lambda_I - \frac{1}{2\sqrt{2}} g \bar{\Omega}^{mn} \Lambda^I \sigma_{mn} \Lambda_I \\
& - \frac{1}{\sqrt{2}} g \Omega^m \bar{\Lambda}_I D_m \bar{\Lambda}^I + \frac{1}{2\sqrt{2}} g \Omega^{mn} \bar{\Lambda}_I \bar{\sigma}_{mn} \bar{\Lambda}^I \\
& + \frac{g^2}{2}\Big([\varphi, \bar{\varphi}] + i\Omega^m D_m \bar{\varphi} - i\bar{\Omega}^m D_m \varphi + i g \bar{\Omega}^m \Omega^n F_{mn} \Big)^2 \\
& - \frac{1}{\sqrt{2}} g \bar{\mathcal{A}}^J{}_I \Lambda^{\alpha I} \Lambda_{\alpha J} - \frac{1}{\sqrt{2}} g \mathcal{A}^J{}_I \bar{\Lambda}_{\dot{\alpha}}{}^I \bar{\Lambda}^{\dot{\alpha}}{}_J \Big].
\end{aligned}
$$

• unbroken supersymmetry (topological symmetry)

For general Ω^{mn} and $\bar{\Omega}^{mn}$, the Lagrangian has one unbroken supersymmetry if we choose

$$\mathcal{A}^I{}_J = -\frac{1}{2}\Omega_{mn}(\bar{\sigma}^{mn})^I{}_J, \quad \bar{\mathcal{A}}^I{}_J = -\frac{1}{2}\bar{\Omega}_{mn}(\bar{\sigma}^{mn})^I{}_J.$$

The corresponding supercharge $\bar{Q}_\Omega$ becomes a scalar after the topological twist. In particular $\bar{Q}_\Omega^2$ becomes

$$\bar{Q}_\Omega^2 = (\text{gauge transf. by } 2\sqrt{2}\varphi) + \underbrace{(\mathrm{U}(1)^2 \text{ rotation by } 2\sqrt{2}\Omega_{mn})}_{\text{equivariant deformation}}.$$

The action can be written as the $\bar{Q}_\Omega$-exact form.

$$\int d^4x\, \mathcal{L}_\Omega = \frac{8\pi^2}{g^2}k + \bar{Q}_\Omega \Xi. \quad (k\text{: instanton number})$$

3. Deformed Instanton Effective Action from String Theory

- setup

– $\mathcal{N}=2$ super Yang-Mills $\leftrightarrow$ N fractional D3-branes
on $\mathbb{R}^2 \times (\mathbb{R}^4/\mathbb{Z}_2)$

– instanton configration $\leftrightarrow$ k fractional D(-1)-branes on the D3's

– ADHM instanton moduli
$\leftrightarrow$ massless states from open strings attached to the branes:

D3-D(-1) $\quad$ ($N \times k$, $k \times N$ matrices)

$w_{\dot\alpha},\ \bar w^{\dot\alpha}$	"size" of instantons
$\mu^I,\ \bar\mu^I$	superpartner of $w_{\dot\alpha},\ \bar w^{\dot\alpha}$

D(-1)-D(-1) ($k \times k$ matrices)

a'_m	"position" of instantons
$\mathcal{M}'^{I}_{\alpha}$	superpartner of a'_m
$\chi,\ \bar{\chi},\ D^c\ (c = 1, 2, 3)$	bosonic auxiliary variables
$\bar{\psi}^{\dot{\alpha}}_I$	fermionic auxiliary variables

– D^c and $\bar{\psi}^{\dot{\alpha}}_I$ are respectively the Lagrange multipliers for the bosonic and fermionic ADHM constraints.

– Each variables have the corresponding vertex operators. For example,

$$V_{a'_m}(z) \sim a'_m \psi^m e^{-\phi}(z).$$

The effective action for D(-1)'s is calculated from the disk amplitudes of the vertex operators.

- effective action for D(-1)-branes (undeformed part)

$$
\begin{aligned}
S^0_{\rm eff} = 2\pi^2 \mathrm{tr}_k \Big[& -2[\bar\chi, a'_m][\chi, a'^m] - \frac{i}{2\sqrt{2}} \mathcal{M}'^{\alpha I}[\bar\chi, \mathcal{M}'_{\alpha I}] + \frac{i}{\sqrt{2}} \bar\mu^I \big(\mu_I \bar\chi - \bar\varphi^0 \mu_I\big) \\
& + \big(\bar\chi \bar w^{\dot\alpha} - \bar w^{\dot\alpha} \bar\varphi^0\big)\big(w_{\dot\alpha}\chi - \varphi^0 w_{\dot\alpha}\big) + \big(\chi \bar w^{\dot\alpha} - \bar w^{\dot\alpha}\varphi^0\big)\big(w_{\dot\alpha}\bar\chi - \bar\varphi^0 w_{\dot\alpha}\big) \\
& - i\bar\psi^{\dot\alpha}_I \big(\bar\mu^I w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot\alpha}]\big) + iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}\big(\bar w^{\dot\beta} w_{\dot\alpha} + \bar a'^{\dot\beta\alpha} a'_{\alpha\dot\alpha}\big) \Big],
\end{aligned}
$$

where $\varphi^0 = \langle\varphi\rangle$ and $\bar\varphi^0 = \langle\bar\varphi\rangle$ are VEVs of the scalars.

- R-R 3-form backgrounds

We neglect the $\mathbb{Z}_2$ orbifolding for a while ($\mathcal{N}=4$, SU(4)$_R$).
We introduce constant R-R field strength $\mathcal{F}^{\alpha\beta AB}$, $\mathcal{F}^{\dot\alpha\dot\beta}{}_{AB}$ and the corresponding vertex operators as $(A,B=1,\ldots,4)$

$$V_{\mathcal{F}^+}(z,\bar z)=(2\pi\alpha')\mathcal{F}^{\alpha\beta AB}S_\alpha S_A e^{-\frac{1}{2}\phi}(z)S_\beta S_B e^{-\frac{1}{2}\phi}(\bar z)$$
$$V_{\mathcal{F}^-}(z,\bar z)=(2\pi\alpha')\mathcal{F}^{\dot\alpha\dot\beta}{}_{AB}S_{\dot\alpha} S^A e^{-\frac{1}{2}\phi}(z)S_{\dot\beta} S^B e^{-\frac{1}{2}\phi}(\bar z)$$

R-R 3-form part is extracted as

$$\mathcal{F}^{(\alpha\beta)[AB]}=\mathcal{F}^+_{mna}(\sigma^{mn})^{\alpha\beta}(\Sigma^a)^{AB},$$
$$\mathcal{F}^{[\alpha\beta](AB)}=\mathcal{F}_{abc}\epsilon^{\alpha\beta}(\Sigma^{[a}\bar\Sigma^b\Sigma^{c]})^{AB}.$$

By orbifold projection, the remaining components are

$$\mathcal{F}^{+}_{mn} \sim \mathcal{F}^{(\alpha\beta)[12]}(\sigma_{mn})_{\alpha\beta}, \qquad \bar{\mathcal{F}}^{-}_{mn} \sim \mathcal{F}^{(\alpha\beta)[34]}(\sigma_{mn})_{\alpha\beta},$$

$$\mathcal{F}^{-}_{mn} \sim \mathcal{F}^{(\dot\alpha\dot\beta)}{}_{[34]}(\bar\sigma_{mn})_{\dot\alpha\dot\beta}, \qquad \bar{\mathcal{F}}^{-}_{mn} \sim \mathcal{F}^{(\dot\alpha\dot\beta)}{}_{[12]}(\bar\sigma_{mn})_{\dot\alpha\dot\beta},$$

$$\mathcal{F}^{(IJ)} \sim \epsilon_{\alpha\beta}\mathcal{F}^{[\alpha\beta](IJ)}, \qquad \mathcal{F}^{(I'J')} \sim \epsilon_{\alpha\beta}\mathcal{F}^{[\alpha\beta](I'J')},$$

$$\bar{\mathcal{F}}_{(IJ)} \sim \epsilon_{\dot\alpha\dot\beta}\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(IJ)}, \qquad \bar{\mathcal{F}}_{(I'J')} \sim \epsilon_{\dot\alpha\dot\beta}\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(I'J')},$$

where $I, J = 1, 2$, $I', J' = 3, 4$ and we choose $\mathcal{F}^{(IJ)} = \mathcal{F}^{(I'J')} = 0$ by hand. $\alpha' \to 0$ limit is taken such that $(2\pi\alpha')^{1/2}\mathcal{F} \equiv C$ is finite.

C^{+}_{mn} and $\bar{C}^{+}_{mn}$ are already introduced in [Billo-Frau-Fucito-Lerda].

• deformed effective action for D(-1)-branes

$$
\begin{aligned}
S_{\mathrm{eff}} = 2\pi^2 \mathrm{tr}_k \Big[& -2\Big([\bar\chi, a'_m] + (\bar C^+_{mn} + \bar C^-_{mn}) a'^n\Big)\Big([\chi, a'^m] + (C^{+mp} + C^{-mp}) a'_p\Big) \\
& - \frac{i}{2\sqrt{2}} \mathcal{M}'^{\alpha I}\Big([\bar\chi, \mathcal{M}'_{\alpha I}] + \frac{1}{2}\bar C^+_{mn}(\sigma^{mn})_\alpha{}^\beta \mathcal{M}'_{\beta I} - \frac{1}{2}\bar C^-_{mn}(\bar\sigma^{mn})^J{}_I \mathcal{M}'_{\alpha J}\Big) \\
& + \Big(\bar\chi \bar w^{\dot\alpha} - \bar w^{\dot\alpha}\bar\varphi^0 + \frac{1}{2}\bar C^-_{mn}(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}\Big)\Big(w_{\dot\alpha}\chi - \varphi^0 w_{\dot\alpha} + \frac{1}{2}C^-_{mn}(\bar\sigma^{mn})^{\dot\gamma}{}_{\dot\alpha} w_{\dot\gamma}\Big) \\
& + \Big(\chi \bar w^{\dot\alpha} - \bar w^{\dot\alpha}\varphi^0 + \frac{1}{2}C^-_{mn}(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}\Big)\Big(w_{\dot\alpha}\bar\chi - \bar\varphi^0 w_{\dot\alpha} + \frac{1}{2}\bar C^-_{mn}(\bar\sigma^{mn})^{\dot\gamma}{}_{\dot\alpha} w_{\dot\gamma}\Big) \\
& + \frac{i}{\sqrt{2}}\bar\mu^I\Big(\mu_I\bar\chi - \bar\phi\mu_I + \frac{1}{2}\bar C^J{}_I \mu_J\Big) \\
& - i\bar\psi^{\dot\alpha}_I\big(\bar\mu^I w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot\alpha}]\big) + iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}\big(\bar w^{\dot\beta} w_{\dot\alpha} + \bar a'^{\dot\beta\alpha} a'_{\alpha\dot\alpha}\big)\Big].
\end{aligned}
$$

The above agrees with the result from the field theory calculation if we identify $C_{mn} = -i\Omega_{mn}$, $\bar C_{mn} = -i\bar\Omega_{mn}$, $\bar C^I{}_J = -i\bar{\mathcal{A}}^I{}_J$.

- deformed supersymmetry

If we choose $\bar{\mathcal{A}}^I{}_J = -\frac{1}{2}\bar{\Omega}_{mn}(\bar{\sigma}^{mn})^I{}_J$, $S_{\rm eff}$ agrees with the deformed instanton effective action given by Nekrasov and preserves one supersymmetry which becomes a scalar after the topological twist ($\mathcal{M}'_{\alpha I} \to \mathcal{M}'_m$, $\mu^I \to \mu_{\dot{\alpha}}$, $\bar{\psi}^{\dot{\alpha}}_I \to \bar{\psi}, \ \bar{\psi}_{mn}$).

$$
\begin{aligned}
\bar{Q}_\Omega a'_m &= \mathcal{M}'_m, \quad & \bar{Q}_\Omega \mathcal{M}'_m &= 2\sqrt{2}i[\chi, a'_m] + 2\sqrt{2}\Omega_m{}^n a'_n,\\
\bar{Q}_\Omega w_{\dot{\alpha}} &= \mu_{\dot{\alpha}}, & \bar{Q}_\Omega \mu_{\dot{\alpha}} &= -2\sqrt{2}i(w_{\dot{\alpha}}\chi - \varphi^0 w_{\dot{\alpha}}) - \sqrt{2}\Omega^-_{mn}(\bar{\sigma}^{mn})^{\dot{\beta}}{}_{\dot{\alpha}} w_{\dot{\beta}},\\
\bar{Q}_\Omega \bar{w}^{\dot{\alpha}} &= \bar{\mu}^{\dot{\alpha}}, & \bar{Q}_\Omega \bar{\mu}^{\dot{\alpha}} &= 2\sqrt{2}i(\chi\bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\varphi^0) + \sqrt{2}\Omega^-_{mn}(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{w}^{\dot{\beta}},\\
\bar{Q}_\Omega \chi &= 0, & &\\
\bar{Q}_\Omega \bar{\chi} &= \bar{\psi}, & \bar{Q}_\Omega \bar{\psi} &= 2\sqrt{2}i[\chi, \bar{\chi}],\\
\bar{Q}_\Omega \bar{\psi}_{mn} &= D_{mn}, & \bar{Q}_\Omega D_{mn} &= 2\sqrt{2}i[\chi, \bar{\psi}_{mn}] + 2\sqrt{2}(\Omega_m{}^p\bar{\psi}_{pn} - \Omega_n{}^p\bar{\psi}_{pm}).
\end{aligned}
$$

Namely, $\bar{Q}_\Omega$ satisfies

$$\bar{Q}_\Omega^2 = (\mathsf{U}(k) \text{ transformation by } 2\sqrt{2}\chi) + \underline{(\mathsf{U}(1)^N \text{ rotation by } 2\sqrt{2}\varphi^0)}$$
$$+ \underbrace{(\mathsf{U}(1)^2 \text{ rotation by } 2\sqrt{2}\Omega_{mn})}_{\text{equivariant deformation}}.$$

$\bar{Q}_\Omega$ coincides with Nekrasov's equivariant BRST operator.

Moreover, S_{eff} can be written as the $\bar{Q}_\Omega$-exact form.

$\to$ The same instanton partiton function is obtained.

4. Summary

summary

1. We have reproduced the instanton effective action in general (two-parameter) Ω-background from the effective action of D(-1)-branes in D3/D(-1) system with the R-R 3-form.

2. The correspondence between R-R 3-form backgrounds and the Ω-background parameters is given as

$$\mathcal{F}^{(\alpha\beta)[IJ]},\ \mathcal{F}^{(\alpha\beta)[I'J']} \leftrightarrow \Omega^{+}_{mn},\ \bar{\Omega}^{+}_{mn} \qquad \text{known}$$

$$\mathcal{F}^{(\dot{\alpha}\dot{\beta})}{}_{[I'J']},\ \mathcal{F}^{(\dot{\alpha}\dot{\beta})}{}_{[IJ]} \leftrightarrow \Omega^{-}_{mn},\ \bar{\Omega}^{-}_{mn} \qquad \text{new result}$$

$$\mathcal{F}^{[\dot{\alpha}\dot{\beta}]}{}_{(IJ)}, \leftrightarrow \bar{A}_{IJ}(= -\tfrac{1}{2}\bar{\Omega}^{-}_{mn}(\bar{\sigma}^{mn})_{IJ}) \qquad \text{new result}$$

3. We have also checked that the deformed supersymmetry coincides with Nekrasov's equivariant BRST symmetry. The deformed instanton effective action can be written as the exact form with respect to the deformed supersymmetry. Hence we can obtain the same instanton partiton function.

future work

• $\mathcal{N} = 4$ (and $\mathcal{N} = 2^*$) case

Since we have no orbifolding, we can introduce more parameters.

parameters : Ω^{mna} $(a = 1, \ldots, 6)$ and R-symmetry Wilson lines

However we should preserve (at least) one supersymmetry.