Maximally Non-Abelian Vortices from Self-dual Yang-Mills Fields

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1 Introduction

Non-Abelian Vortex : plays an important role in

Dual Confinement

Cosmic String

Moduli gives **Effective Fields** on the soliton

Moduli Space describes **Dynamics of Non-Abelian Vortices**

Non-Abelian Vortices in $\boldsymbol{U(N)}$ gauge theory :

Moduli Matrix Approach

No Exact solutions

Exactly Solvable Vortex : $\boldsymbol{U(1)}$ **Vortex on a Hyperbolic Plane**

Equivalent to Instantons along a Line

Dimensional Reduction of Instantons to Hyperbolic Plane $\longrightarrow$ Vortices

Witten, Phys.Rev.Lett.**38**, 121 (1977)

Our Pourpose: Find Exactly Solutions of **Non-Abelian Vortices**

2 $SO(3)$ Invariant Instantons

Pure $SU(2)$ Gauge Theory in Euclidean 4 dimensions

Instantons as Solutions of **Self-duality** Equations

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$$

Instantons along a line (Let's call it τ axis)

Invariant under **Rotations $SO(3)$** around τ axis

($SU(2)$ gauge transformations can be accompanied)

Take spherical polar coordinates r, θ, ϕ for S^3

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$SO(3)$ invariant configurations : functions of τ, r (independent of θ, φ)

Complex coordinates (Stereographic projection of S^2)

$$z = \tau + ir, \quad y = \tan\frac{\theta}{2}e^{i\varphi}$$

$$ds^2 = dzd\bar{z} + (\mathrm{Im}z)^2\left(\frac{4}{(1+y\bar{y})^2}dyd\bar{y}\right)$$

Conformally equivalent to **hyperbolic plane** and sphere $\Sigma \times S^2$

$$ds^2 = \frac{(\mathrm{Im} z)^2}{2}\left(\frac{2}{(\mathrm{Im} z)^2} dz d\bar{z} + \frac{8}{(1+y\bar{y})^2} dy d\bar{y}\right)$$

Yang-Mills Theory and Self-Duality is **Conformally invariant**

$SO(3)$ invariant Instantons are equivalent to

$U(1)$ vortices on a hyperbolic plane Σ

Witten Ansatz : $SO(3)$ of S^2 is embedded into $SU(2)$

$$A_j^a = \frac{\varphi_2 + 1}{r^2}\epsilon_{jak}x_k + \frac{\varphi_1}{r^3}(\delta_{ja} - x_j x_a) + A_1 \frac{x_j x_a}{r^2}, \quad A_0^a = \frac{A_0 x^a}{r}$$

Only $U(1) \in SU(2)$ gauge symmetry is intact

A gauge transformation gives ($A_r = A_1, H = -\varphi_1 - i\varphi_2$)

$$\mathcal{A}_j = A_i(\tau, r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \tau, r,$$

$$\mathcal{A}_\theta = \begin{pmatrix} 0 & \bar{H}(\tau, r) \\ H(\tau, r) & 0 \end{pmatrix}, \quad \mathcal{A}_\varphi = \begin{pmatrix} -\cos\theta & -i\bar{H}(\tau, r)\sin\theta \\ iH(\tau, r)\sin\theta & \cos\theta \end{pmatrix}$$

$A_i(\tau, r)$: **2 Dimensional gauge fields** for $U(1)$ (I_3 of $SU(2)$)

$\boldsymbol{H(\tau, r)}$: **charged complex scalar** field

Self-Duality

$$\mathcal{F}_{\tau r} = \frac{1}{r^2 \sin\theta}\mathcal{F}_{\theta\varphi}, \quad \mathcal{F}_{\tau\theta} = \frac{1}{\sin\theta}\mathcal{F}_{\varphi r}, \quad \mathcal{F}_{r\theta} = \frac{1}{\sin\theta}\mathcal{F}_{\tau\varphi}$$

Reduces to **BPS equations for Vortices on a Hyperbolic Plane**

$$D_\tau H = i D_r H, \qquad F_{\tau r} = \frac{1}{2r^2}(1 - |H|^2)$$

3 General $SO(3)$ Invariant Gauge Fields

Metric on $\Sigma \times S^2$ ($\sigma = \frac{2}{(\mathrm{Im} z)^2}$, if Σ is the hyperbolic plane)

$$ds^2 = \sigma(z, \bar{z}) dz d\bar{z} + \frac{8}{(1 + y\bar{y})^2} dy d\bar{y}$$

Field configuration should be invariant under

a combined spatial $SO(3)$ rotation and gauge $SO(3)$ rotation

General Embedding of $SO(3)$ into Non-Abelian Group G

Isotropy generator $SO(2)$ is mapped to an $SO(2)$ generator Λ in $\mathcal{G}$

Most general $SO(3)$ invariant gauge potential

$$\mathcal{A}_z = \mathrm{A}_z(z, \bar{z}), \quad \mathcal{A}_{\bar{z}} = \mathrm{A}_{\bar{z}}(z, \bar{z})$$

$$\mathcal{A}_y = \frac{1}{1+y\bar{y}}(-\Phi(z,\bar{z}) - i\Lambda\bar{y}), \quad \mathcal{A}_{\bar{y}} = \frac{1}{1+y\bar{y}}(\bar{\Phi}(z,\bar{z}) + i\Lambda y)$$

$SO(2) = U(1)$ invariance (generators are anti-hermitian matrix)

$$[\Lambda, \mathbf{A}_z] = [\Lambda, \mathbf{A}_{\bar{z}}] = 0$$

$$[\Lambda, \Phi] = -i\Phi\,, \quad [\Lambda, \bar{\Phi}] = i\bar{\Phi}$$

Self-Duality ($\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$)

$$\mathcal{F}_{z\bar{y}} = 0, \quad \mathcal{F}_{\bar{z}y} = 0, \quad \frac{8}{(1+y\bar{y})^2}\mathcal{F}_{z\bar{z}} = \sigma \mathcal{F}_{y\bar{y}}$$

$$D_z\bar{\Phi} = 0, \quad D_{\bar{z}}\Phi = 0, \quad \mathbf{F}_{z\bar{z}} = \frac{\sigma}{8}\left(2i\Lambda - [\Phi, \bar{\Phi}]\right)$$

Finite Energy Solutions $\longrightarrow$ **Vacuum** ($\mathbf{F}_{z\bar{z}} = 0$) at $z \longrightarrow \infty$

Vacuum value Φ_0 of Φ forms $SO(3)$ algebra

$$[\Lambda, \Phi_0] = -i\Phi_0, \quad [\Lambda, \bar{\Phi}_0] = i\bar{\Phi}_0, \quad [\Phi_0, \bar{\Phi}_0] = 2i\Lambda$$

Boundary Condition at $r = \mathrm{Im}z = 0$: Fields approach vacuum values

4 Maximally Non-Abelian Vortices

Take $\boldsymbol{SU(2N)}$ gauge group : $\boldsymbol{\Lambda}$ can be taken in Cartan subalgebra

$$\boldsymbol{\Lambda = i \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_{2N} \end{pmatrix}, \quad \sum \Lambda_\alpha = 0}$$

$$\boldsymbol{[\Lambda, \Phi] = -i\Phi \longrightarrow \Lambda_\beta - \Lambda_\alpha = 1 \;\; \text{if} \;\; \Phi_{\alpha\beta} \neq 0}$$

Maximally Non-Abelian case

$$\boldsymbol{\Lambda = \frac{i}{2} \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}}$$

$$\boldsymbol{SU(2N) \longrightarrow SU(N) \times \widetilde{SU(N)} \times U(1)} \text{ gauge symmetry}$$

$\boldsymbol{SO(3)}$ **invariant** gauge fields on $\boldsymbol{\Sigma \times S^2}$

$$\boldsymbol{\mathcal{A}_z = \begin{pmatrix} A_z & 0 \\ 0 & \widetilde{A}_z \end{pmatrix}, \quad \mathcal{A}_{\bar{z}} = \begin{pmatrix} A_{\bar{z}} & 0 \\ 0 & \widetilde{A}_{\bar{z}} \end{pmatrix}}$$

$$\boldsymbol{\Phi = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \quad \bar{\Phi} = \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix}}$$

$A_z, (\tilde{A}_z)$: $SU(N)$ ($\widetilde{SU(N)}$) gauge field

H : A Higgs scalar in **Bi-fundamental** of $SU(N) \times \widetilde{SU(N)}$

Bogomolny equations for non-Abelian Vortices on Hyperbolic Plane

$$D_z H^\dagger = 0, \quad D_{\bar{z}} H = 0$$

$$F_{z\bar{z}} = \frac{\sigma}{8}\left(-1_N + H^\dagger H\right), \quad \widetilde{F}_{z\bar{z}} = \frac{\sigma}{8}\left(1_N - H H^\dagger\right)$$

$$D_{\bar{z}} H = \partial_{\bar{z}} H + \widetilde{A}_{\bar{z}} H - H A_{\bar{z}}, \quad F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$$

Vacuum Solutions

$$H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad A_z = 0\,, \quad \widetilde{A}_z = 0$$

Unbroken local gauge symmetry : $SU(N)_{\mathrm{d}}$ diagonal gauge group

If $SU(2N) \to SU(N_1) \times SU(N_2) \times U(1)$, $N_1 \neq N_2$,

$F_{z\bar{z}} = 0$ vacuum does not exist

Exact Vortex Solutions

$$\boldsymbol{H} = \begin{pmatrix} h^{(1)} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \boldsymbol{A}_{\bar z} = -\widetilde{\boldsymbol{A}}_{\bar z} = \begin{pmatrix} i a_{\bar z}^{(1)} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Bogomolny equations reduce to ($f_{z\bar z}^{(1)} = \partial_z a_{\bar z}^{(1)} - \partial_{\bar z} a_z^{(1)}$)

$$\partial_{\bar z} h^{(1)} - 2 i a_{\bar z}^{(1)} h^{(1)} = 0, \quad i f_{z\bar z}^{(1)} = \frac{\sigma}{8}\left(-1 + |h^{(1)}|^2\right)$$

= **Witten's equation for $U(1)$ vortices** on hyperbolic plane

Exactly solved by mapping to the Liouville equation

We found **exact solutions** in the diagonal $U(1)^N$ subgroup

Genuine **non-Abelian** vortices (fractional $U(1)$ and $SU(N)$ winding)

Solutions with complete orientational moduli remain to be worked out

Moduli Matrix and Master Equations

Solution of the first BPS equation

$$A_{\bar z} = S^{-1}\partial_{\bar z} S - \partial_{\bar z}\psi \mathbf{1}_N, \quad \widetilde{A}_{\bar z} = \widetilde{S}^{-1}\partial_{\bar z}\widetilde{S} + \partial_{\bar z}\psi \mathbf{1}_N$$

$$H(z,\bar{z}) = e^{\frac{1}{2}\psi(z,\bar{z})}\widetilde{S}^{-1}(z,\bar{z})H_0(z)S(z,\bar{z})$$

Moduli matrix $H_0(z)$, Master equations ($\Omega \equiv SS^\dagger$, $\widetilde{\Omega} \equiv \widetilde{S}\widetilde{S}^\dagger$)

$$\partial_z\partial_{\bar{z}}\psi = \frac{\sigma}{4}\left(-1 + \frac{1}{N}e^\psi \mathrm{Tr}(\widetilde{\Omega}^{-1}H_0\Omega H_0^\dagger)\right)$$

$$\partial_z(\Omega^{-1}\partial_{\bar{z}}\Omega) = \frac{\sigma}{8}e^\psi\left(H_0^\dagger\widetilde{\Omega}^{-1}H_0\Omega - \frac{1}{N}1_N\mathrm{Tr}(\widetilde{\Omega}^{-1}H_0\Omega H_0^\dagger)\right)$$

$$\partial_z(\widetilde{\Omega}^{-1}\partial_{\bar{z}}\widetilde{\Omega}) = -\frac{\sigma}{8}e^\psi\left(\widetilde{\Omega}^{-1}H_0\Omega H_0^\dagger - \frac{1}{N}1_N\mathrm{Tr}(\widetilde{\Omega}^{-1}H_0\Omega H_0^\dagger)\right)$$

5 Conclusion

1. $SO(3)$ symmetric instantons of $SU(2N)$ gauge group gives **non-Abelian vortices** on a **hyperbolic plane**.
2. **Maximally non-Abelian** case gives non-Abelian vortices in $SU(N) \times \widetilde{SU(N)} \times U(1)$ gauge group.
3. The maximally non-Abelian vortices possess **unbroken non-Abelian gauge symmetry** $SU(N)_\mathrm{d}$.
4. **Exact solutions** of $U(1)^N$ subgroup are completely obtained, but the **orientational moduli** remain to be worked out.